AMERICAN

Journal of Mathematics.

EDITED BY J. J. SYLVESTER.

PUBLISHED UNDER THE AUSPICES OF THE

JOHNS HOPKINS UNIVERSITY.

Πραγμάτων ἔλεγχος οὐ βλεπομένων.

VOLUME V.

BALTIMORE: PRESS OF ISAAC FRIEDENWALD.

AGENTS:

B. WESTERMANN & CO., *New York.*
D. VAN NOSTRAND, *New York.*
E. STEIGER & CO., *New York.*
FERREE & CO., *Philadelphia.*
CUSHINGS & BAILEY, *Baltimore.*
A. WILLIAMS & CO., *Boston.*

TRÜBNER & CO., *London.*
GAUTHIER-VILLARS, *Paris.*
A. ASHER & CO., *Berlin.*
MAYER & MÜLLER, *Berlin.*
ULRICO HOEPLI, *Milan.*
KARL J. TRÜBNER, *Strassburg.*

1882.

INDEX.

ERRATA.

TABLES OF THE SYMMETRIC FUNCTIONS OF THE TWELFTHIC.

Page	line	column	for	read
47	$(5^2 1^2)$	$[2^6]$	60	66
50	$(2^4 1^4)$	$[521^5]$	530640	53640
51	$(3^2 2^2 1^2)$	$[4^2 21^2]$	370	376
51	$(3^2 21^4)$	$[4^2 2^2]$	420	424
51	$(2^4 1^4)$	$[43^2 2]$	7398	7308
55	$[4^2 21^2]$	(63^2)	9	-9
58	$[6^2$	(532^2)	-19	-18

SOME ELLIPTIC FUNCTION FORMULÆ.

r equation 1 read

$$\frac{\ldots \operatorname{sn}^n u}{\ldots u^2} = n(n-1)\operatorname{sn}^{n-2}u - n^2(1+k^2)\operatorname{sn}^n u + n(n+1)k^2\operatorname{sn}^{n+2}u.$$

t du under first integral sign.

A CONSTRUCTIVE THEORY OF PARTITIONS.

After * * insert *.

oot. For *untractile* read *contractile.*

ot of text. For *and that those* read *and those.*

owing the words "we obtain the equation" for $1+ax.1+ax$

d $1+ax.1+ax^4$.

age *dele* $1+ax^3$. preceding $x^{18}a^4$.

nber of insert *improper fractions with.*

Between *into* and *termed* insert *what I have elsewhere.*

For 2^θ read 2θ.

uracy read *precision,* and for *method* read *result.*

$\frac{\ldots 2}{\ldots 2.3} + \frac{12}{3.5}$ read $\frac{n}{2.3} + \frac{n}{3.5}$.

Page 306, line 10 from foot. For *lemma* read *the remark made.*
" " lines 8 and 12 from foot. For S_j read $S_{j'}$.
" 325, Paragraph 3 is quite unintelligible as it stands and will be corrected hereafter.
" 330, lines 2 and 4 below the diagram. For the words following *consequently* and preceding *be a*, substitute *no similar contour obtained by treating any one of the three nodes which it contains as a centre of similitude will.*
" " line 5 below the diagram. After the word *origin* insert *in such contour.*
" " lines 6 and 5 from foot. *Dele* from *so* to *sign* inclusive and supply what follows as a parenthesis: (Points in a plane arranged in any order of sequence, such that the successive determinants formed by their trilinear coordinates are of uniform sign, are said to be in a normal order. Rays of a conical pencil arranged in any order of sequence, such that their intersections by a plane satisfy the above condition, are also said to be in a normal order: see privately printed syllabus of my lectures on Partitions, 1857, or M. Halphen's theory of *Aspects.*)

ON NON-EUCLIDEAN PROPERTIES OF CONICS.

Page 375, line 10 before *conics,* line 11 before *ellipse,* and line 12 before *circle,* insert *rea*

DI UN NUOVO TEOREMA, ETC.

Page 382, for *Dominico* read Domenico.

Quelques Applications de la Théorie des Formes Binaires aux Fonctions Elliptiques.

BY M. FAÀ DE BRUNO, *Turin.*

Après avoir pris connaissance, presque à l'hasard, d'une note de M. Klein sur les fonctions elliptiques, insérée dans les *Mathematische Annalen*, il m'est venue la pensée qu'on pourrait d'une façon élémentaire arriver aux résultats de M. Klein sur l'emploi de l'invariant absolu dans la détermination des éléments elliptiques. J'espère avoir atteint le but sans passer par les séries hypergéométriques peu connues, et qui du côté pratique laissent beaucoup à désirer dans le cas qui nous occupe. Ce qui était pour ainsi dire entrevu par de grands génies, trouve ici une réalité facile d'exécution. Il m'est arrivé dans cette recherche de trouver des séries que je croyais nouvelles, et que ensuite j'ai rencontrée dans quelques auteurs; il en reste pourtant quelques nouvelles. Ce qui peut-être pourra paraître intéressant, c'est une série d'une convergence vertigineuse qui donne à l'instant la valeur de la période une fois calculée la valeur du module, pour n'importe quelle quartique par l'emploi de l'invariant absolu.

§ I.

De l'Invariant absolu dans les Formes Quartiques.

Toute quartique peut être reduite par une transformation linéaire à la forme canonique

$$x^4 + 6\mu x^2 y^2 + y^4. \tag{1}$$

Ainsi en appellant I_2, I_3, Δ, les invariants quadratiques, cubiques, et le discriminant de la quartique donnée, et en observant que ceux de la quartique (1) sont $1+3\mu^2$, $\mu-\mu^3$, $(1-9\mu^2)^2$, il viendra

(2) . . . . $I_2 = (1 + 3\mu^2)\delta^4,\ I_3 = (\mu - \mu^3)\delta^6,\ \Delta = I_2^3 - 27 I_3^2 = (1 - 9\mu^2)^2 \delta^{12}$,
δ designant le module de la transformation. Nous avons dejà vu (*American Journal*, Tome III) que μ peut s'exprimer en fonction de l'invariant absolu. Dans les recherches actuelles nous préciserons davantage le rôle que joue l'invariant absolu. Rappellons nous à cet effet que la canonisante de la quartique est

(3) $$\lambda^3 - I_2\lambda + 2I_3 = 0,$$

équation, qui, ayant égard aux valeurs (2), peut être mise sous la forme

$$\lambda^3 - \lambda\delta^4(1 + 3\mu^2) + 2(\mu - \mu^3)\delta^6 = 0.$$

Il s'ensuit, que en posant $\lambda = \lambda'\delta^2$, il viendra

(4) $$\lambda'^3 - \lambda'(1 + 3\mu^2) + 2(\mu - \mu^3) = 0;$$

dont les racines sont, comme nous avons fait voir en 1875 (*Théorie des formes binaires*),

(5) $$\lambda'_1 = 2\mu,\ \lambda'_2 = 1 - \mu,\ \lambda'_3 = -(1 + \mu);$$

ainsi les racines de (3) seront

(6) $$\lambda_1 = 2\mu\delta^2,\ \lambda_2 = (1 - \mu)\delta^2,\ \lambda_3 = -(1 + \mu)\delta^2.$$

Or on a par les valeurs (2)

(7) $$\frac{\mu - \mu^3}{1 + 3\mu^2}\delta^2 = \frac{I_3}{I_2},$$

et les 3 racines λ deviendront

(8) $$\lambda_1 = 2\frac{I_3}{I_2}\frac{1 + 3\mu^2}{1 - \mu^2},\ \lambda_2 = \frac{I_3}{I_2}\frac{1 + 3\mu^2}{\mu(1 + \mu)},\ \lambda_3 = -\frac{I_3}{I_2}\frac{1 + 3\mu^2}{\mu(1 - \mu)};$$

observont maintenant que des équations (2) on deduit que

(9) $$\frac{\mu(\mu^2 - 1)}{9\mu^2 - 1} = B.\ \text{où } B = \sqrt{\frac{I_3^2}{\Delta}};\ \text{ou } \mu^3 - 9B\mu^2 - \mu + B = 0,$$

équation qui peut se mettre sous la forme

(10) $$\mu\frac{\mu - 1}{3\mu + 1}\frac{\mu + 1}{3\mu - 1} = B.$$

Sous cette forme il est assez naturel de considerer μ, $\frac{\mu - 1}{3\mu + 1}$, $\frac{\mu + 1}{3\mu - 1}$ comme les 3 racines, puisque le premier membre est au signe près le produit aussi des racines, et de songer à substituer $\frac{\mu - 1}{3\mu + 1}$ à μ; et alors on voit que l'équation (10)

est satisfaite tout de même, car on a

$$\frac{\frac{\mu-1}{3\mu+1}-1}{3\,\frac{\mu-1}{3\mu+1}+1}=-\frac{\mu+1}{3\mu-1}, \qquad \frac{\frac{\mu-1}{3\mu+1}+1}{3\,\frac{\mu-1}{3\mu+1}-1}=-\mu\,;$$

d'où il s'ensuit, le produit demeurant le même, que si μ est une racine, $\frac{\mu+1}{3\mu-1}$, $\frac{\mu-1}{3\mu+1}$ le seront pareillement. On verra ainsi que on peut poser

$$\lambda_2=2\,\frac{I}{I_2}\,\frac{1+3\left(\frac{\mu+1}{3\mu-1}\right)^2}{1-\left(\frac{\mu-1}{3\mu+1}\right)^2}=\frac{I}{I_2}\,\frac{1+3\mu^2}{\mu(1+\mu)}, \tag{11}$$

$$\lambda_3=2\,\frac{I_3}{I_2}\,\frac{1+3\left(\frac{\mu+1}{3\mu-1}\right)^2}{1-\left(\frac{\mu+1}{3\mu-1}\right)^2}=\frac{I_3}{I_2}\,\frac{1+3\mu^2}{\mu(\mu-1)}. \tag{12}$$

Réciproquement, en posant généralement $\alpha_i=\frac{\lambda_i}{2}\,\frac{I_2}{I_3}$, on deduira de (11), (12) que

$$\mu^2=\frac{\alpha_1-1}{\alpha_1-3},\quad \left(\frac{\mu-1}{3\mu+1}\right)^2=\frac{\alpha_2-1}{\alpha_2+3},\quad \left(\frac{\mu+1}{3\mu-1}\right)^2=\frac{\alpha_3-1}{\alpha_3+3}, \tag{13}$$

d'où

$$\mu^2=\sqrt{\frac{I_2\lambda_1-2I_3}{I_2\lambda_1+6I_3}},\quad \frac{\mu+1}{3\mu-1}=\sqrt{\frac{I_2\lambda_2-2I_3}{I_2\lambda_2+6I_3}},\quad \frac{\mu-1}{3\mu+1}=\sqrt{\frac{I_2\lambda_3-2I_3}{I_2\lambda_3+6I_3}}, \tag{14}$$

et puisque en dernier lieu tout depend de la valeur de μ, les 3 racines μ_1, μ_2, μ_3 seront encore

$$\mu_1=\sqrt{\frac{I_2\lambda_1-2I_3}{I_2\lambda_1+6I_3}},\quad \mu_2=\frac{1+\sqrt{\frac{I_2\lambda_2-2I_3}{I_2\lambda_2+6I_3}}}{1-3\sqrt{\frac{I_2\lambda_2-2I_3}{I_2\lambda_2+6I_3}}},\quad \mu_3=\frac{1+\sqrt{\frac{I_2\lambda_3-3I_3}{I_2\lambda_3+6I_3}}}{1+3\sqrt{\frac{I_2\lambda_3-2I_3}{I_2\lambda_3+6I_3}}}. \tag{15}$$

Les racines λ de l'équation canonisante sont liées avec celles de la quartique d'une manière remarquable comme a montré M. Hermite.* En les appelant

---

* Crelle, T. 52.

$\alpha, \beta, \gamma, \delta$, et designant par a_0 le premier coefficient de la quartique, on a

$$\lambda_1 = \frac{a_0}{6}[(\alpha-\delta)(\beta-\gamma)-(\alpha-\gamma)(\beta-\delta)],$$

$$(17)\qquad \lambda_2 = \frac{a_0}{6}[(\alpha-\beta)(\delta-\gamma)+(\alpha-\gamma)(\delta-\beta)],$$

$$\lambda_3 = \frac{a_0}{6}[(\alpha-\delta)(\gamma-\beta)+(\alpha-\beta)(\gamma-\alpha)];$$

et si l'on pose

$$(18)\quad l_1 = a_0(\alpha-\beta)(\gamma-\delta),\ l_2 = a_0(\alpha-\delta)(\beta-\gamma),\ l_3 = a_0(\alpha-\gamma)(\delta-\beta),$$

il viendra

$$(19)\qquad \lambda_1 = \frac{1}{6}(l_2-l_3),\ \lambda_2 = \frac{1}{6}(l_3-l_1),\ \lambda_3 = \frac{1}{6}(l_1-l_2),$$

et

$$(20)\qquad \lambda_2-\lambda_3 = -\frac{1}{2}l_1,\quad {}_1-\lambda_3 = -\frac{1}{2}l_2,\ \lambda_1-\lambda_2 = -\frac{1}{2}l_3.$$

On observera que les λ et les l sont liées par le relations suivantes:

$$(21)\qquad \lambda_1+\lambda_2+\lambda_3 = 0,\quad l_1+l_2+l_3 = 0,$$

d'où

$$\Sigma\lambda_1\lambda_2 = -\frac{1}{2}\Sigma\lambda^2,\ \Sigma l_1 l_2 = -\frac{1}{2}\Sigma l^2.$$

Rappellons (v. *Théorie des formes binaires*) qu'on a

$$(22)\quad I_2 = \frac{a_0^2}{24}[(\alpha-\beta)^2(\gamma-\delta)^2+(\alpha-\gamma)^2(\beta-\gamma)^2] = \frac{1}{24}(l_1^2+l_2^2+l_3^2),$$

ce qu'on pourra vérifier pour l'équation canonisante. En effet on doit avoir

$$(22)\qquad I_2 = -\frac{1}{36}(\lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_2\lambda_3) = \frac{1}{36}\,\frac{1}{2}\Sigma\lambda^2$$

$$= \frac{1}{36}(l_1^2+l_2^2+l_3^2-l_1l_2-l_1l_3-l_2l_3) = \frac{1}{24}(l_1^2+l_2^2+l_3^2).$$

Pareillement on aura

$$I_3 = -\frac{1}{632}(l_1-l_2)(l_1-l_3)(l_3-l_1),\ 2I_3 = -\lambda_1\lambda_2\lambda_3.$$

Il viendra aussi pour le discriminant

$$\Delta = \frac{a_0^6}{4^4}\Pi(\alpha-\beta),\ \Delta = \frac{1}{4^4}l_1^2l_2^2l_3^2;$$

d'où

$$(23)\qquad l_1l_2l_3 = 16\sqrt{\Delta}.$$

D'ailleurs

$$l_1 l_2 + l_2 l_3 + l_1 l_3 = -\frac{1}{2}\Sigma l^2 = -12 I_2\,;$$

ainsi
$$\frac{1}{4}\Sigma l_1 l_2 = -3 I_2\,;$$

et puisque $\Sigma l = 0$, il s'ensuit que l'équation dont $\frac{l_1}{2}$, $\frac{l_2}{2}$, $\frac{l_3}{2}$ sont la racines aura la forme

(24)
$$L^3 - 3 I_2 L - 2\sqrt{\varDelta} = 0\,,$$

dont le discriminant $\mathfrak{D}$ par rapport aux racines sera

$$\mathfrak{D} = -4(\Delta - I_2^3) = +108 I_3^2\,.$$

Pour se rendre compte de ce resultat il faut noter que le produit des carrés des differences des racines pour la cubique à coefficients binôme est $\{(ad - bc)^2 - 4(ac - b^2)(bd - c^2)\}$; c'est à dire—le discriminant. Ce resultat nous prouverait au premier abord que l'équation (24) a toutes ses racines réelles, ce qui est impossible, car les racines α, β, γ, δ de la quartique ne sont pas toujours réelles. Mais observons que cette deduction ordinaire tirée du signe du discriminant n'est lêgitime, qu'autant que le coefficients sont réels, ce qui n'est pas le cas lorsque la quartique n'a qu'un couple de racines imaginaires. Dans ce cas les 3 racines l ont la forme

$$l_1 = Ai;\ l_2 = C + Di,\ l_3 = -(C - Di);$$

d'où

$$(l_1 - l_2)^2 (l_2 - l_3)^2 (l_3 - l_1)^2 = (2C)^2 \{-C + (A - D)i\}^2 \{-C + (D - A)i\}^2;$$

ainsi le discriminant sera positif, quoique les racines de la cubique soient toutes fois imaginaires; mais alors aussi Δ est negatif, $= -\Delta'$, Δ' étant essentiellement positif, et l'équation (24) devient

$$L^3 - 3 I_2 L - 2i\sqrt{\varDelta'} = 0.$$

Mais si nous changeons L in $\frac{L'}{i}$, il viendra

(24)'
$$L'^3 + 3 I_2 L' + 2\sqrt{\varDelta'} = 0\,,$$

dont les 3 racines seront

$$l'_1 = -A\,,\quad l'_2 = -D + Ci\,,\quad l'_3 = -D - Ci\,,$$

et il faudra que le discriminant $\mathfrak{D}'$ de (24)' soit negatif, ce qui arrive, car on a

$$\mathfrak{D}' = -L(\Delta' + I_2^3) = -L(-\Delta + I_2^3) = -108 I_3^2.$$

Quant aux racines λ, à cause des relations (19), on voit que les λ seront réelles pour le cas des racines toutes réelles ou toutes imaginaires, et que pour le cas d'un seul couple de racines imaginaires de la quartique on a

$$\lambda_1 = \frac{1}{6}(2C), \ \lambda_2 = \frac{1}{6}(-C + (\Delta - D)i), \ \lambda_3 = \frac{1}{6}(-C - (\Delta - D)^2);$$

et alors la cubique (3) aura aussi un couple de racines imaginaires. Dans ce cas en effet le discriminant est

$$\mathfrak{D} = -\left[4I_3^2 - 4\frac{I_2^3}{27}\right] = \frac{4}{27}(I_2^3 - 27I_3^2) = \frac{4}{27}\Delta,$$

et il sera negatif avec Δ.

En résumé, donc, les racines λ et l seront toujours conjointement réelles si les racines de la quartique sont toutes réelles ou imaginaires ; λ et $l\sqrt{-1}$ auront un couple de racines imaginaires si la quartique a un couple de racines imaginaires, ou si $\Delta < 0$, à savoir $\frac{I_2^3}{I_3^2} < 27$, ou $I < 27$, en appelant I l'invariant absolu $\frac{I_2^3}{I_3^2}$.

Cela posé, pour la determination des racines λ et l nous aurons recours aux formules données dans la *Théorie des formes binaires*, p. 113. En remarquant que le discriminant de la canonisante par rapport aux coefficients est $= -\frac{4}{27}(I_2^3 - 27I_3^2) = -\frac{4}{27}\Delta$, ou bien $4I_3^2\left(1 - \frac{I}{27}\right)$, nous trouverons que

$$\lambda = \sqrt[3]{I_3}\left(\sqrt[3]{-1 + \sqrt{1 - \frac{I}{27}}} + \sqrt[3]{-1 - \sqrt{1 - \frac{I}{27}}}\right)$$

Par conséquent, en posant

$$\lambda = \sqrt[3]{I_3}\, y, \tag{27}$$

la valeur (14) de μ^2 deviendra

$$\mu^2 = \frac{\sqrt[3]{I}\, y - 2}{\sqrt[3]{I}\, y + 6} \tag{28}$$

resultat qui nous apprend que le paramètre μ de la quartique canonique ne depend que de l'invariant absolu I et en fournit une expression très simple. Pareillement on trouvera

$$L = \sqrt[6]{27}\ \sqrt[3]{I_3}\left(\sqrt[3]{1 + \sqrt{\frac{I}{27} - 1}} + \sqrt{-1 + \sqrt{\frac{I}{27} - 1}}\right) \tag{29}$$

Nous verrons maintenant comment on peut developper les racines λ et l en série suivant les puissances ascendantes de l'invariant absolu. Observons d'abord qu'en posant

(30) $$\lambda = \sqrt[3]{2I_3}\, x$$

l'équation (3) devient

$$x^3 = \sqrt[3]{\frac{I_2^3}{4I_3^2}}\, x + 1\,,$$

ou en rappellant la valeur

$$I = \frac{I_2^3}{I_3^2}\,,$$

(31) $$x^3 - \sqrt[3]{\frac{I}{4}}\, x + 1 = 0\,.$$

Ainsi toute canonisante d'une quartique peut-être reduite à ne dépendre que de l'invariant absolu de la quartique. Ce resultat nous suggère une reflexion. Supposant qu'on ait construit des tables des racines de la cubique $x^3 - ax + 1$ (ce que p. ex. la *British Association* pourrait très bien entreprendre) pour une certain étendue des valeurs de a, on aurait alors tout a la fois les racines de toute cubique par un simple transformation linéaire et les racines de la quartique canonique à l'aide de la formule (28) par une simple resolution d'une biquadratique.

Posons maintenant pour abréger $\frac{I}{4} = T$, l'équation (31) pourra être mise sous la forme

(32) $$x = \sqrt[3]{T} + \sqrt[3]{Tx^3};$$

et en y appliquant la formule de Lagrange* pour les équations trinômes, il viendra

(33) $$x = \sqrt[3]{T}\left[1 + T + \frac{6}{1.2}T^2 + \sum_{p=2}^{p=\infty} \frac{3p(3p-1)\ldots(3p-p+2)}{1.2.3\ldots p}\, 2^{2p+1} T^p\right].$$

Pour la convergence de la série il faudra d'après des théorèmes connus que

$$\sqrt[3]{T} < \frac{4}{27\sqrt[3]{T^2}}\,, \text{ ou } I > 27\,.$$

Le cas correspond à celui de racines réelles dans la canonisante. En effet le discriminant, qui comme nous savons est égal $-\frac{4}{27}(I_2^3 - 27I_3^2)$, doit être alors > 0. Or l'inégalité

(34) $$I_2^3 - 27I_3^2 > 0$$

* Voir la note 1e à la fin.

révient à $I > 27$. Alors le developpement (33), ou

$$(35)\quad \frac{\lambda}{\sqrt[3]{2I_3T}} = 1 + T + \frac{6}{1.2}T^2 + \sum_{p=2}^{p=\infty} \frac{3p(3p-1)\ldots(3p-p+2)}{1.2.3\ldots p} 2^{2p+1} T^{2p},$$

correspondra à la racine réelle du plus petit module. Posons

$$(36)\quad P = 1 + T + \frac{6}{1.2}T^2 + \ldots$$

il sera aisé de voir que l'on aura, en se reportant à la valeur (14) de μ^2,

$$(37)\quad \mu^2 = 2\frac{P\sqrt[3]{T^2} - 1}{P\sqrt[3]{T^2} + 6}.$$

Ainsi μ^2 est donné explicitement en fonction de l'invariant absolu T. Passons au cas où le discriminant est negatif, et où par consequent $I < 27$. Alors, pour abréger, considérons l'équation

$$y^3 - ay + 1 = 0,\text{*}$$

(38) et posons $y = a\sqrt[3]{z} - 1 = 0$, ou $z = 1 + a\sqrt[3]{z}$. D'après des théorèmes connus la condition de convergence de la série de Lagrange appliquée à la forme $z = 1 + a\sqrt[3]{z}$ sera

$$a < \frac{\left(-\frac{2}{3}\right)^{-\frac{2}{3}}}{\left(\frac{1}{3}\right)^{\frac{1}{3}}}, \text{ ou } a < \sqrt[3]{\frac{27}{4}}.$$

Mais dans notre cas on a

$$a = \sqrt[3]{\frac{I}{4}};$$

donc $I < 27$, comme il fallait. La série de Lagrange donnerait

$$(39)\quad z = 1 + a + \frac{\frac{2}{3}}{1.2}a^2 + \sum_{p=4}^{p=\infty} \frac{\frac{p}{3}\left(\frac{p}{3} - 1\right)\ldots\left(\frac{p}{3} - p + 2\right)}{1.2.3\ldots p} a^p;$$

mais $z = 1 + a\sqrt[3]{z}$; donc

$$(40)\quad y = 1 + \frac{1}{3}a + \sum_{p=4}^{p=\infty} \frac{\frac{p}{3}\left(\frac{p}{3} - 1\right)\ldots\left(\frac{p}{3} - p + 2\right)}{1.2.3\ldots p} a^p;$$

---

* Voir la note 2^e^ à la fin.

et il est aisé de voir que pour les valeurs de p multiple de 3, il y aura des termes qui disparaîtront. En particulier on aura

$$y = 1 + \frac{1}{3}\alpha - \frac{1}{81}\alpha^3 + \frac{1}{35}\alpha^4 - \frac{4}{38}\alpha^6 + \frac{5}{39}\alpha^7 + \ldots \tag{41}$$

et si on remplace y par sa valeur et α par $\sqrt[3]{\frac{I}{4}}, = \sqrt[3]{T}$, il viendra

$$\frac{\lambda}{\sqrt[3]{2I_3}} = 1 - \frac{1}{81}T - \frac{4}{81^2}T^2 + \ldots + \frac{1}{3}\sqrt[3]{T}\left[1 + \frac{1}{81}T + \frac{5}{81^2}T^2 + \ldots\right]; \tag{42}$$

série très convergente, en se rappellant que $I = 4T < 27$, et qui fournira encore la racine du plus petit module. Il s'ensuit qu'en posant

$$Q = 1 - \frac{1}{81}T - \frac{4}{81^2}T^2 + \ldots + \frac{1}{3}\sqrt[3]{T}\left(1 + \frac{1}{81}T + \frac{5}{81^2}T^2 + \ldots\right) \tag{43}$$

on trouvera

$$\mu^2 = \frac{Q\sqrt[3]{4T} - 2}{Q\sqrt[3]{4T} + 6}, \tag{44}$$

et μ^2 sera pareillement exprimé explicitement en fonction de l'invariant absolu T.

Passons à l'équation en L (24). En posant

$$L = \sqrt{2}\,\sqrt[6]{\Delta}\,x \tag{45}$$

elle pourra être mise sous la forme

$$x^3 - \frac{3I_2}{\sqrt[3]{4}\,\sqrt[3]{\Delta}}x - 1 = 0 \tag{46}$$

Posons

$$\alpha = \frac{\sqrt[3]{4}\,\sqrt[3]{\Delta}}{3I_2} \text{ et } x = -y,\ \alpha = \sqrt[3]{\frac{4}{27}}\sqrt{1 - \frac{27}{I}}, \tag{46'}$$

et appliquons la série de Lagrange à l'équation

$$y = \alpha + \alpha y^3; \tag{47}$$

on aura

$$\frac{y}{\alpha} = 1 + \alpha^3 + \frac{6}{1.2}\alpha^6\,\frac{9.8}{1.2.3}\alpha^9 + \ldots + \frac{3p(3p-1)\ldots(3p-p+2)}{1.2.3\ldots p}\alpha^{3p} + \ldots \tag{48}$$

série, qui en posant

$$H = \frac{\Delta}{I_2^3}, \quad \alpha^3 = \frac{4}{27}H, \tag{49}$$

deviendra

$$\frac{y}{\alpha} = 1 + \frac{4}{27}H + \frac{6}{2}\left(\frac{4}{27}H\right)^2 + \frac{9.8}{2.3}\left(\frac{4}{27}H\right)^3 + \ldots = T. \tag{50}$$

Maintenant on a

$$L=\frac{l}{2}=-\sqrt[3]{2}\,\sqrt[6]{\Delta},\ y=-\frac{2}{3}\frac{\sqrt[3]{\Delta}}{I_2},\ \sqrt[6]{\Delta}\ T=-\frac{2}{3}\frac{\sqrt{\Delta}}{I_2}\,T.$$

Ainsi

$$(51)\quad l=-\frac{4}{3}\sqrt[3]{\Delta}\ \sqrt[3]{H}\left(1+\frac{4}{27}H+\frac{6}{2}\left(\frac{4}{27}H\right)^2+\frac{9.8}{2.3}\left(\frac{4}{27}H\right)^3+\ldots\right)$$

où H est un autre invariant absolu lié avec le premier I par la relation

$$H=1-\frac{27}{I}.$$

Pour la condition de convergence il faut que

$$\alpha<\frac{4}{27\alpha^2};\ \text{ou}\ \Delta<I_2^3.$$

Mais comme $\Delta=I_2^3-27I_3^2$, cette condition se reduit à

$$1-\frac{27}{I}<1;$$

et par conséquent $I>27$. Or, dans ce cas nous avons vu que le discriminant de la canonisante est positif, et en ayant égard aux relations (19, 20), les racines l seront toutes réelles. Lorsque $I<27$, les racines de la canonisante sont imaginaires et l'équation (24)', en posant

$$L'=\sqrt[3]{2}\,\sqrt[6]{\Delta'}\,y,\ a=\frac{3I_2}{\sqrt[3]{4}\,\sqrt[3]{\Delta}}$$

deviendra

$$y^3-ay+1=0,$$

équation semblable à la (38) et qui fournira la série (40),

$$y=\frac{1}{3}+\frac{1}{3}a+\sum\frac{\frac{p}{3}\left(\frac{p}{3}-1\right)\ldots\left(\frac{p}{3}-p+2\right)}{1.2.3\ldots p}a^{p-1},$$

qui, developpée, deviendra

$$y=1\ \frac{1}{81}a^3-\frac{4}{38}a^6+\ldots+a\left\{\frac{1}{3}+\frac{1}{35}x^3+\frac{5}{39}a^6+\ldots\right\},$$

où $a^3=\frac{27}{4}k$, $k=\frac{I_2^3}{\Delta}=\frac{1}{H}\cdot$ Par conséquent il viendra

$$(51)'\quad \frac{l\sqrt{-1}}{\sqrt[3]{16\sqrt{\Delta}}}=1-\frac{1}{81}\left(\frac{27}{4}k\right)-\frac{4}{81^2}\left(\frac{27}{4}k\right)^2+\ldots$$

$$+\sqrt[3]{\frac{27}{4}k}\left(\frac{1}{3}+\frac{1}{3^5}\left(\frac{27}{4}k\right)+\frac{5}{3^9}\left(\frac{27}{4}k\right)^2+\ldots\right);$$

et on trouvera que la condition pour la convergence de la série se reduit à l'inégalité $I < 27$ comme pour (38). Ainsi dans tous les cas les expressions de λ et l sont connues et exprimées par des invariants absolus.

Nous verrons dans le paragraphe suivant la partie avantageuse que l'on peut tirer des expressions des racines l. Mais avant de passer outre, nous remarquerons que les fonctions elliptiques nous offrent un moyen de verification des formules (39) et (44). On sait que si dans une équation differentielle elliptique ordinaire, où figure une quartique quelconque dont les racines sont $\alpha, \beta, \gamma, \delta$, on opére une transformation linéaire* pour reduire la quartique à la forme typique, $(1-x^2)(1-k^2x^2)$, k le module aura cette expression

$$= \frac{\sqrt{(\alpha-\delta)(\gamma-\beta)} - \sqrt{(\alpha-\gamma)(\delta-\beta)}}{\sqrt{(\alpha-\delta)(\gamma-\beta)} + \sqrt{(\alpha-\gamma)(\delta-\beta)}};$$

d'où l'on deduira, en égard aux équations (20),

$$k + \frac{1}{k} = 2\frac{l_2 - l_3}{l_2 + l_3} = \frac{1.2\lambda_1}{-l_1} = 6\frac{\lambda_1}{\lambda_2 - \lambda_3}.$$

Or par les valeurs (6) il vient immédiatement

$$6\frac{\lambda_1}{\lambda_2 - \lambda_3} = 6\mu,$$

ou

$$k + \frac{1}{k} = 6\mu;$$

comme cela doit être, puisque de la quartique $(1-x^2)(1-k^2x^2)$ on passe à la canonique $1 + 6\mu x^2 + x^4$ en posant $kx^2 = y^2$ et $k + \frac{1}{k} = 6\mu$. Ainsi les systemes, que l'on choisira d'après le caractère du discriminant Δ,

$$(52) \qquad \mu = \frac{1}{6}\left(k + \frac{1}{k}\right) = \sqrt{2\frac{P\sqrt[3]{T^2} - 1}{P\sqrt[3]{T^2} + 6}},$$

$$(53) \qquad \mu = \frac{1}{6}\left(k + \frac{1}{k}\right) = \sqrt{\frac{Q\sqrt[3]{4T} - 2}{Q\sqrt[3]{4T} + 6}},$$

* D'après cela l'intégrale $u = \int \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$ deviendra $\sqrt{k}\,u = \int \frac{dy}{\sqrt{1-(k+\frac{1}{k})y^2+y}}$

resoudront complétement la question, à savoir, d'exprimer le paramêtre de la canonique ou le module en fonction de l'invariant absolu. Mais ce sujet peut être encore envisagé sous un autre point de vue. Supposons que par une transformation linéaire on ait transformé une quartique quelconque dans celle-ci

$$(1-x^2)(1-k^2x^2).$$

Les invariants quadratique et cubique de cette forme seront respectivement

$$\frac{1+14k^2+k^4}{12}, \quad \frac{1+k^2}{6}, \quad \frac{(1-34k^2+k^4)}{36}.$$

Par conséquent l'invariant absolu de la quartique proposée sera

$$I=\frac{I_2^3}{I_3^2}=27\,\frac{(1+14k^2+k^4)^3}{(1-34k^2+k^4)^2(1+k^2)^2}.$$

Or nous allons voir que cette sextique, d'où depend le module k se réduit précisément à la canonisante (31). Posons en effet

$$k^2=l, \quad l+\frac{1}{l}=y;$$

il viendra
$$\frac{(y+14)^3}{(y+2)(y-34)^2}=\frac{I}{27}.$$

Faisons
$$\frac{y+14}{y-34}=\frac{t}{3};$$

nous obtiendrons
$$\frac{t^3}{1+t}=\frac{I}{4}=T,$$

équation qui, en posant $t=-\sqrt[3]{T}x$, revient à $x^3-\sqrt[3]{T}x+1=0$.

Ainsi nous retombons sur la canonisante connue, resultat remarquable, qui prouve une fois de plus comment l'étude de la canonisante fournit la clef de la susdite question, à savoir, *d'exprimer les élements des fonctions elliptiques en fonctions de l'invariant absolu.*

De l'équation (53) on tire

$$\frac{l^2+14l+1}{l^2-34l+1}=\frac{t}{3},$$

d'où

$$k^2=\frac{-21+17\sqrt[3]{T}x\pm k\sqrt{(1-\sqrt[3]{T}x)(3-2\sqrt[3]{T}x)}}{3-\sqrt[3]{T}x},$$

x designant une racine de l'équation (31), racine que nous avons après à calculer par la série de Lagrange. Ainsi k serait exprimée en fonction de l'invariant absolu.

§2. *Application aux Fonctions Elliptiques.*

Supposons qu'en posant

$$(54)\qquad X=(1-x^2)(1-k^2x^2),$$

$$(55)\qquad Y=(y-\alpha)(y-\beta)(y-\gamma)(y-\delta),$$

$$(56)\qquad k=\frac{P\mp P'}{P\pm P'},\quad P=\sqrt{(\alpha-\delta)(\gamma-\beta)},\quad P'=\sqrt{(\alpha-\gamma)(\delta-\beta)},\quad \rho=\frac{P\pm P'}{2};$$

on ait transformée l'integrale quelconque $\int\frac{dy}{\sqrt{Y}}$ en $\frac{1}{\rho}\int\frac{dx}{\sqrt{X}}$ par une transformation linéaire. Or en ayant égard aux équations (18) on voit que la valeur des k sera

$$(57)\qquad k=\frac{\sqrt{-l_2}-\sqrt{l_3}}{\sqrt{-l_2}+\sqrt{l_2}}.$$

En prenant le carré, on aura

$$(58)\qquad k=\frac{(-l_2+l_3)\sqrt{l_1}-2i\sqrt{l_1l_2l_3}}{(-l_2+l_3)\sqrt{l_1}+2i\sqrt{l_1l_2l_3}}.$$

En observant maintenant que $l_1l_2l_3=16\sqrt{\Delta}$, $-l_2+l_3=6\lambda_1$, il viendra

$$k^2=\frac{6\sqrt{\lambda_1^2l_1}-8i\sqrt[4]{\Delta}}{6\sqrt{\lambda_1^2l_1}+8i\sqrt[4]{\Delta}}.$$

Mais on verra aisément que

$$\lambda_1^2=\frac{(24I_2-l_1^2)\,l_1-32\sqrt{\Delta}}{36l_1},\quad \frac{l_1^3}{8}=3I_2\frac{l_1}{2}+\frac{l_1}{2}\,2\sqrt{\Delta},$$

et par conséquent l'expression de k^2 se reduira à

$$k^2=\frac{\sqrt{12\sqrt{\Delta}-3I_2l_1}+4\sqrt[4]{\Delta}}{\sqrt{12\sqrt{\Delta}-3I_2l_1}-4\sqrt[4]{\Delta}}.$$

En remplaçant, par exemple, l_1 par sa valeur, $l_1=-\frac{4}{3}\sqrt[6]{\Delta}\sqrt[3]{H}\,T$ (éq. 50), nous aurons

$$k_2 = \frac{\sqrt{12+4T}+4}{\sqrt{12+4T}-4} = \frac{\sqrt{3+T}+2}{\sqrt{3+T}-2}. \tag{59}$$

En posant

$$T = \frac{1}{4}\left\{\frac{4}{27}H + \frac{6}{2}\left(\frac{4}{27}H\right)^2 + \frac{9.8}{2.3}\left(\frac{4}{27}H\right)^3 + \ldots\right\} \tag{60}$$

il viendra encore l'expression très simple

$$k^2 = \frac{\sqrt{1+T}-1}{\sqrt{1+T}+1}. \tag{61}$$

Par conséquent

$$k'^2 = \frac{2}{1+\sqrt{1+T}}, \tag{62}$$

qu'on pourra facilement calculer à l'aide du seul invariant H. Quant à ρ nous aurons

$$\rho = \frac{P-P'}{2} = \frac{\sqrt{-l_2}-\sqrt{l_3}}{2},$$

d'où

$$4\rho^2 = -l_2 + l_3 - 2\sqrt{l_2 l_3} = \frac{1}{\sqrt{l_1}}\left((-l_2+l_3)\sqrt{l_1} - 2i\sqrt{l_1 l_2 l_3}\right),$$

$$4\rho^2 = -\frac{2i}{\sqrt{l_1}}\left(\sqrt{12\sqrt{\Delta} - 3l_2 l_1} + 4\sqrt[4]{\Delta}\right),$$

$$\rho^2 = -\frac{i}{\sqrt{l_1}}\sqrt{3+T} + 2)\sqrt[4]{\Delta},$$

$$\rho^2 = -\sqrt{\frac{3}{4}}\sqrt[6]{\frac{\Delta}{H}}\frac{\sqrt{3+T}+2}{\sqrt{T}},$$

et enfin

$$\rho = i\sqrt[4]{\frac{3}{4}}\sqrt[12]{\Delta}\left\{\frac{\sqrt{3+T}+2}{\sqrt{T}\sqrt[6]{H}}\right\}^{\frac{1}{2}} = i\sqrt[4]{3}\sqrt[12]{\frac{\Delta}{H}}\left\{\frac{1+\sqrt{1+T}}{\sqrt{1+4T}}\right\}^{\frac{1}{2}}; \tag{63}$$

par où l'on voit que, au facteur près $\sqrt[12]{\Delta}$, le multiplicateur ρ est fonction de l'invariant absolu H, conclusion qui sera confirmée par ce qui va suivre. En résumant ce qui précede nous arriverons à ce théorème general :

Soient

$$V = a_0 y^4 + 4a_1 y^3 + 6a_2 y^2 + 4a_3 y + a_4, \tag{64}$$

(65) $I_2 = a_0 a_4 - 4a_1 a_3 + 3a_2, \quad I_3 = \begin{vmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \end{vmatrix}, \quad \Delta = I_2^3 - 27 I_3^2 > 0,$

(66) $I = \frac{I_2^3}{I_3^2}, \quad H = \frac{\varDelta}{I_2^3} = 1 - \frac{27}{I},$

(67) $T = \frac{1}{4}\left\{\left(\frac{4}{27}H\right) + \frac{6}{2}\left(\frac{4}{27}H\right)^2 + \frac{9.8}{2.3}\left(\frac{4}{27}\cdot H\right)^3 + \ldots\right\},$

(68) $k^2 = \frac{\sqrt{1+T}-1}{\sqrt{1+T}+1},$

(69) $\rho = i\sqrt[4]{3}\sqrt[12]{\frac{\varDelta}{H}}\left\{\frac{1+\sqrt{1+T}}{\sqrt[6]{1+4T}}\right\}^{\frac{1}{2}};$

on aura

$$\int\frac{dy}{\sqrt{Y}} = \frac{1}{\rho}\int\frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}},$$

où, au facteur près $\sqrt[12]{\varDelta}$, le second membre est seulement fonction de l'invariant absolu H ou I.

Ce resultat, par lequel $\sqrt[12]{\varDelta}\int\frac{dy}{\sqrt{Y}}$ est fonction de l'invariant absolu pourrait se prévoir, comme a observé M. Stein (*Mathematische Annalen*, T. 13),* mais en y arrivant par une autre voie que celle que nous allons exposer.

Soit à cet effet une quartique quelconque

(71) $$f(x, 1) = a_0x^4 + 4a_1x^3 + 6a_2x^2 + 4a_3x + a_4.$$

On sait (v. notre *Théorie des formes binaires*, edit. française (140), ou allemande, p. 232) d'après Hermite qu'en appellant I_2, I_3 les invariant quadratique et cubique de la quartique, ainsi que $I = \frac{I_2^3}{I_3^2}$: l'invariant absolu, on a

(72) $$\int\frac{dx}{\sqrt{f(x,1)}} = -i\sqrt{\frac{1}{8}}\int\frac{d\lambda}{\sqrt{\lambda^3 - I_2\lambda + 2I_3}},$$

où $\lambda^3 - \lambda I_2 + 2I_3$ est la canonisante. Posons $\lambda = y\sqrt[3]{2I_2}$; il viendra

$$\int\frac{dx}{\sqrt{f(x,1)}} = \frac{-i}{\sqrt{8}\sqrt[6]{2I_3}}\int\frac{dy}{\sqrt{y^3 - \sqrt[3]{\frac{1}{4}}y + 1}};$$

équation qui nous prouve que $\sqrt[6]{I_2}\int\frac{dx}{\sqrt{f(x,1)}}$

* There seems to be some error in this reference.—Ed.

est fonction de l'invariant absolu seulement. Or de l'équation

(73) $$\Delta = I_3^2(I - 27)$$

on tire

$$\sqrt[6]{I_3} = \frac{\sqrt[12]{\Delta}}{\sqrt[12]{I - 27}};$$

donc $\sqrt[12]{\Delta}\int\frac{dx}{\sqrt{f(x,1)}}$ sera ainsi une fonction de l'invariant absolu seulement. Il en sera de même de $\sqrt[4]{I_2} = \sqrt[12]{\Delta}\sqrt[12]{\frac{I}{I-27}}$. Il s'ensuit que si on appelle K, K' les périodes de $\int\frac{dx}{\sqrt{f(x,1)}}$, les expressions $\sqrt[4]{I_2}\,K$, $\sqrt[4]{I_2}\,K'$; $\sqrt[6]{I_3}\,K$, $\sqrt[6]{I_3}\,K'$; $\sqrt[12]{\Delta}\,K$, $\sqrt[12]{\Delta}\,K'$ seront fonctions uniquement de l'invariant absolu I.

Si on avait pris pour invariant absolu $T' = \frac{I_2}{\Delta}, = \frac{1}{H}$, la conclusion aurait été la même; car par les équations $I = \frac{I_2^3}{I_3^2}$, $\Delta = I_3^2(I - 27)$, on a $I = \frac{27T}{T-1} = \frac{27}{1-H}$. Ainsi les fonctions $\sqrt[12]{\Delta}$, $\sqrt[4]{I_2}$, $\sqrt[6]{I_3}$ rendront l'intégrale elliptique générale fonction de l'invariant absolu quelconque.

Pour determiner tous les élements des fonctions elliptiques en fonctions de l'invariant absolu, il nous reste à exprimer les periodes K, K' et l'argument Jacobien q en fonction de I ou de H. Nous donnerons de cette question deux solutions, une analytique et l'autre infiniment approchée. Rappellons nous à cet effet que l'on a

(74) $$\sqrt{k'} = \frac{1 - 2q + 2q^4 - 2q^9 - \ldots}{1 - 2q + 2q^4 + 2q^9 - \ldots},$$

d'où l'on tire, en posant

(75) $$\alpha = \frac{1}{2}\cdot\frac{1 - \sqrt{k'}}{1 + \sqrt{k'}},$$

quantité qui sera toujours $< \frac{1}{2}$,

(79) $$\alpha = \frac{q + q^9 + q^{25} + \ldots}{1 + 2q^9 + 2q^{16} + \ldots}.$$

Observons maintenant que par le changement de q en $-q$ et en iq, α se change en $-\alpha$ et en $i\alpha$. Cela nous prouve que la fonction α par laquelle on expri-

mera q en α sera impaire et ne contiendra des puissances de α que de la forme $4p+1$. Nous pouvons donc poser

$$q = \alpha + B\alpha^6 + C\alpha^9 + D\alpha^{13} + E\alpha^{17} + \ldots \tag{77}$$

Alors, par la méthode des coefficients indéterminés on trouvera aisément

$$B = 2, \quad C = 15, \quad D = 150, \quad E = 1707;$$

les équations de conditions seraient

$$\begin{aligned} B - 2 &= 0, & 8C + 12B^2 - 9B - D &= 0, \\ 8B - C - 1 &= 0, & 8D + 8B^3 + 24BC + 2 - 9C - 36B^2 - E &= 0; \end{aligned}$$

et par conséquent

$$q = \alpha + 2\alpha^5 + 15\alpha^9 + 150\alpha^{13} + 1707\alpha^{17} + \ldots \tag{78}$$

Mais nous savons aussi que

$$\sqrt{\frac{2K}{\pi}} = 1 + 2q + 2q^4 + 2q^9 + \ldots \tag{75}$$

d'où par la série (78) nous obtiendrons après quelques reductions faciles

$$\sqrt{\frac{2K}{\pi}} (1 + 2\alpha)(1 + 2\alpha^4 + 16\alpha^8 + 176\alpha^{12} + 1986\alpha^{16} + \ldots). \tag{80}$$

Comme on peut mettre l'équation (76) sous la forme

$$q = \alpha \frac{1 + 2q^4 + 2q^{16} + \ldots}{1 + q^8 + q^{24} + \ldots}, \tag{82}$$

il s'ensuit, d'après la série de Lagrange, que l'on aura

$$q = \alpha + \sum \frac{\alpha^{4p+1}}{1.2.3\ldots(4p+1)} D_q^{4p} \left(\frac{1 + 2q^4 - q^8 - 2q^{12} + \ldots}{1 + q^8 + q^{24}}\right), \tag{83}$$

qu'on mettra encore sous la forme

$$q = \alpha + \sum \frac{\alpha^{4p+1}}{1.2.3\ldots(4p+1)} D_q^{4p} (1 + 2q^4 - q^8 - 2q^{12} + 3q^{16} + \ldots), \tag{84}$$

et on retrouvera ainsi la série (78). Pareillement la série (80) que nous venons de trouver par la methode des coefficients indéterminées peut se ramener à une formule générale par la série de Lagrange. Il est aisé de voir en effet par les deux séries qui expriment $\sqrt{\frac{2K}{\pi}}$ et $\sqrt{\frac{2k'K}{\pi}}$ que

$$\sqrt{\frac{2K}{\pi}} (1 - \sqrt{k'}) = 4F(q); \tag{85}$$

en posant $$F(q) = q + q^2 + q^{25} + \ldots$$

De là on tire, en comparant avec (76),

$$(86)\qquad F(q) = \alpha(1 + 2q^4 + 2q^{16} + 2q^{36} + \ldots)$$

Donc d'après la série de Lagrange

$$F(2) = F(x) + tF'(x)f(x) + \frac{t^2}{1.2}\frac{d}{dx}\left(F(x)f^2(x)\right) + \ldots,$$
$$+ \frac{t^n}{1.2\ldots} D_x^{n-1}\left(F'(x)f^n(x)\right) + \ldots$$

où $2 = x + tf(2)$, il viendra $\frac{1}{4}\sqrt{\frac{2K}{\pi}}(1 - \sqrt{k'}) =$

$$\alpha + \sum \frac{\alpha^{4p+1}}{1.2\ldots(4p+1)} D_q^{4p}\left[F(q)\left(\frac{1 + 2q^4 + 2q^{16} + \ldots}{1 + q^8 + q^{24} + \ldots}\right)^{4p+1}\right]_{q=0},$$

ou, en remarquant que

$$\frac{4\alpha}{1 - \sqrt{k'}} = \frac{2}{1 + \sqrt{k'}} = (1 + 2\alpha),$$

$$(87)\qquad \sqrt{\frac{2K}{\pi}} = (1 + 2\alpha)\left[1 + \sum \frac{\alpha^{4p}}{1.2.3\ldots(4p+1)}\right.$$
$$\left. D_q^{4p}\left[F(q)(1 + 2q^4 + q^8 + 2q^{12} + 3q^{16} + \ldots)^{4p+1}\right]\right]$$

La convergence de la série (84) depend comme l'on sait de celle qui donnerait q au lieu de $F(q)$ par l'équation

$$(89)\qquad q = \alpha(1 + 2q^4 + 2q^{16} + \ldots).$$

Or cette série sera convergente pour tout module de α inférieur a celui pour lequel l'équation en q acquiert deux racines égales. À cet effet, limitons l'équation (85) à celle-ci

$$(89)\qquad q = \alpha(1 + 2q^4),$$

ce qui est admissible, car le terme $2\alpha q^{16}$ est inférieur a $\frac{1}{2316}$. Alors en considérant la derivée $1 = 8\alpha q^3$ nous obtiendrons

$$q = \sqrt[4]{\frac{1}{6}},$$

et par suite

$$(90)\qquad \alpha = \frac{6}{8}\frac{1}{\sqrt[4]{6}}.$$

En remontant à la valeur de α il faudra que

$$\frac{1-\sqrt{k'}}{2(1+\sqrt{k'})} < \frac{3}{4} \cdot \frac{1}{\sqrt[4]{5}},$$

ou à peu près $\quad k' > \frac{1}{9}$ et $k^2 < \frac{80}{81}$;

condition que embrassera la majorité des cas, car k est toujours < 1. Dans le cas où cette condition ne serait pas remplie, en changeant entre eux les modules k et k', on aurait recours a k'. Ainsi la série (84) peut dans tous les cas être censée convergente. On pourra donc l'appliquer à déterminer la période K après le calcul préalable de α à qu'on fera par la valeur du module k donnée precédement. Une fois K trouvé K' se determinera par la relation $e^{-\pi\frac{K}{K'}} = q$, puisque q été déjà exprime en fonction de α. Donc les elements k, k', K, K' preuvent être censés genéralement exprimables en série à l'aide d'une seule constante, a savoir, l'invariant absolu de la quartique, sans être obligé de la resoudre pour la reduire a la forme $(1-x^2)(1-k^2x^2)$, comme l'on croyait auparavant. De la formule (80) en élevant au carré, on deduit

$$K = \frac{\pi}{2}(1+2\alpha)^2(1+4\alpha^4+36\alpha^8+400\alpha^{12}+4900\alpha^{16}+\ldots)$$

ou

$$(91) \quad K = \frac{2\pi}{(1+\sqrt{k'})^2}\left(1+\frac{1}{4}\left(\frac{1-\sqrt{k'}}{1+\sqrt{k'}}\right)+\frac{9}{64}\left(\frac{1-\sqrt{k'}}{1+\sqrt{k'}}\right)^8+\frac{25}{256}\left(\frac{1-\sqrt{k'}}{1+\sqrt{k'}}\right)^{12}+\frac{(35)^2}{(128)^2}\left(\frac{1-\sqrt{k'}}{1+\sqrt{k'}}\right)^{16}+\ldots\right)$$

On peut mettre cette série sous la forme

$$K = \frac{2k'}{(1+\sqrt{k'})^2}\left[1+\left(\frac{1}{2}\right)^2\left(\frac{1-\sqrt{k'}}{1+\sqrt{k'}}\right)^4+\left(\frac{1.3}{2.4}\right)^2\left(\frac{1-\sqrt{k'}}{1+\sqrt{k'}}\right)^8+\left(\frac{1.3.5}{2.4.6}\right)^2\left(\frac{1-\sqrt{k'}}{1+\sqrt{k'}}\right)^{12}+\left(\frac{1.3.5.7}{2.4.6.8}\right)^2\left(\frac{1-\sqrt{k'}}{1+\sqrt{k'}}\right)^{16}+\ldots\right]$$

série dont le loi est évidente. Si on la compare à celle qu'a donnée M. Gauss (*Math. Werke*, Bd. III., p. 367) et que exprime la reciproque de la moyenne arithmetico-geometrique $M(1+x, 1-x)$, à savoir,

$$\frac{1}{M(1+x, 1-x)} = 1+\left(\frac{1}{2}\right)^2x^2+\left(\frac{1.3}{2.4}\right)^2x^4+\left(\frac{1.3.5}{2.4.6}\right)^2x^{12}+\ldots,$$

on trouve que

$$K = \frac{\pi}{M(1+k', 2\sqrt{k'})},$$

formule qui pourrait être encore employée pour le calcul de K.

Revenant à la série (91) nous obtenons une série rapidement convergente qui fournit la période une fois le module k' connue, ce à quoi on pourra réussir au moyen de l'invariant absolu $H = \frac{\Delta}{I_2^3}$. En effet nous avons par l'équation (62) $k'^2 = \frac{2(\sqrt{1+T}-1)}{T}$, ce qui fournit k, k' indifféremment par la rélation

$$k^2 = 6\left(\frac{4}{27}H\right)\left(\frac{1}{32} + \frac{29}{8.32}\left(\frac{4}{27}H\right) + \frac{43}{128.32}\left(\frac{4}{27}H\right)^2 + \ldots\right),$$

série convergente pour $H < 1$ et pour $T < 1$. Nous avons acquis ainsi pour ce cas deux séries qui resolvent complétement le problème à elles seules; c'est à dire, d'exprimer l'intégrale K, sous autre calcul préalable que celui de l'invariant absolu. Mais du coté pratique à cause de la convergence rapide des séries au q, nous pouvons donner une solution approchée de la question que suffira surabondamment dans tous les cas.

Remarquons que dans l'équation $q = e^{-\pi\frac{K'}{K}}$ on peut toujours supposer $\frac{K'}{K} < 1$, ou tout au plus $= 1$, car on pourrait en cas contraire échanger la modules k, k' entr'eux, et par suite K, et K'. Par conséquent on pourra toujours admettre que $q \leq e^{-\pi}$; c'est à dire, que la valeur maximum q est à peu près $\frac{1}{23}$. Cela nous apprend que lè série procédant par les puissances de q sont exactes à partir de q^9 jusqu'à la 13^e décimale. Les termes seulement en q^{16} seront déjà moindres que $\frac{1}{1022}$. Par les séries connues en effet

$$\Theta(0) + \Theta_1(0) = \sqrt{\frac{2K}{\pi}}(1+\sqrt{k'}) = 2(1 + 2q^4 + 2q^{16} + \ldots),$$

(92) $$\Theta\left(\frac{K}{2}\right) = 1 - 2q^4 + 2q^{16} - 2q^{36} + \ldots,$$

on obtient la série nouvelle

$$(93) \qquad \Theta(0) + \Theta_1(0) + 2\Theta\left(\frac{K}{2}\right) = 4 + 8\,(q^{16} + q^{64} + q^{144} + \ldots),$$

qui est d'une convergence enormément rapide. Pour avoir la valeur du premier membre, nous aurons recours à la formule connue de Jacobi, qui nous fournira $\Theta\left(\frac{K}{2}\right)$, à savoir,

$$(94) \qquad \Theta(u+v)\;\Theta(u-v) = \left(\frac{\theta(u).\,\theta(v)}{\theta(0)}\right)^2 \left(1 - k^2 s^2(u) s^2(v)\right).$$

Posons $u = \frac{K}{2}$, $v = \frac{K}{2}$, il viendra, en notant que

$$\frac{\theta(K)}{\theta(0)} = \frac{1}{\sqrt{k'}}, \quad s\left(\frac{K}{2}\right) = \sqrt{\frac{1}{1+k'}},$$

$$(95) \qquad \Theta\left(\frac{K}{2}\right) = \sqrt{\frac{2K}{\pi}}\sqrt[4]{\frac{1}{2}(1+k')\sqrt{k'}}\,.$$

Par conséquent

$$(96) \qquad \Theta(0) + \Theta_1(0) + 2\,\Theta\left(\frac{K}{2}\right) = \sqrt{\frac{2K}{\pi}}(1 + \sqrt{k'} + \sqrt[4]{1+k'}\,\sqrt[8]{64k'}\,.$$

On aura donc la formule remarquable

$$(97) \qquad \sqrt{\frac{K}{8\Pi}}(1 + \sqrt{k'} + \sqrt[4]{1+k'}\,\sqrt[8]{64k'}) = 1 + 2\,(q^{16} + q^{64} + q^{144} + \ldots)$$
$$= 1 + 2\sum_{1}^{\infty} q^{(4p)^2}.$$

Si on compare cette formule avec celle-ci

$$\sqrt{\frac{2K}{\pi}} = 1 + 2q + 2q^4 + 2q^9 + \ldots;$$

on voit qu'en appellant K_{16} ce que devient K si q se change en q^{16}, on aura

$$\sqrt{\frac{K}{8\pi}}(1 + \sqrt{k'} + \sqrt[4]{1+k'}\,\sqrt[8]{64k'}) = \sqrt{\frac{2K_{16}}{\pi}}.$$

A moins donc de $\frac{1}{1022}$, nous aurons

P 11184

$$(98) \qquad \sqrt{\frac{K}{8\pi}} = \frac{1}{1+\sqrt{k'}+\sqrt[4]{1+k'}\,\sqrt[8]{64k'}}$$

Par cette formule, qui est exacte jusqu'à la 22^e chiffre décimale, en substituant au lieu de k' sa valeur exprimée en fonction de l'invariant absolu, on obtient de suite la période relative à la quartique donnée. Nous pouvons donner un exemple frappant de cette grande approximation. Supposons qu'on ait $k' = \frac{1}{2}$, ou $k = \frac{\sqrt{3}}{2} = \sin 60°$. On trouvera

$$\log \int_0^1 \frac{dx}{\sqrt{(1-x^2)\left(1-\frac{3}{4}x^2\right)}} = 0.33375\ 26136\ 78325\ 44512.$$

N'ayant pas à la main du logarithmes à 20 décimales, on a du construire ceux qui etaient necessaires en s'aidant pour ce que l'on pourrait des tables de Callet appropriée à ce but. Voici par exemple quelques logarithmes qui serviront de contrôle au lecteur.

$$\log 2 = 0.\ 30102\ 99956\ 63981\ 19521$$
$$\log \pi = 0.\ 49714\ 98716\ 94133\ 85435$$

$$\frac{1+\sqrt{k'}}{2} = \frac{2+\sqrt{2}}{4} = 0.\ 55358\ 33905\ 93298\ 99379$$

$$\log \sqrt[8]{\frac{9}{32}} = 1.\ 93111\ 05663\ 89927\ 36271$$

Ce resultat donne d'un coup seul les 12 chiffres de Legendre dans ses Tables (page 234, Tome 2 de ses *Fonctions Elliptiques*). N'ayant pas par faute de tables étendues, pousser plus loin nos calculs nous ne pouvons assurer que les 17 premiers chiffres; les 3 derniers pointillés sont douteux. On voit néanmoins par cet exemple comment cette formule remarquable (97) puisse fournir pour n'importe quelle valeur de k (en cela d'accord avec le théorème) une valeur extrêmement approchée, et en ne partant si l'on veut que de l'invariant absolu, valeur que ne diffère de la valeur exacte que par une quantité presque inappreciable, representée par une série en q d'une convergence vertigineuse.

APPENDICE.

Note 1^er^. En supposant qu'on ait

$$z = a + tz^m$$

il vient (v. Serret *Algèbre* 3^e^ edit.) pour la série de Lagrange

$$z = a + t^m + \frac{2m}{1.2} a^{2m-1} t^2 + \frac{3m(3m-1)}{1.2.3} a^{3m-2} t^3 + \ldots,$$

série qui sera convergente pour tous module de t par lequel on a

$$\text{mod } t < \text{mod } \frac{(m-1)^{m-1}}{m^m a^{m-1}}$$

Note 2^e^. Comme les canonisantes de la quartique se reduisent en général à l'expression

$$y^3 - ay + 1 = 0$$

il sera bon de voir comment se comportant les racines sous cette forme. On sait par le discriminant que les racines seront réelles ou imaginaires selon que

$$(a) \qquad \frac{4a^3}{27} - 1 \gtrless 0$$

ou selon que environ $\alpha \gtrless 2$. Considerons les deux cas successivement.

1^er^ *Cas.—Racines réelles.*—On aperçoit bientôt qu'il y a deux racines positives; une entre 0 et 1; l'autre entre 1 et α; puis une racine negative entre -1 et $-\alpha$.

2^e^ *Cas.—Racines imaginaires.*—Nous observerons d'abord qu'il ne peut pas y avoir des racines positives, car sans cela, en se reportant un terme connu $+1$, on conclurait que le module des racines imaginaires serait imaginaire, ce qui est absurde. En changeant par conséquent y en $-y$, on voit que la racine positive de

$$y^3 + ay - 1 = 0$$

est comprise entre $\sqrt{a}$, $\sqrt{a} + 1$, car les resultats des substitutions seront de signe contraire. Une fois trouvée la racine y, la relation $-\rho^2 y = -1$, ρ designent le module, par la valeur du module $\rho = \sqrt{\frac{1}{y_1}}$, il sera facile ensuite d'en deduire

l'argument des racines imaginaires. Les restes de Sturm seraient selon Cayley

$$(C) \qquad y^3 - ay + 1\,, \quad y^2 - \frac{a}{3}\,, \quad \frac{2}{3}\,ay - 1\,, \quad \frac{4}{27}\,a^3 - 1$$

et fourniraient une autre preuve de ces limites ; car il viendrait,

$y^3 - ay + 1$	+	−	−	+	+	−	+	+	+	−
$y^2 - \frac{a}{3}$	−	−	−	+	−	+	−	+	+	+
$\frac{2}{3}\,ay - 1$	−	+	+	+	−	−	−	+	−	−
$\frac{4}{27}\,a^3 - 1$	+	+	+	+	+	+	−	−	−	−

D'après ce type général on saurait de suite dans les cas particuliers, quelles sont les limites qu'on doit assigner aux racines.

The Intersection of Circles and the Intersection of Spheres.

By Benjamin Alvord, *Brig. Gen. U. S. A.*

1. In this paper I propose to discuss the various problems of the Intersections of Circles and the Intersections of Spheres.

The first problem is to draw a circle which shall make a certain given angle with three given circles. The next is to draw a sphere which shall cut each of four given spheres at a given angle. Afterwards I shall take up the two problems (proposed by Steiner, 1st Vol. of Crelle, 1826, page 163) to draw a circle which shall cut four given circles at the same angle (said angle being unknown): also to describe a sphere which shall cut each of five given spheres under the same angle (said angle being unknown). All these problems are solved geometrically.

2. In each case I shall give the number of solutions. It has been long known that to draw a circle tangent to or intersecting three given circles at the same angle, there are 8 solutions; also that there are 16 solutions to the question to draw a sphere intersecting or touching four given spheres at a given angle. But I think it has not been known that there are 96 solutions to the question to draw a circle cutting four given circles at the same angle, or that there are 640 solutions to the problem to draw a sphere to cut five given spheres at equal angles. But in many of them the intersections, either in circles or spheres, will be imaginary.

3. I expect to show that the central geometrical principle from which all these solutions are evolved is that of the radical centre, or centre of the orthogonal circle, which was the basis of the solution of the Problem of the Tangencies of Circles and of Spheres given by me, and published in 1855 in the 8th Volume of the Smithsonian Contributions. In this dissertation the tangency will be regarded as the case in which the intersection is at an angle of 0 or 180°.

In general we can say that two circles make with each other an angle β, viz., the angle which the tangents at the point of intersection make with each

other, or the supplement of the angle which the two radii drawn to the points of intersection, make with each other.

4. In discussing the intersection of circles, there may in a certain sense be said to be ten different cases (the number of the cases propounded by Appolonius in reference to tangencies): but all of them are really embraced in the single one, to draw a circle cutting each of three given circles at a given angle: some of those circles can be reduced to points, *i. e.* circles with radius 0, or to right lines, *i. e.* circles with radii ∞, and in this way 10 sub-cases arise. The table giving the various questions in tangencies, at page 4 of my Smithsonian memoir, would answer for the intersections, substituting for the word "tangent," a certain angle β, &c. And as in that memoir, Problem 4 is the central problem containing the gist of the whole, so Problem 4 in the theory of intersections bears a like relation to the rest: this is to draw a circle through two given points cutting a given circle at a given angle.

5. I premise that I shall consider as known to the reader the principles of the polar line and polar plane; of the radical axis, centres or planes of similitude, the radical centre and orthogonal circles or spheres, as given in the so-called modern geometry. But I shall be compelled to reproduce a few of them of most frequent use in this paper, but only so far as may be necessary to throw light on their subsequent development.

6. I have found the most lucid way to consider, and to construct, the radical axis of two given circles, is to draw any circle cutting both; the right lines joining the two points of intersection in each, those two lines will intersect each other in the radical axis; then by another secant circle finding in like manner another such point, the radical axis is determined. If from any point of this axis as a centre, and the tangent line as a radius, a circle be drawn, it will cut each of the two given circles orthogonally.

It is of frequent use to remember, that if we mark the two points in which one of said orthogonal circles cuts the line joining their centres, *all* circles drawn through those two points will be orthogonal to the two given circles and have their centres on the radical axis. Also, if it is proposed to find a circle orthogonal to a given circle and having its centre on a given line, we may proceed as follows: let fall a perpendicular from the centre of the given circle upon the line; draw at random from any point of it, a tangent to the given circle, and use it as a radius to draw an orthogonal circle thereto: it will intersect the perpendicular in two points. *Any* circle whatever drawn through those points will be orthogonal to the given circle.

Thus in Figure 1, RR' is the radical axis of circles A and B. From any point of R draw a tangent RM, and with it as a radius describe a circle orthogonal to them. It will cut the line AB at Q and Q'. *Any* circle through Q and Q' will be orthogonal to circles A and B; as $AQ \times AQ' = AM^2$, whatever may be the position of M in the circumference.

If as in Figure 2 it is required to draw through Q a circle orthogonal to circle A: join Q with the centre A; find the point Q' on line joining the centre A such that $AQ \times AQ' = AE^2$. *All* circles through Q and Q' will be orthogonal to the given circle. Or given a right line RR', to draw a circle orthogonal to circle A, having its centre on said line; draw *any* circle orthogonal to circle A and having its centre on RR', and mark the points Q and Q' in which it cuts AK drawn from the centre perpendicular to RR'. Any circle as before through Q and Q' will fulfil the conditions.

7. If as in Figure 3 it is required to draw a circle orthogonal to circle A, touching the circle D, and having its centre on a given line FT: find as in the last article the two points Q and Q' through which all circles must pass orthogonal to circle A and having their centres on FT. Then by Problem 4, (Tangencies of Circles,) through the two points Q and Q' draw a circle tangent to circle D. It will also be orthogonal to circle A. Draw any secant circle as QHQ' to circle D. The chord $H'H$ will give O the "centre of converging chords." Then draw from O to circle D two tangent lines OM, OM'. There are thus two such circles having their points of contact at M and M', and their centres at F and F'.

8. As in the preceding article I shall have frequent occasion to speak of the centre of converging chords, or to refer to "the principle of converging chords"; by this I mean only the well-known principle of the radical centre. The latter term is applied generally to three circles, but the former to a system or group of any number of circles. The three radical axes of three given circles, it is well known, meet in the same point, called the radical centre, or the centre of the circle orthogonal to the three given ones.

In Figure 4, let the three circles first considered be the circles C, ABF and ABH. Their three radical axes are the lines AB, FF', HH', all uniting at O, their radical centre. But if a fourth or any secant circle, as $ABII'$, is added: the chord $I'I$ will also pass through the same point O, which I call the centre of converging chords. The principle of converging chords can be enunciated as follows: If a fixed circle is cut by any circle or system of circles, which pass

through two given points in the plane of the given circle, the chords pass through a fixed point on the line passing through the two points. Or to use language still more general: Any system of circles having a common radical axis (whether they intersect or not) will cut a fixed circle in chords converging towards a fixed point on such axis. In Figures 3 and 4 they intersect. But in Figure 5 suppose the case in which they do not intersect—as the system of circles FF', GG', HH', orthogonal to a fixed circle AQ, having their centres on QQ', and having AO for a common radical axis. Mark the intersections of this system with a given circle C; the chords $D'D$, $E'E$, $L'L$ will all converge to a point O on the radical axis, the centre of converging chords, in reference to circle C, of the whole series. It will be noted that in this figure the whole series of circles passing through QQ' having the line QQ' for a radical axis, are orthogonal to each circle in the other series above named having AO at right angles to it, for a radical axis.

9. If then the problem is proposed to draw one of this last series, viz. a circle tangent to a given circle C in Figure 5, having its centre on a given line QQ', and orthogonal to circle AQQ': draw at will any such orthogonal circle as HH', join the points D and D' in which it intersects said circle by a line meeting the radical axis in O. This will be the "centre of converging chords." From O draw tangents to the circle C, these will determine W and W', the two points of contact of the pair of required circles. The circles X and Y in the figure will be each tangential to C and orthogonal to A.

10. The well-known principle of the polar line and its pole is exhibited in Figure 6. If through any point E, interior or exterior to circle A, any number of chords (or secants) be drawn, and from the extremities tangents be drawn, they will all unite in the same line PX. For draw one chord BB' through E as well as $H'H$, the shortest chord through E, and their pairs of tangents, the latter uniting at P, the former at X, $AD \times AX = R^2 = AE \times AP$. Therefore the triangles ADE and APX are similar, and the angle APX is a right angle, whatever may be the position of X. Thus if chord II' be drawn, the point X' will be on the same line PX. A glance at the diagram will show that XX' is also the locus of circles orthogonal to circle A passing through a given point Q, as in Figure 2. All such circles will pass through Q and Q', and the chords can be regarded as "the converging chords" of the circles having E for its centre. Thus the principle of the pole and its polar line may be regarded as derived from that of the converging chords, or of the radical centre.

11. I will now proceed to discuss the question referred to in Article 4, as follows: *To draw a circle through two given points to cut a given circle at a given angle* β.

The solution is reduced to a question in tangency, and to finding a circle orthogonal to one auxiliary circle and tangent to another.* In Figure 7, through the given points D and E to draw a circle cutting the given circle C at a given angle β.

Draw the triangle CMN with angle $CMN = \beta$, $MN = \cos\beta$. With $CN =$ sine β, draw a circle concentric to the given circle. Through E, one of the given points, draw the circle EI with a radius equal to $MN =$ cosine of β. Find as in the last article the circle orthogonal to auxiliary circle CN and tangent to auxiliary circle EI, and having its centre on the line BA perpendicular to DE through its middle point. There are two such circles XH and YF. Circle X is orthogonal at H to circle CH. Mark the point X' in which the tangent line XH cuts the given circle. $EI = X'H = MN = \cos\beta$. Therefore $XE = XX'$, and the point X is the centre of the required circle making an angle $CX'H = \beta$ with the given circle. In like manner circle Y is another answering the conditions, as $FY'' = MN$. If $\beta = 0$ or the required circle is to be a tangent to the given circle, find S, the "centre of converging chords," and draw two tangent lines to circle C; they will give the two points of contact K and K' of the pair of tangent circles, G and G' being their centres.

12. I will now explain some properties of the orthogonal circle not belonging immediately to the discussion, but which will be hereafter referred to. Join the two points of contact K', K, and from A where the line cuts BX, draw two tangents AP and AP'. $AK \times AK' = AP^2$, and $SP \times SP' = SK^2$ (or AKK' is the polar line of S as pole in reference to circle C). The circle DPP' is orthogonal to circle C, or answers to the case in which β is 90°.

13. This orthogonal circle bisects the angle which the two required circles DEX', DEY', make with each other. But I will first show that it bisects the angle which the two tangent circles DEK, DEK' make with each other at the point E. Extend $K'KA$ to R. Extend CK' to G'', making $K'G'' = GK$. Join GR and GG'', EG, EA and EX. GG'' is parallel to AK', and the triangle $G'AK'$ similar to $G'GG''$, and

$$G'K' : K'G'' :: G'A : AG. \quad \text{But } K'G'' = KG = GE \text{ and } G'K' = G'E.$$

* *Note by Prof. Cayley.*—Thus it can be shown that if two circles, radii R and C cut at angle β, then the concentric circles, radii $R + C\cos\beta$ and $C\sin\beta$ will cut at right angles.

Therefore $G'E : EG :: G'A : AG$. Thus in the triangle GEG' as the line EA divides the base so that its segments are proportional to the other two sides, it follows that the radius EA of the orthogonal circle bisects the angle GEG', or the angle which the two tangent circles make with each other.

Also A, the centre of the orthogonal circle, is a centre of similitude of the two tangent circles. The angles ARG, AKG, CKK', $CK'K$ being all equal, the radii $G'K'$ and GR are parallel, and therefore the line RKK' passes through the "external centre of similitude."

By a similar chain of reasoning it will easily be seen that AE bisects the angle XEY which the two required circles (cutting the given circle at angle β) make with each other, and the lines $AX'Y'$, $AX''Y''$ are straight lines, and A the centre of the orthogonal circle is also the "internal centre of similitude" of the two required circles X, Y.

Also $AE^2 = AP^2 = AK \times AK' = AX' \times AY'$; or the radius of the orthogonal circle is a mean proportional to the distance of its centre from the pair of tangent circles, or from the pair of required secant circles.

Suppose the two tangent circles DEK, DEK' to remain fixed, and the circle C to change, remaining tangent to them: the right line joining the points of contact will continue to pass through A. Draw at will any right line through A, mark the points in which it cuts the two tangent circles and the orthogonal circle, draw tangents at each of those points, the two first will make equal angles with the last, or the curve of the orthogonal circle bisects the angle which at those points the curves of the two tangent circles make with each other.

It will be found that in the whole series of problems regarding intersections, the circumference and the centre of the orthogonal circle enjoy similar properties to those above explained.

14. I will now investigate the problem: *Through a given point to draw a circle cutting two given circles at a given angle β.*

In Figure 8 let P be the given point and A and B the given circles. Let TO be the radical axis of those circles; Q and Q' the points (as in Article 6) through which all circles orthogonal to those two circles must pass. Through P, Q, Q' pass a circle, it will be the orthogonal circle through the given point to those circles, having of course its centre O on the radical axis. Prolong the radii BE and AK to meet at Y. The triangles OYK, OYE are equal. Therefore the angles AKR' and BRE are equal, and the radii BR, AK are parallel. There-

fore the lines EK or $E'K'$ must pass through S the "external centre of similitude" of the two given circles. The point P' in which SP cuts the orthogonal circle is determined by means of the relations $SN \times SN' = SL \times SL' = SP \times SP' = SE \times SK = SE' \times SK'$. Then by Article 11, through the points P and P' draw a circle making the required angle with circle A, it will make the same angle with circle B. $PP'HD$ with centre at (1) is one of those required circles, $PP'VM$ another, with centre at (2). Join HD and produce it to W on circle A. Join (1), the centre of first required circle, with Y', the point of intersection of the two radii BH, AD. The triangle (1) HD is isosceles. The angle (1) $DY' =$ (1) $HY' = \beta$. Therefore the triangle $Y'HD$ is isosceles. And the angle $Y'DH = Y'HD = AWD$. Therefore the radii BH and AW are parallel, and the line HW must pass through S, the "external centre of similitude" of the two circles.

When the angle β is zero (the case of tangency), join the points of contact with each circle of the pair touching each of the given circles in the *same* manner, the line will pass through the "external centre of similitude." The other pair of tangent circles will touch the given circle in a *different* manner, and the line joining their points of contact will pass through the "internal centre of similitude."

Join P with S' the "internal centre of similitude," and mark P'' the point in which PS' cuts the orthogonal circle. Then through P and P'' (as in Article 11) pass a circle cutting circle A at the given angle β, it will also cut circle B at the same angle. There are two such circles, and thus there are four in all fulfilling the conditions of the problem. The centres of these two circles are at (3) and (4) in Figure 8.

Join the centres of the first two (1) and (2), the line must pass through O the centre of the orthogonal circle, and be perpendicular to the axis of similitude SPP'. Join the centres (3) and (4) and the line passes through the same centre and is perpendicular to the axis of similitude $PS'P''$. This will be found to be a general rule, extending to the tangent circles. To three given circles it is well known there are (Article 18) four axes of similitude passing through their centres of similitude in groups of three each. In this case, where one of the three circles is the point P (or a circle of radius zero), there are but two axes of similitude, as above indicated.

15. Thus the four required circles through the given point intersect each other in pairs in the circumference of the orthogonal circle, at the points in

which the two axes of similitude cut it. And it will readily be seen by a method similar to that employed in Article 13, that the orthogonal circle bisects the angles which each pair of the required secant (or tangent) circles make with each other. Through the centre of the orthogonal circle (or radical centre) draw at will any line cutting the circumferences of each pair, and mark the distances to each circumference along that line from the centre—the radius of the orthogonal circle is a mean proportional to these distances.

The radical centre is also *one* of the centres of similitude of each pair of the required circles. Call $\frac{2}{4}$ the point in which circles 2 and 4 intersect, and $\frac{1}{3}$ the point in which circles 1 and 3 intersect; the right line $\frac{2}{4}\,\frac{1}{3}$ joining those two points passes through the radical centre, and the angle at which 2 and 4 intersect, it is easily proved, is the same as the angle at which 1 and 3 intersect. So also the right line $\frac{2}{3}\,\frac{1}{4}$ will pass through the same centre, and the angle at which 2 and 3 intersect is the same as the angle at which 1 and 4 intersect.

16. It should be noted that any circle through P and P' will cut the two given circles at the same angle, and that a mate to it will be found cutting them at a like angle. The same is true of any circle through P and P''. If it should not meet the circles, the intersections will be imaginary, and each pair will be situated in a similar manner in reference to each of the given circles, and the orthogonal circle will bisect the angle each of a pair make with one another.

17. It should also be noted that if from S, the external centre of similitude of the two given circles, a tangent line be drawn to one of the required circles, as, SZ, we shall have $SZ^2 = SP \times SP' = SD' \times SH' = SE \times SK$. Hence the said circle will be orthogonal to the pair of required circles cutting or tangent to the given circles in a similar manner. [This is called by Steiner in the 1st Volume of Crelle the "power" circle, "Puissance" circle, and is useful in certain exceptional cases.]

In Figure 11 draw a circle orthogonal to circles A and B, cutting them at P and P', and with any point X on the radical axis as a centre, $P'P$ will, we have seen, pass through S the external centre of similitude. The "power" circle ZZ' will have a radius SZ tangential to PP', and $SZ^2 = SP \times SP' = SO \times SO' = ST \times ST'$. Thus XY is a common radical axis to circles A and B and also to the "power" circle. This power circle is orthogonal to the whole

series of circles either tangent to or making equal angles with A and B. We have before seen how the orthogonal circle bisects the angles which two circles make with each other when they intersect. But it may also in a certain sense be said to bisect the angle which circles A and B make with each other *though they do not intersect.* This might be named their *angle of approach.* Draw any straight line through S, the centre of the power circle, cutting the circles A and B, the radius of the power circle is a mean proportional to the distances along that line to each circumference.

But this power circle bisects the angle of approach in another sense (as in Article 11.) Draw any line as SOO' through S, cutting A and B at O and O' and the power circle at D. Draw tangents at O, O' and D. Those from O and O' will unite at Y on the radical axis. Join FY. HDH' is parallel to FY. The angle $FYO = FYO' = DHO = DKO' = \frac{1}{2}\ OYO'$. Or the curves at O and O' make equal angles with the curve at D.

18. We shall pass over the problems "to draw a circle which shall cut a given circle and two given right lines at the same angle," and, "to draw a circle which shall cut a given line and two given circles at the same angle," which are readily solved by like principles, and pass to the 10th and last problem in the series, which indeed is *the* general problem of this investigation, and comprehends all.

To draw a circle which shall cut each of three given circles at a given angle.

Let A, B, C, Figure 10, be the three given circles. Find the three external centres of similitude N', M', S', and the three internal centres of similitude N, M, S of said circles. The three external are in one straight line, called the external axes of similitude of the three circles, and three other axes are found by joining each of the external centres of similitude with the internal centres of similitude. There are thus four axes of similitude to the three circles. (See *Rouché et Comberousse, Traité de Géometrie élémentaire*, Art. 389.) As in Article 13 it will be found that each of these four axes of similitude is the radical axis of a pair of the required circles (whether secant or tangent) or eight in all. In Figure 10 draw the circle whose centre is O orthogonal to the three given circles A, B, and C. Call the centres of the required circles (1) (2) (3) (4) (5) (6) (7) (8). Let $\frac{7}{8}\ \frac{7}{8}$ be the points in which circles 7 and 8 intersect, and so

on; the line $\frac{7}{8}\frac{7}{8}$ will be one of the axes of similitude; the lines $\frac{5}{6}\frac{5}{6}$, $\frac{3}{4}\frac{3}{4}$, and $N'M'S'$ being the other axes of similitude. This is supposing the circles 1 and 2 do not intersect as in this diagram; the case in which they do intersect will be treated separately in the next article.

Thus the solution of this problem consists in finding the two points in which any axis of similitude cuts the orthogonal circle; through those two points draw, as in Article 14, a circle cutting one of the given circles at the given angle β (or zero in case of tangency), and it will cut the two other given circles at the same angle.* Indeed, *any* circle through those two points will cut the three given circles at the same angle, or if it chances not to intersect it (the intersection being imaginary) it will (Article 16) be situated in a similar manner towards each. Thus these points $\frac{1}{2}\frac{1}{2}$, $\frac{3}{4}\frac{3}{4}$, $\frac{5}{6}\frac{5}{6}$, $\frac{7}{8}\frac{7}{8}$ might be called the key points of the problem. They are referred to by Poncelet and Salmon for other purposes, and are called by them "limiting points."

19. We will now consider the exceptional case in which the two required circles as 1 and 2 do not intersect in the circumference of the orthogonal circle, but will have as a radical axis the external axis of similitude, sometimes named the axis of "direct similitude." In Figure 12, let A, B, C be the given circles, NMS the external axis of similitude, FEE' the orthogonal circle. Through F the radical centre let fall FF' a perpendicular to said axis, it must contain the centres of the two required circles. In circles B and C draw auxiliary circles CH', BL', with radii equal to $\sin\beta$ in each circle. That is, make the angle $CHH' = \beta$. Lay off $HI = \cos\beta$ in circle B, and draw the auxiliary circle CI, with centre at C.

Find the points Q and Q' through which (see Article 7) all circles must pass orthogonal to circle BL' and having their centres on FF'. Through Q as a centre draw the auxiliary circle QD' with a radius equal to $H'I =$ cosine of CIH', a known angle.

Draw a circle, as in Article 9, having its centre on FF', tangent to circle QD' and orthogonal to circle CH', intersecting the latter at W. Its centre X is the centre of the required circle. For, from X draw the right line XW tangent

---

* *Note by Prof. Cayley.*—The theorem seems well worthy of an independent statement, viz.: Take any circle O and three circles A, B, C, each cutting O at right angles. Take any one of the four axes of similitude meeting O in two points X and Y (these points are imaginary for the exterior axes, but real for each of the other three), then the theorem is that any circle whatever through X and Y cuts each of the circles A, B, C at the same angle β.

to circle CH', and $XL'L$ tangent to circle BL'; circle $QQ'L'$ will be orthogonal at L' to circle BL'. Mark the points Z and L''. These are points in the required circle; for

$$QD' = WO'' = H'I = LL'$$
$$ZW = HH' = \cos.\ \beta \text{ in circle } C.$$
$$L'L'' = ZO'' = HI = \cos.\ \beta \text{ in circle } B. \quad Q.\ E.\ D.$$

Find the other circle having its centre on FF' tangent to circle Q, and orthogonal to circle BL'. The centre of this circle Y is the centre of the mate to KZ, cutting the given circle at the required angle.

Let KK', RR' be the arcs of intersection of the two required circles with circle A. The two lines FKR, $FK'R'$ will be straight lines. Mark the point P in which the line EE' meets $F'S$. P is "the centre of converging chords" of all circles having their centres on FF' and the radical axis $F'S$. Therefore $K'K$ and $R'R$ will pass through P. Draw the two tangents from P to circle A. The points of contact will be the points of contact of the pair of circles tangent to the three given circles, or the case of $\beta = 0$.

More generally draw at will through P any straight line cutting circle A. Mark the two points of intersection. Pass through these two points a circle having its centre on FF', it will cut the three given circles at the same angle, but not necessarily at the given angle β. Again, draw, with S (the external centre of similitude of A and C) as a centre, the "power circle" $P'T$, which is orthogonal to all the circles cutting A and C at the same angle, and in like manner as in Article 17. Then if with *any* point X of FF' as a centre and a radius equal to XX', drawn tangent to this "power circle," a circle be drawn, it will cut the three given circles at equal angles.

20. I append a *resumé* of some of the curious relations of the required circles to each other and to the orthogonal circle and the radical centre. Although not belonging strictly to the problem, they have arisen from time to time in the course of the investigation and cannot well be omitted. The first three heads in this recapitulation have been well known for many years, but the remaining four heads set forth new properties, so far as I can ascertain. After long and patient research I find no allusion to the circumference of the orthogonal circle bisecting the angle between the pairs of secant or tangent circles, or to its centre being a point from which radiate such a multitude of lines.

RECAPITULATION OF THE THEOREMS WHICH OCCUR IN THE SOLUTION OF THE PROBLEM TO DRAW ALL THE CIRCLES CUTTING THREE GIVEN CIRCLES AT ANY GIVEN ANGLE.

1st. The required secant circles are eight in number, distributed in pairs, the radical axis of each pair being one of the "axes of similitude" of the three given circles.

2d. Each pair intersect in the circumference of the orthogonal circle; or if a pair do not intersect, the orthogonal circle has with each of them the same radical axis—one of the axes of similitude.

3d. The line joining the centres of each pair of the required circles passes through the radical centre. And the radical centre is the "internal centre of similitude" of each pair of the required circles, and thus in the case of tangency the line joining the points of contact passes through the radical centre.*

4th. If a pair of the required circles intersect in the circumference of the orthogonal circle, that circumference bisects the angle which they make with each other. If they do not intersect, the orthogonal circle bisects what may be called their *angle of approach;* the radius of the orthogonal circle is a mean proportional between the distances from its centre on any straight line to the circumferences of the circles; and the orthogonal circle (Article 17) makes equal angles with the curves at the points of intersection of the line.

5th. In any pair of the required circles join the corresponding points of their intersections with any one of the given circles; each line of junction will pass through the radical centre. Furthermore, the radius of the orthogonal circle is a mean proportional between the distances along those lines from the radical centre to each of these points of intersection, or to the points of contact of any pair of the required circles, in the case of tangency.

6th. Join the corresponding points of intersection of any one secant circle with any pair of the given circles, the line of junction will pass through one of the centres of similitude of that pair (Article 14).

*This last fact is the one employed in the elegant solution of the problem of Tangencies by Gergonne in Volume 4, Annales de Mathématiques, 1814. This, concisely stated, is as follows. Find in each of the given circles the pole of each axis of similitude. Join it with the radical centre, the joining line will pass through a pair of the required points of contact in each circle. This solution is justly admired. But it is only one of the last and crowning facts of the development. The writer aims at the opposite method, to commence at the beginning and evolve the entire series from an elementary geometrical principle.

7th. The pairs of required circles above referred to are those having the same radical axis, viz. one of the axes of similitude. Find the *other* intersections of the required circles, those not in the circumference of the orthogonal circle. If the intersection farthest from the radical centre is joined with the intersection nearest, the line will pass through the radical centre. In that way we shall find 24 such lines passing through the radical centre, and these circles in pairs intersect each other in equal angles (Article 15).

No. of Group.	24 lines which pass through the centre of the orthogonal circle.		
I	$\frac{2}{3}$ $\frac{1}{4}$	$\frac{2}{3}$ $\frac{1}{4}$	
	$\frac{2}{4}$ $\frac{1}{3}$	$\frac{2}{4}$ $\frac{1}{3}$	
II	$\frac{2}{5}$ $\frac{1}{6}$	$\frac{2}{5}$ $\frac{1}{6}$	
	$\frac{2}{6}$ $\frac{1}{5}$	$\frac{2}{6}$ $\frac{1}{5}$	
III	$\frac{2}{7}$ $\frac{1}{8}$	$\frac{2}{7}$ $\frac{1}{8}$	Circles 2 and 3 intersect at the same angle as circles 1 and 4, and so on through the list.
	$\frac{2}{8}$ $\frac{1}{7}$	$\frac{2}{8}$ $\frac{1}{7}$	
IV	$\frac{3}{5}$ $\frac{4}{6}$	$\frac{4}{6}$ $\frac{3}{5}$	
	$\frac{3}{6}$ $\frac{4}{5}$	$\frac{4}{5}$ $\frac{3}{6}$	
V	$\frac{4}{7}$ $\frac{3}{8}$	$\frac{3}{8}$ $\frac{4}{7}$	
	$\frac{4}{8}$ $\frac{3}{7}$	$\frac{3}{7}$ $\frac{4}{8}$	
VI	$\frac{6}{7}$ $\frac{5}{8}$	$\frac{5}{8}$ $\frac{6}{7}$	
	$\frac{6}{8}$ $\frac{5}{7}$	$\frac{5}{7}$ $\frac{6}{8}$	

THE INTERSECTION OF SPHERES.

21. It will be found that the problems in relation to spheres can be solved by a similar mode of investigation to that which has been followed in those concerning circles. By altering the table on page 10 in the Memoir by the writer in the 8th Volume of Smithsonian Contributions, substituting for "tangent to" the words "*making a given angle β with*," it will be seen that there are fifteen such problems for the intersections as there are for the tangencies of spheres. Of course two spheres intersecting make an angle β with each other when the right-lined elements of the cones drawn tangent to each sphere through the common circle of intersection, make that angle with each other (or the supplement to it).

If two spheres intersect, their radical plane is the plane of the small circle in which they intersect. Draw pairs of spheres tangent in every conceivable manner to them. It will be found that the radical plane of each pair will pass through the *external* centre of similitude if they are tangent in a *similar* manner, that is, both concave or both convex towards the given spheres. If they are tangent in a *different* manner, their radical plane will pass through the *internal* centre of similitude of the pair of spheres. The same is true of the pairs of secant spheres in this problem, being distributed in like manner in reference to the centres of similitude.

22. *To draw a sphere through three given points and making a given angle with a given sphere.*

Pass a circle through the three points, it will form a small circle of the required sphere. From the centre of the sphere let fall a perpendicular to the plane of said circle. Pass through this perpendicular and the centre of the circle a plane. It will cut the circumference of the small circle in two points, and it will cut the given sphere in a great circle. Then by Article 11, through these two points draw a circle making the required angle β with said great circle, it will form a great circle of the required sphere. Thus two such can be found, except when the given angle is a right angle, in which case there is but one.

23. The radical plane of two spheres is the plane from any point of which if tangent lines are drawn to each sphere they will be equal. Any such point is the vertex of a pair of cones tangent to each sphere, having elements of equal length, and it is the centre of a sphere orthogonal to the pair of spheres. And as in Circles—see Figure 1—two points Q and Q' can be found on the line

joining the centres, through which all spheres must pass which are orthogonal to the pair of spheres.

When three spheres are given, as A, B and C (Figure 9), whose centres are in the plane of the paper, the three radical planes will unite at O the radical centre of these great circles of the spheres. Find QQ', HH', EE', the points above referred to. The perpendicular through O to the plane of the paper may be called the common radical axis of the three spheres. Any sphere orthogonal to the three spheres must have its centre on this perpendicular, and pass through the circle drawn through the six points QQ', HH', EE'. Therefore if at will *any* sphere whatever is passed through said circle, it must be orthogonal to the three spheres.

24. *To find the sphere orthogonal to four spheres.*

If a fourth sphere D is given whose centre, in Figure 9, is above the plane of the paper, draw the radical plane between A and D, the point in which said plane cuts the radical axis through O, above described, is the centre of the required orthogonal sphere, or the radical centre of the four spheres. The six radical planes which can be drawn to the four given spheres in pairs will all pass through the radical centre. There are sixteen axes of similitude to the four spheres, four external, twelve internal. There are eight planes of similitude, one external containing the four external axes of similitude.

25. *To draw a sphere which shall cut four given spheres at a given angle.*

It will be found that there are sixteen solutions to this question, as in the Tangency of Spheres, distributed in pairs, each of the eight planes of similitude being the radical plane of a pair. Thus the required spheres intersect each other in pairs in the surface of the orthogonal sphere, the small circle of intersection being their radical plane, which must be one of the planes of similitude of the four given spheres. If any pair do not intersect they still have one of the planes of similitude for a radical plane, and the solution will be an exceptional case, as in Article 19 in Intersection of Circles, and solved in an analogous manner.

Thus to obtain one of the required spheres, find graphically, as in descriptive geometry, the small circle in which one of the planes of similitude intersects the orthogonal sphere. Then proceed as in Article 22: pass a sphere through the circle which will cut one of the given spheres at the given angle β; it will cut the other three spheres at the same angle.

26. The circle referred to has this remarkable property. Any sphere whatever passed through it will cut the four given spheres in the same angle. Or if it does not meet them the intersection becomes imaginary; but it is (as in Circles, Article 16) situated in a similar manner towards each of the given spheres.

27. I might append a complete recapitulation of the peculiar relations of the orthogonal sphere and its centre to the required spheres similar to that given for Circles at Article 20. I subjoin a portion of the same.

If a pair of the required spheres intersect in the surface of the orthogonal sphere, the latter bisects the angle which they make with each other. If they do not intersect, the orthogonal sphere bisects their "angle of approach": also its radius is a mean proportional to the distances from its centre on any straight line to the surfaces of the pair of spheres, and the surface of the orthogonal sphere makes equal angles with their surfaces, estimated at the points in which the straight line meets them.

28. With reference to the number of lines passing through the radical centre of the four spheres:—As regards a pair of required spheres intersecting in the surface of the orthogonal sphere, the line joining the centres of the pair will pass through the radical centre and be perpendicular to the corresponding plane of similitude. If tangential, the line joining the points of contact of the pair with any one of the given spheres will pass through the radical centre. If secant, find the two circles of intersection of such pair with any one of the given spheres, from the radical centre as a vertex pass a cone through the nearest circle, its surface produced will pass through the farthest circle.

29. Again, the intersections of the required spheres (whether tangent or secant) with each other which do not occur in the surface of the orthogonal sphere are distributed in pairs and will obey a law similar to that described in the case of Circles, Article 20, or paragraph 7 in the Recapitulation. Draw a cone from the radical centre as a vertex to the nearest circle of intersection. Its surface produced will pass through the circle of intersection farthest from that centre. And the angle of intersection of the first pair will be equal to the angle of intersection of the second pair of the required spheres.

30. In conclusion it may be stated that articles 11, 18 and 25 set forth the main principles of the generalized theory. Article 11 shows that the solution of the questions in intersections is reduced to one in tangencies, and in the case

of orthogonal circles, all evolved from the principle of the converging chords, or rather of the radical centre.*

31. *To draw a circle which shall cut each of four given circles at the same angle.*

We found in the problem, Article 18, to draw a circle to cut each of three given circles at the same angle, that after finding the two points as K^1 and K^2 in which one of the axes of similitude cuts the orthogonal circle, *any* circle drawn through those two points (Figure 13) will cut the three given circles A, B and C at the same angle. Introduce in the figure a fourth circle D. Find the external and internal centres of similitude of B and D and call them N and N^1. First proceed as if to find in reference to B and D (as in Article 14) a point conjugate to K^1, through which all circles which can be drawn will cut B and D at equal angles. Join K^1 with N, and find the point R on NK^1 such that $NK^1 \times NR = NL \times NL^1$, a constant quantity. Thus those three points K^1, K^2 and R will enjoy these properties—all circles through K^1 and K^2 will intersect circles B and C at equal angles, and all circles through K^1 and R will intersect B and D at equal angles. Hence if we draw through K^1, K^2 and R a circle, it must cut the *four* given circles at one and the same angle.

In like manner join K^1 with N^1 the internal centre of similitude of B and D, and find R^1 a point conjugate to K^1 such that $N^1K^1 \times N^1R^1 = N^1L \times N^1T$. A circle drawn through the three points K^1, K^2 and R^1 will also cut the four circles at the same angle. Thus two solutions are found by the use of the *key* points K^1, K^2. [It must be remarked that if K^2 had been joined with N and afterwards with N^1 in like process, the same identical pair of required circles would be reached, not two new ones.]

In like manner two solutions can be obtained by combining the key points K^1, K^2 with the two centres of similitude of C and D, and two more with those of A and D, or six in all by the use of those points, as shown in the table following:

*It is proper to add that the writer, in the 4th Volume of Johnson's Cyclopedia (published in 1878), under the head of "Tangencies," gave a history of the problem of the Tangencies. The principal addition which now should be made to it is to refer to the important paper by Plücker in Vol. 18 of Gergonne's Annales de Mathématiques, 1827, page 29. Also an article by Poncelet in Vol. 11 of same publication. The title of Plücker is "Memoir upon the contacts and the intersections of circles." It contains a very curious dissertation, but has not anticipated the writer, in the present generalization. In the intersections he takes an entirely different process. My paper in the Smithsonian Contributions, 1855, on "Spheres," as indeed the whole treatment of the subject is believed to be entirely novel. But Plücker in "Circles" gives an analytical solution founded on increasing the radius of each given circle by a certain quantity.

Let M and M^1 be the centres of similitude of circles C and D, P and P^1 of A and D.

By combining K and K^1 with	N and N^1	 2	solutions.
" " "	M and M^1	 2	"
" " "	P and P^1	 2	"
		6	

But there are four axes of similitude of A, B and C, and thus four pairs of key points, or points in which each axis will cut the orthogonal circle. We have called K^1 and K^2 the first pair, and will name the other pairs K^3 and K^4, K^5 and K^6, K^7 and K^8. Treating each pair of key points as above in combination with the centres of similitude of A, B and C respectively as paired with D, there will be 6 solutions for each of the four pairs, or 24. Tabulated as follows:

The centres of similitude of A, B and C as paired with D joined with	K^1, K^2 ...	6
" " " "	K^3, K^4 ...	6
" " " "	K^5, K^6 ...	6
" " " "	K^7, K^8 ...	6
		24

But thus far we have only investigated the number of solutions when A, B and C are combined with D. Combine by the same process any three with a fourth and 24 solutions will be obtained. There are four axes of similitude of *each* group of three of the given circles, each giving a pair of key points to be combined as above with the centres of similitude of a fourth circle. Accordingly there results from

			No. of Solutions.
A, B, C	combined with two centres of similitude of each combined with	D ...	24
A, B, D	" " " "	C ...	24
A, C, D	" " " "	B ...	24
B, C, D	" " " "	A ...	24
		Total	96

There are thus in general ninety-six distinct answers to the problem of finding a circle cutting four given circles at the same angle, which, of course, will usually be different for each solution.

But it is to be noted that some of the intersections will be imaginary; as Salmon (in Conic Sections, page 105) discusses the case of an imaginary intersection of two circles, but that the pair continue after separated to have a common radical axis. But the required circle in such cases would still be situated (Article 16) in a similar manner towards each of the four given circles.

32. *To draw a sphere which shall cut each of five given spheres, A, B, C, D and E, at the same angle (said angle being unknown).*

I need hardly premise that a sphere becomes known when two of its small circles are found, as its centre can be obtained by the intersection of two perpendiculars to the planes of the circles, erected from their centres.

Therefore find the sphere orthogonal to A, B, C and D, and also one of their planes of similitude: it will as explained above cut the orthogonal sphere in a small circle, which (Article 26) enjoys the property that any sphere drawn through it will cut those four spheres at the same angle.

In like manner find the sphere orthogonal to A, B, C and E and also one of their planes of similitude, and the small circle in which it cuts said orthogonal sphere. As any sphere passed through this last circle will cut each of A, B, C and E at the same angle, if a sphere is drawn through these two small circles it must be one of the required spheres, and must cut each of the *five* spheres at the same angle.

To ascertain how many solutions can be obtained to this problem, we must ascertain how many different pairs of such small circles can be found;* or, which is the same thing, we must find how many different pairs of planes of similitude

---

* I will add that I was indebted to a paper by Mr. R. J. Adcock in the "Analyst" for 1877, which impelled me to undertake a geometrical solution of this problem as a sequence of my former investigations. Mr. Adcock had given the equations for an analytical solution. It was followed by a solution of the same equations by Dr. Craig, of the Johns Hopkins University, which appeared in the number of the "Analyst" for January, 1880.

I was also indebted to Mr. Marcus Baker of the U. S. Coast Survey, who, in the number of the "Analyst" for July, 1877, gave out the question at my instance, as it had attracted our attention in Vol. 1st of Crelle, 1826. It was Steiner who proposed it in that number, without a solution. I have never been able to find Steiner's solution, if he ever gave one. I will add that after I made known to Mr. Baker this solution of the problem, we in the hunt for the number of solutions independently reached the same conclusion, that there must be six hundred and forty solutions.

It should be stated that all of this memoir except the two last problems, were completed and sent to the Smithsonian Institute in January, 186[illegible], from Fort Vancouver, Washington Territory, but the manuscript was burned in January, 1865, when the upper story of the Smithsonian building was on fire. The two last problems (from Article 31) were solved in 1878, and read in November of that year to the National Academy of Science at its meeting in New York.

can be obtained for the five given spheres. The five spheres A, B, C, D and E can be grouped in five sets of four each, as

1. A, B, C, D.
2. A, B, C, E.
3. A, B, D, E.
4. A, C, D, E.
5. B, C, D, E.

Take A, B, C, D, each of their 8 planes of similitude can be paired with each of the 8 planes of similitude of

	A, B, C, E	64 pairs.	
Also of	A, B, D, E	64	
"	A, C, D, E	64	
"	B, C, D, E	64	256

Each of A, B, C, E with each of the 8 planes of those after it in the above table, viz.

With	A, B, D, E	64	
	A, C, D, E	64	
	B, C, D, E	64	192

Each of A, B, D, E, with those of

A, C, D, E	64	
B, C, D, E	64	128

The eight of A, C, D, E with eight of B, C, D, E	64	64
Grand total		640

Thus there are six hundred and forty different spheres which can be drawn cutting the five given spheres at the same angle.

As for Circles (Article 31) some of these intersections will be imaginary, but the sphere obtained graphically will be situated in a similar manner towards the given spheres.*

* Since writing the above my attention has been called to a paper by M. Darboux, in p. 323 of Vol. 1st, 2d series of *Annales de L'École Normale Supérieure*, Paris, 1872, upon "the relations between the groups of points of circles and of spheres in a plane and in space," in which will be found solutions of the leading questions in this paper.

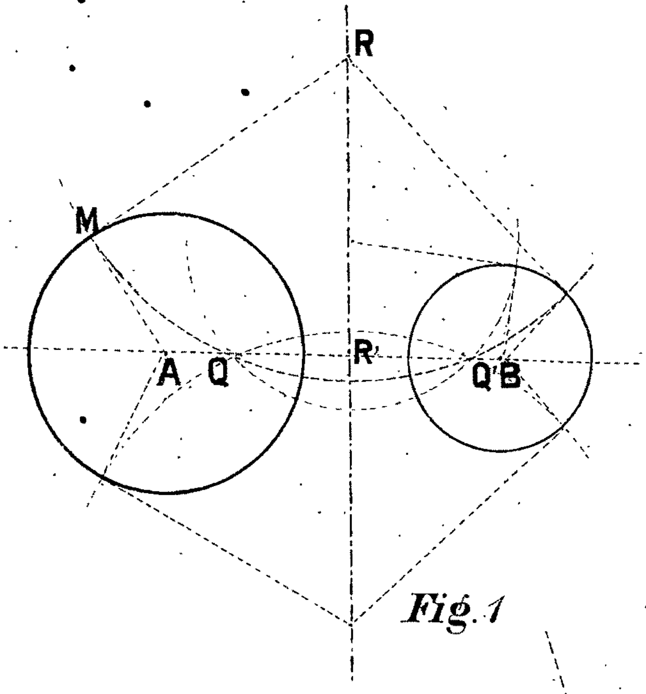


Fig. 1

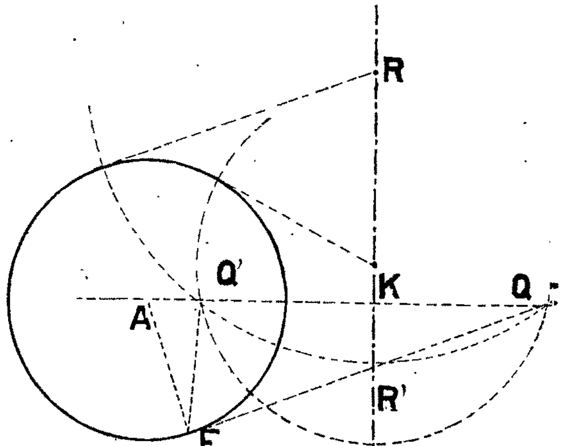


Fig.

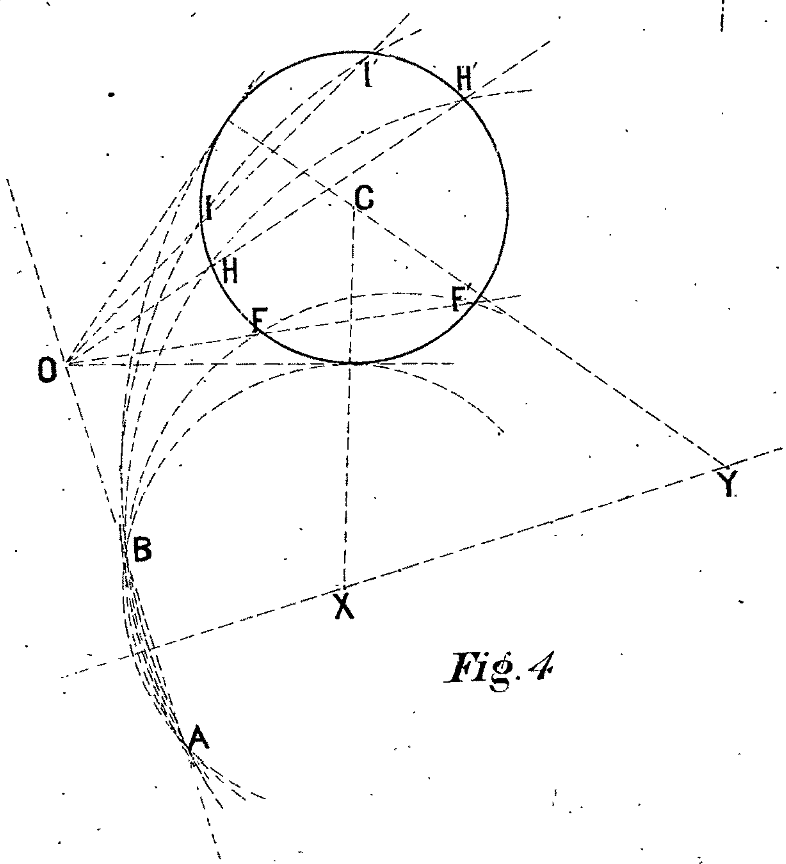


Fig. 4

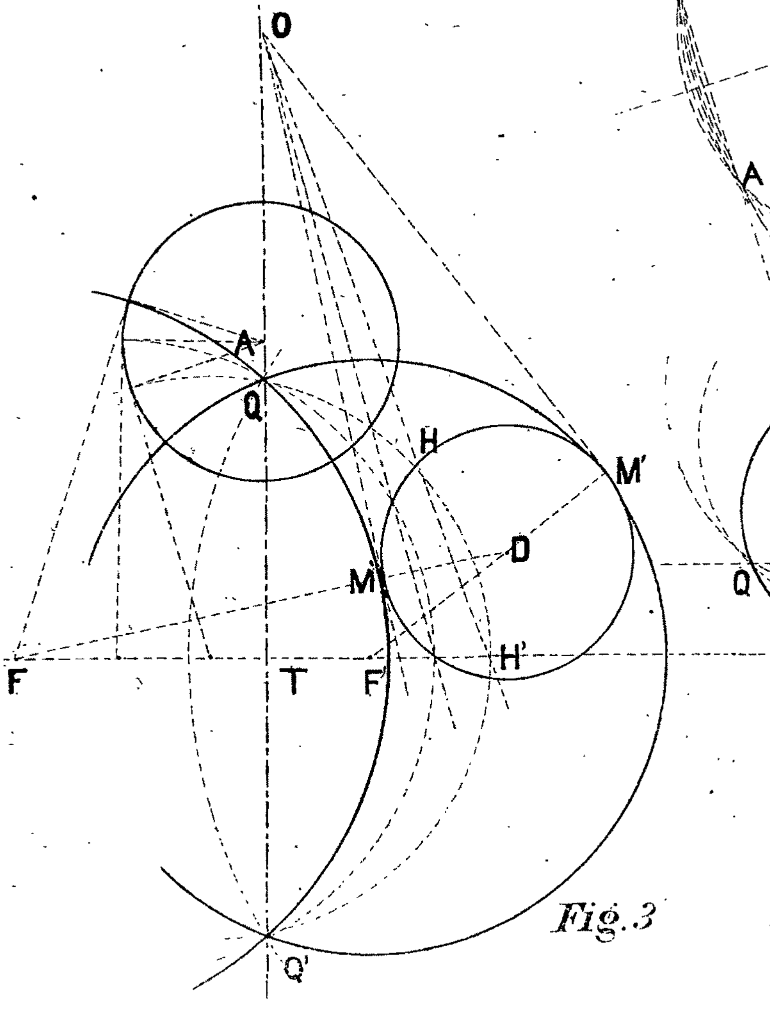


Fig. 3

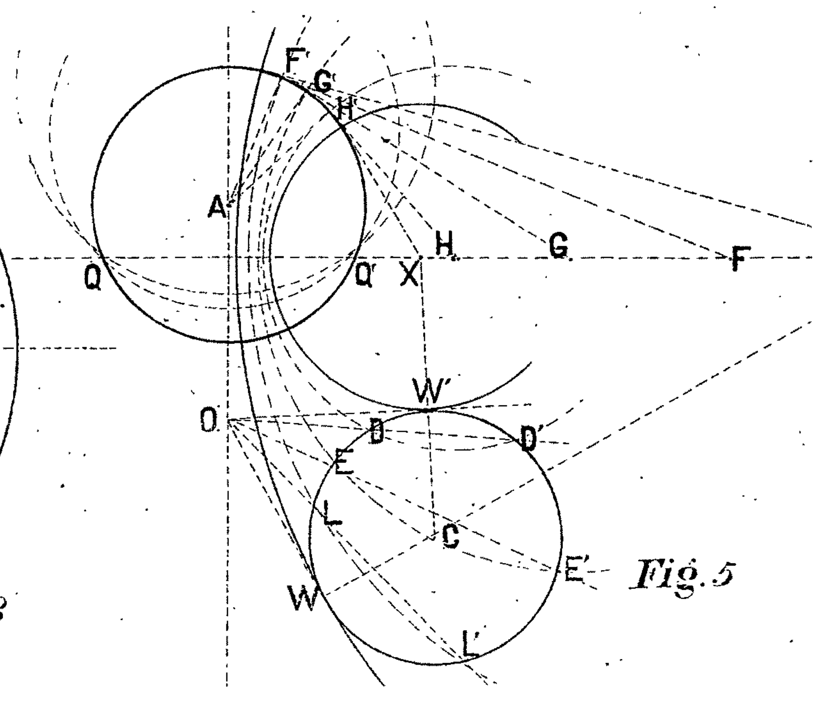


Fig. 5

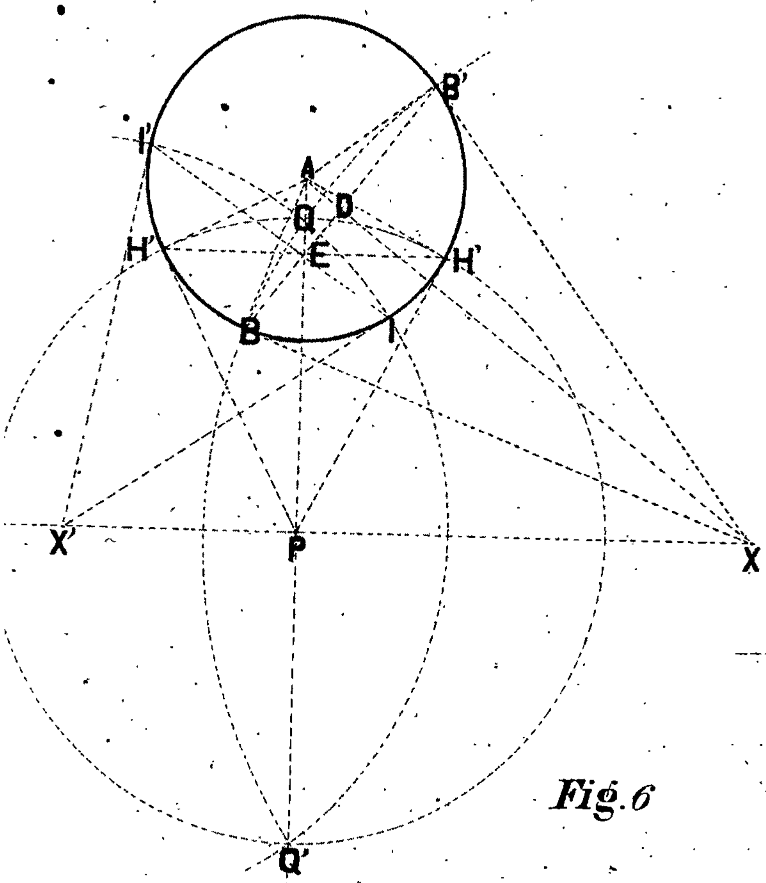

B'
I'
A
D
Q
E
H'
H'
B
I
X'
P
X
Q'

Fig. 6

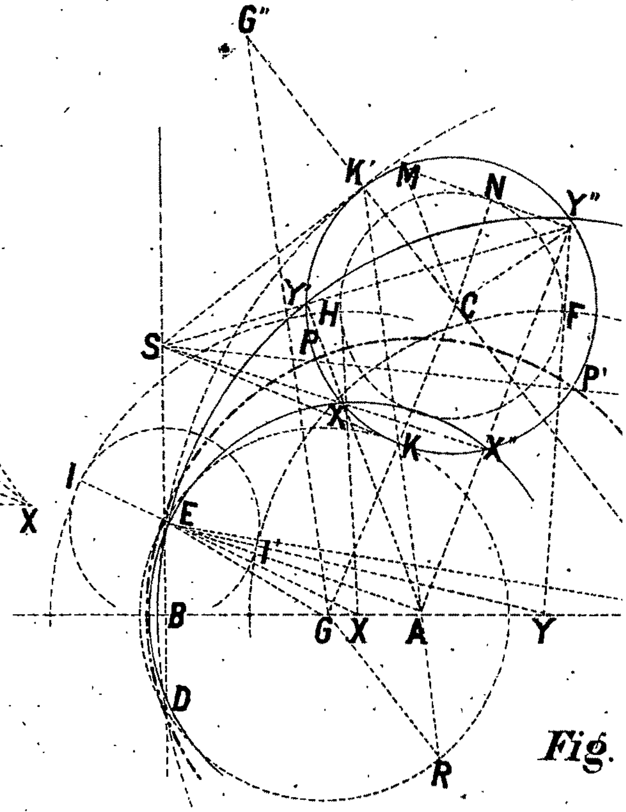
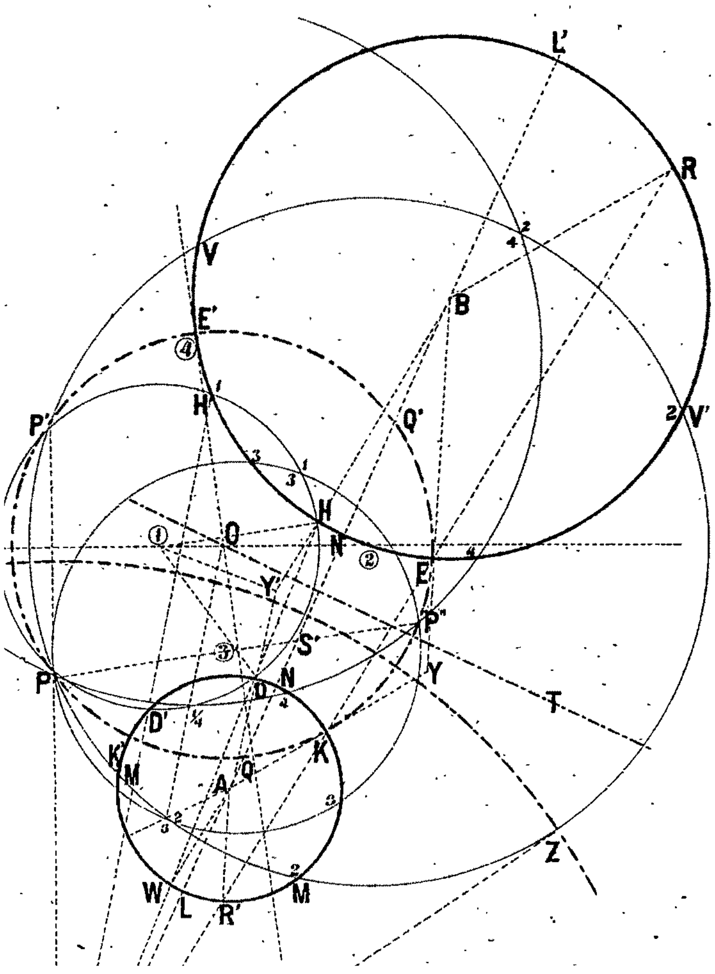
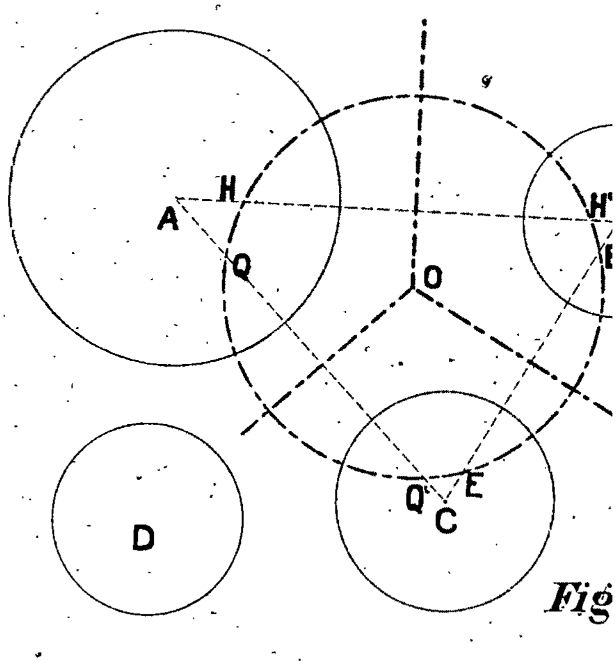

G"
K'
M
N
Y"
Y'
H
C
F
S
P
P'
X
K
X"
I
E
I'
B
G
X
A
Y
D
R

Fig.


L'
R
V
B
E'
P'
H'
Q'
V'
O
H
N
E
Y'
P"
S'
P
D
N
Y
T
D'
K'
M
A'
Q
K
W
L
R'
M
Z


A
H
H'
Q
O
D
Q'
C
E

Fig

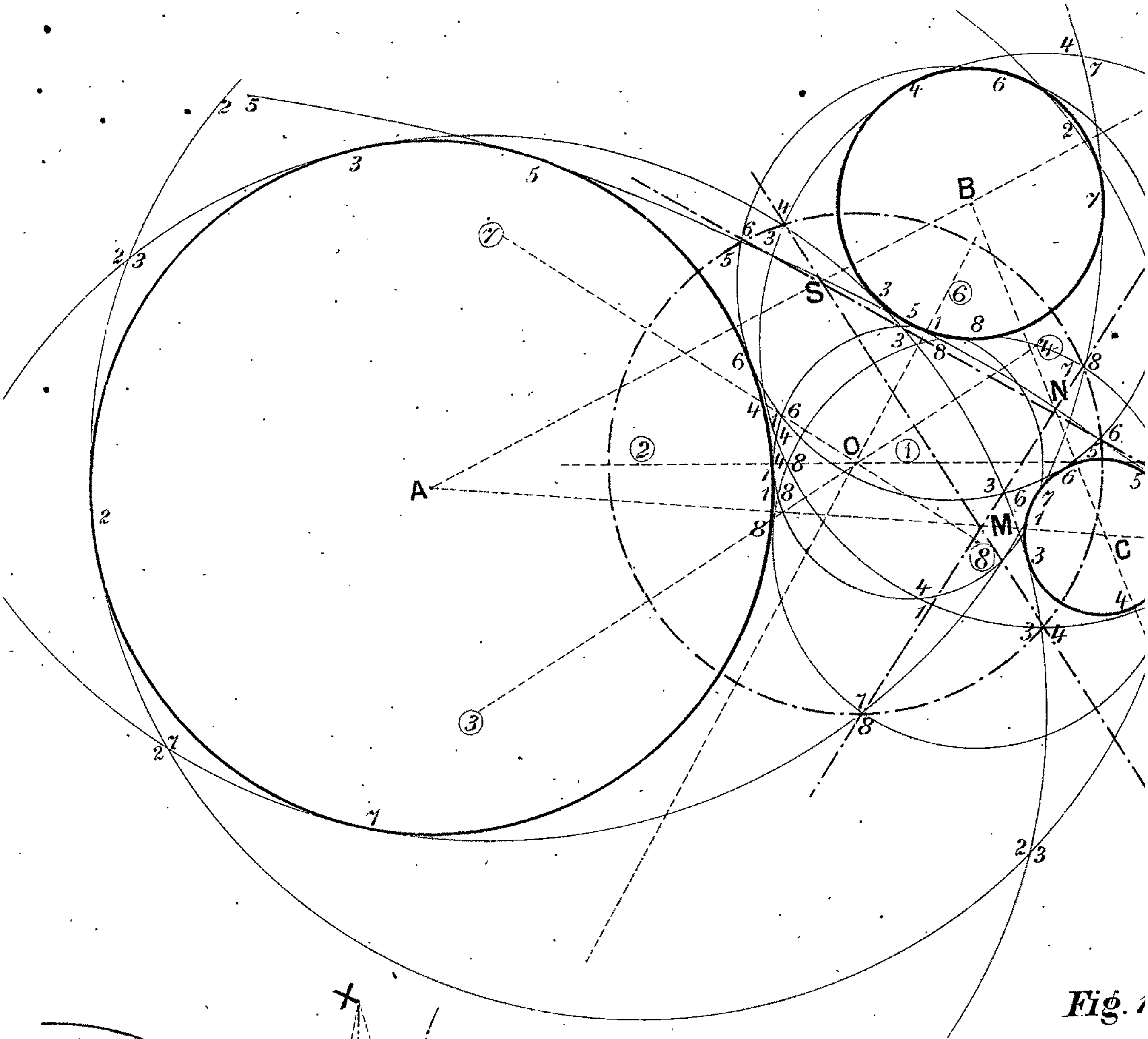
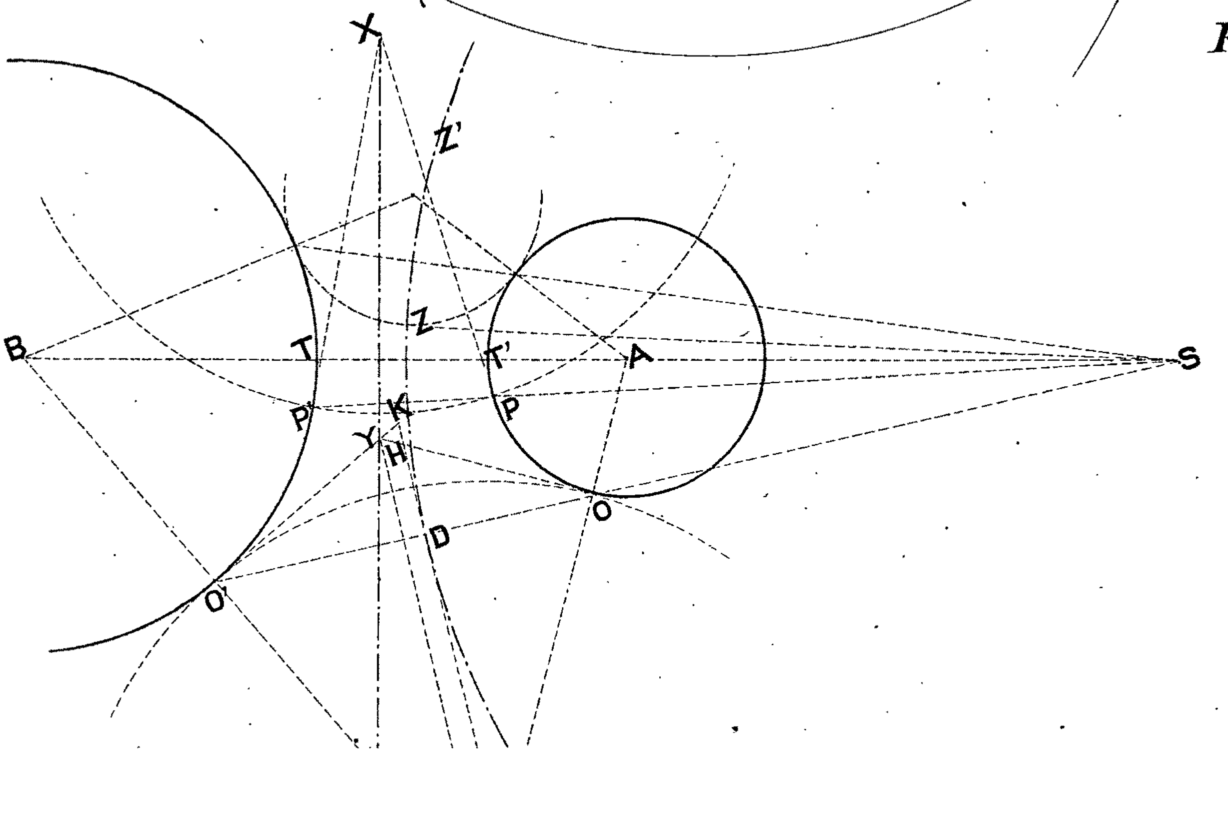


Fig.



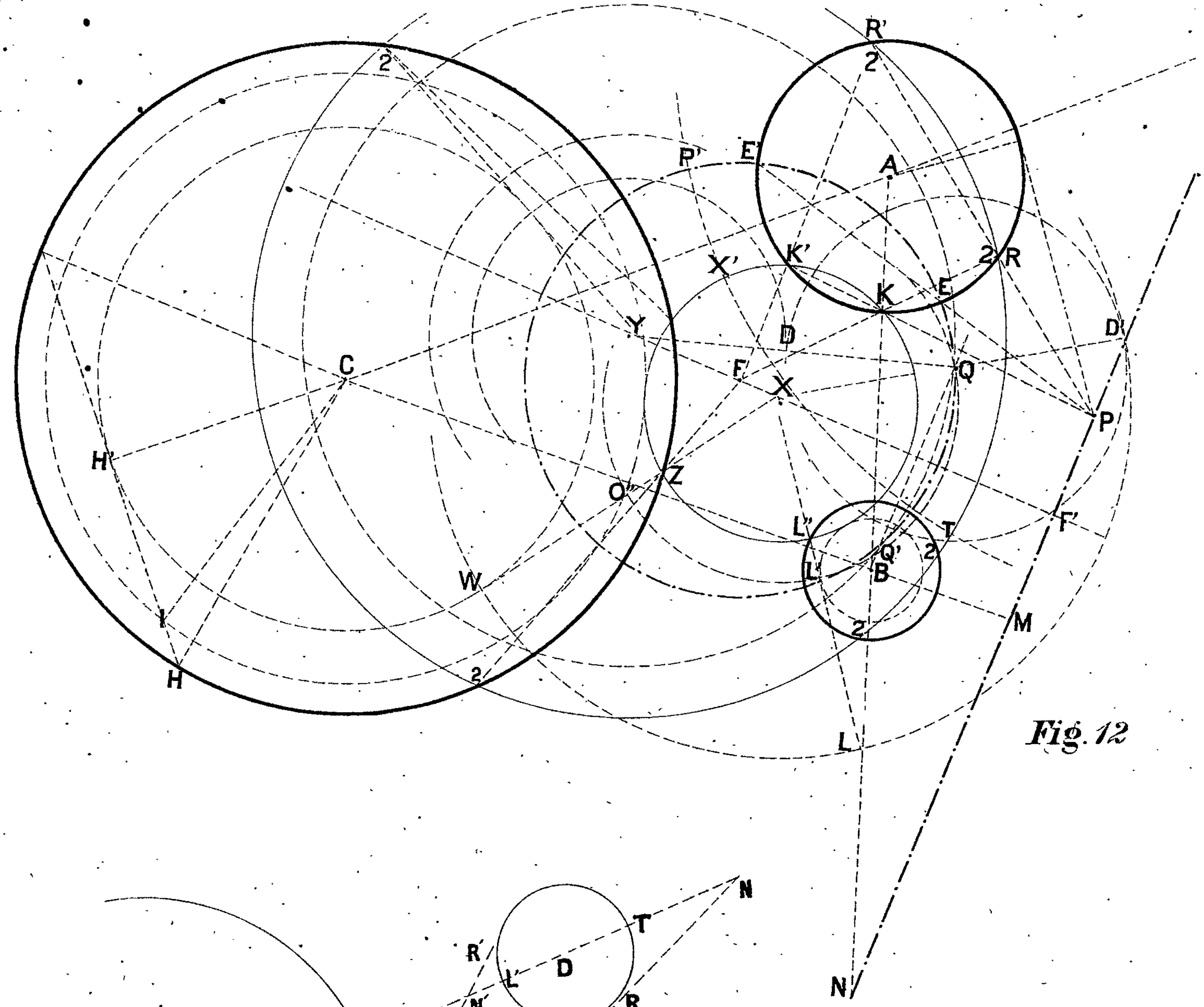

R'
2
A
E'
P'
K'
2
R
K
E
X'
Y
D
D'
F
X
Q
C
P
H'
Z
O''
L''
T
F'
Q'
2
L'
B
W
I
2'
M
H
2
L
N

Fig. 12

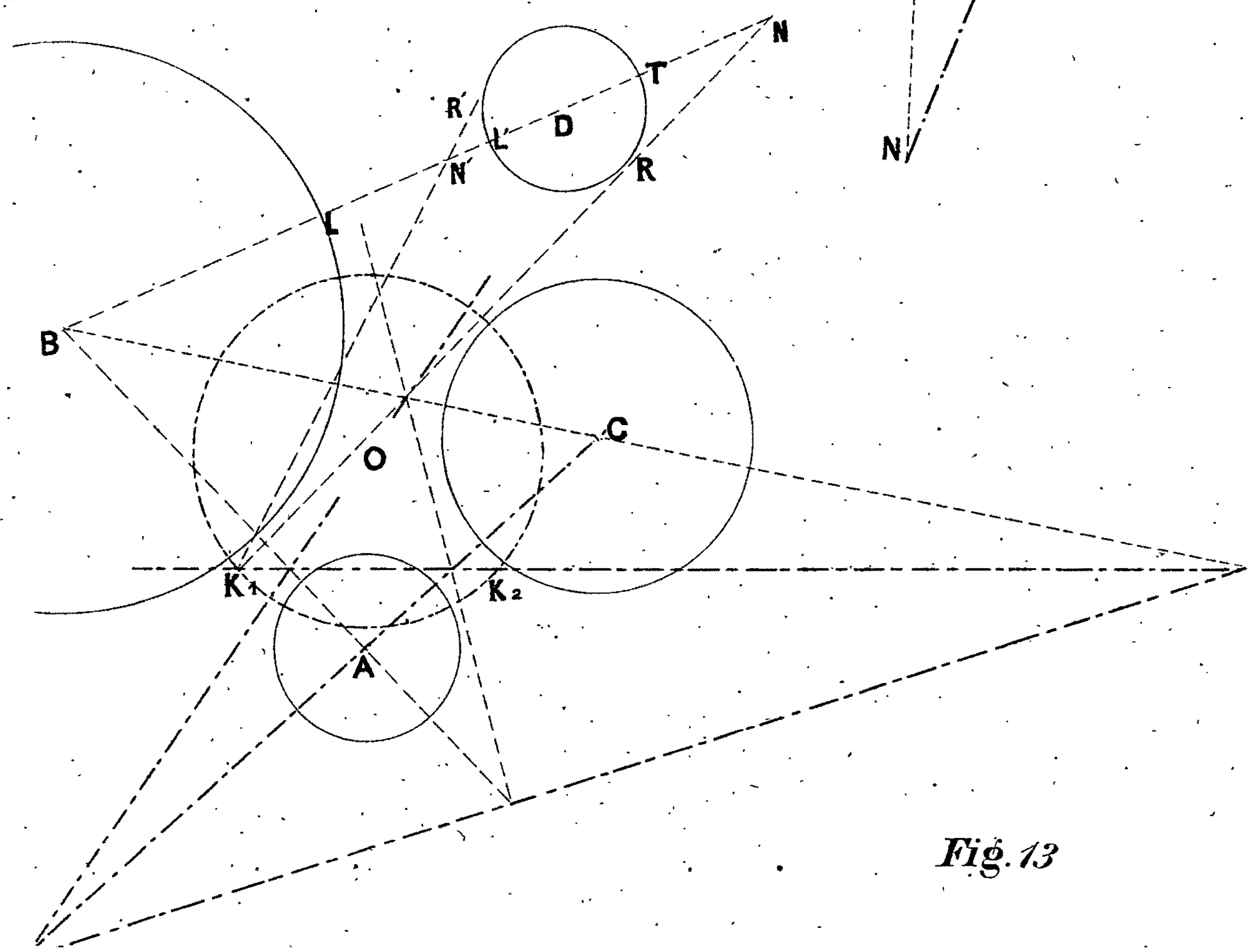

N
T
R'
L'
D
N'
R
L
B
O
C
K1
K2
A

Fig. 13

Tables of the Symmetric Functions of the Twelfthic.

BY W. P. DURFEE, *Fellow of the Johns Hopkins University.*

In the following pages is given in tabular form, 1st the values of the functions (of weight twelve) of the coefficients of the twelfthic $(1, -p_1, p_2, - \ldots p_{12})(x, 1)^{12}$ in terms of the symmetric functions of its roots, and 2nd, the values of the symmetric functions of the roots in terms of the coefficients. The notation used is that of Hirsch. For example: $(4^2 21^2) = p_4^2 p_2 p_1^2$; $[4^2 21^2] = \alpha^4 \beta^4 \gamma^2 \delta \varepsilon$ where $\alpha, \beta, \gamma, \delta$, &c., are the roots of the given quantic. The value of any function is found from the table by taking the sum of the results obtained by multiplying each coefficient in its row by the expression at the head of the corresponding column.

The symmetry of the results was made use of as a check, the method followed being to calculate at the same time the value of a function, say $[\alpha^a \beta^b \gamma^c]$, and the coefficients of the corresponding function $(\alpha^a \beta^b \gamma^c)$ in the value of each of the root functions. In other words, I calculated simultaneously a row and its symmetric column. An error in a constituent of a row would not affect any other of the row, but would affect those following in its column, while a corresponding error in a column would affect the remainder of that column but no other column. Since the rows and columns agree throughout, the existence of errors is almost impossible. As a final test I made use of the well-known expression for the sum of the homogeneous symmetric functions of the roots:

$$H_n = \begin{vmatrix} p_1 & p_2 & \cdots & \cdots & p_n \\ 1 & p_1 & \cdots & \cdots & p_{n-1} \\ & 1 & p_1 & \cdots & p_{n-2} \\ & & \cdots & \cdots & \\ & & & 1 & p_1 \end{vmatrix}$$

If each coefficient p_1, p_2, &c., be made equal to unity, H_n reduces to the sum of its numerical coefficients, which from the resulting form of the determinant is seen to be equal to zero. On summing the coefficients of the second table I found their sum zero as it should be.

TABLE I. (upper half): Coefficients in Terms of Roots.

	$[1^{12}]$	$[21^{10}]$	$[2^21^8]$	$[31^9]$	$[2^31^6]$	$[321^7]$	$[41^8]$	$[2^41^4]$	$[32^21^5]$	$[3^21^6]$	$[421^6]$	$[51^7]$	$[2^51^2]$	$[32^31^3]$	$[3^221^4]$
$(\overline{12})$	1														
$(\overline{11}-1)$	12	1													
$(\overline{10}-2)$	66	10	1												
(93)	220	45	8		1										
(84)	495	120	28		6			1							
(75)	792	210	56		15			4					1		
(6^2)	924	252	70		20			6					2		
$(\overline{10}-1^2)$	132		2	1											
(921)	660	145	26	9	3	1									
(831)	1980	525	128	36	27	7		4	1						
(82^2)	2970	810	201	72	42	15		6	2	1					
(741)	3960	1170	336	84	93	21		24	5	0			5	1	
(732)	7920	2460	736	252	207	70		52	17	6			10	3	1
(651)	5544	1722	532	126	165	35		52	10	0			17	3	0
(642)	13860	4620	1526	504	498	161		160	50	15			50	15	4
(63^2)	18480	6300	2128	756	707	252		228	81	30			70	24	9
(5^22)	16632	5670	1932	630	660	210		228	70	20			81	24	6
(543)	27720	9870	3528	1260	1266	455		456	165	60			165	60	22
(4^3)	34650	12600	4620	1680	1710	630		639	240	90			240	93	36
(91^3)	1320	300	54	28	6	3	1								
(821^2)	5940	1665	418	189	87	38	8	12	5	2	1				
(731^2)	15840	5040	1528	624	432	168	28	108	40	12	6		20	7	2
(72^21)	23760	7740	2392	1080	681	302	56	168	72	32	13		30	12	5
(641^2)	27720	9450	3164	1218	1041	378	56	336	115	30	15		105	34	8
(6321)	55440	19740	6888	3024	2337	1008	168	760	320	132	51		230	93	37
(62^3)	83160	30240	10752	5040	3696	1722	336	1206	552	262	108		360	159	72
(5^21^2)	33264	11592	4004	1512	1380	490	70	480	160	40	20		172	54	12
(5421)	83160	30872	11396	4914	4188	1771	280	1536	335	250	95		565	228	86
(53^21)	110880	42000	15848	7056	5952	2632	420	2220	976	420	150		820	357	154
(532^2)	166320	64260	24696	11592	9417	4410	840	3552	1652	776	315		1320	606	279
(4^231)	138600	53550	20720	9240	8040	3570	560	3132	1390	600	210		1225	546	238
(4^22^2)	207900	81900	32270	15120	12720	5950	1120	5022	2340	1090	440		1990	924	424
(43^22)	277200	111300	44800	21420	18075	8680	1680	7308	3525	1710	690		2960	1434	699
(3^4)	369600	151200	62160	30240	25680	13600	2520	10656	5380	2640	1080		4440	2220	1128
(81^4)	11880	3420	870	468	180	94	33	24	12	6	4	1			
(721^3)	47520	15840	4968	2520	1422	703	204	348	166	78	46	7	60	27	12
$(631\ )$	110880	40320	14280	6888	4890	2289	588	1596	722	306	174	21	480	208	84
(62^21^2)	166320	61740	22288	11340	7737	3578	1064	2532	1541	608	331	42	750	354	167

TABLE I. (upper half continued): Coefficients in Terms of Roots.

	$[42^21^4]$	$[431^5]$	$[521^5]$	$[61^6]$	$[2^6]$	$[32^41]$	$[3^22^21^2]$	$[3^31^3]$	$[42^31^2]$	$[4321^3]$	$[4^21^4]$	$[52^21^3]$	$[531^4]$	$[621^4]$	$[71^5]$
$(\overline{12})$															
$(\overline{11}-1)$															
$(\overline{10}-2)$															
(93)															
(84)															
(75)															
(6^2)					1										
$(\overline{10}-1^2)$															
(921)															
(831)															
(82^2)															
(741)															
(732)															
(651)					6	1									
(642)					15	4	1								
(63^2)					20	6	2	1							
(5^22)					30	9	2	0							
(543)					60	22	8	3							
(4^3)					90	36	15	6							
(91^3)															
(821^2)															
(731^2)	1														
(72^21)	2	1													
(641^2)	4	0			30	9	2	0	1						
(6321)	14	5			60	22	8	3	3	1					
(62^3)	30	16			90	36	15	6	6	3	1				
(5^21^2)	6	0			60	20	4	0	2	0	0				
(5421)	32	10			210	83	30	9	11	3	0				
(53^21)	53	20			300	128	54	24	18	7	0				
(532^2)	114	55			480	216	96	42	39	18	4				
(4^231)	80	30			480	216	96	42	31	12	0				
(4^22^2)	172	80			795	368	168	72	68	30	6				
(43^22)	284	140			1200	584	286	141	117	58	12				
(3^4)	468	240			1860	936	480	256	204	108	24				
(81^4)															
(721^3)	7	3	1												
(631^3)	47	15	5		120	48	18	6	10	3	0	1			
(62^21^2)	90	43	11		180	78	34	15	18	8	2	2	1		

TABLE I. (upper half continued): Coefficients in Terms of Roots.

	[3²2³]	[3³21]	[42⁴]	[432²1]	[43²1²]	[4²21²]	[52³1]	[3⁴]	[43²2]	[4²2²]	[4²31]	[53²1]	[5421]	[4³]	[543]	[5²2]
$(\overline{12})$																
$\overline{1}$—1)																
$\overline{0}$—2)												. . .	. . .		. . .	. . .
(93)																
(84)																
(75)																
(6²)																
—1²)																
(921)																
(831)																
(82²)																
(741)																
(732)																
(651)																
(642)																
(63²)																
(5²2)	1															
(543)	3	1														
(4³)	6	3						1								
(91³)																
(821²)																
(731²)																
(72²1)																
(641²)																
(6321)																
(62³)																
(5²1²)	2	0	1													
(5421)	11	3	4	1												
(53²1)	18	7	6	2	1											
(532²)	31	12	12	5	2	1										
(4²31)	39	18	12	5	2	0		4	1							
(4²2²)	68	30	28	12	4	2		6	2	1						
(43²2)	117	58	48	24	12	5		12	5	2	1					
(3⁴)	204	108	90	48	28	12		24	12	6	4			1		
(81⁴)																
(721³)																
(631³)																
32²1²)																

TABLE I. (lower half): Coefficients in Terms of Roots.

	$[1^{12}]$	$[21^{10}]$	$[2^21^8]$	$[31^9]$	$[2^31^6]$	$[321^7]$	$[41^8]$	$[2^41^4]$	$[32^21^5]$	$[3^21^6]$	$[421^6]$	$[51^7]$	$[2^51^2]$	$[32^31^3]$	$[3^221^4]$
(541^3)	166320	63000	23604	11088	8766	3983	966	3240	1420	570	320	35	1200	507	192
(5321^2)	332640	131040	51128	25704	19722	9793	2548	7500	3670	1762	926	105	2800	1344	632
(52^31)	498960	200340	79632	41580	31221	16212	4536	12012	6177	3192	1713	210	4500	2277	1153
(4^221^2)	415800	166950	66780	33390	26640	13160	3360	10620	5180	2460	1280	140	4245	2046	954
(43^21^2)	554400	226800	92680	47040	37860	19110	4900	15468	7780	3840	1950	210	6320	3174	1570
(432^21)	831600	346500	144200	75600	59925	31360	8680	24864	12980	6760	3575	420	10300	5361	2779
(42^4)	1247400	529200	224280	120960	94860	51240	15120	40002	21600	11760	6420	840	16800	9054	4900
(3^321)	1108800	470400	199920	105840	85140	45150	12600	36336	19320	10260	5400	630	15540	8289	4428
(3^22^3)	1663200	718200	310800	168840	134760	73500	21840	58536	32040	17660	9630	1260	25470	13986	7728
(71^5)	95040	32400	10320	5760	2970	1620	600	720	380	200	135	36	120	60	30
(621^4)	332640	126000	46200	25200	16200	8680	2940	5316	2780	1440	910	196	1560	786	392
(531^4)	665280	267120	105840	56448	41310	21644	6888	15840	8132	4056	2489	476	5940	2976	1442
(52^21^3)	997920	408240	164808	90720	65412	35658	11844	25380	13656	7308	4446	882	9540	5037	2640
(4^21^4)	831600	340200	138180	73080	55800	28980	9030	22464	11440	5620	3420	630	9060	4524	2160
(4321^3)	1663200	705600	298200	163800	125550	68495	22260	52668	28530	15270	9110	1715	22020	11841	6284
(42^31^2)	2494800	1077300	463680	260820	198765	111510	37800	84780	47385	26460	15975	3150	35940	19986	11085
(3^31^3)	2217600	957600	413280	228480	178380	98280	31920	77040	42360	23040	13590	2520	33300	18300	9972
$(3^22^21^2)$	3326400	1461600	642320	362880	282360	159460	54040	124176	70120	39520	23720	4620	54640	30858	17404
(32^41)	4989600	2230200	997920	574560	446940	257880	90720	200328	115800	67200	40980	8400	89850	52020	30220
(2^6)	7484400	3402000	1549800	907200	707400	415800	151200	323460	190800	113400	70200	15120	148140	87660	52200
(61^6)	665280	257040	95760	55440	33930	19320	7560	11160	6210	3440	2355	672	3240	1740	930
(521^5)	1995840	831600	341040	196560	137070	78120	29400	53640	30130	16800	11085	2856	20220	11130	6070
(431^5)	3326400	1436400	616560	352800	263070	149100	54600	111600	62600	34600	22425	5460	47100	26130	14250
(42^21^4)	4989600	2192400	958440	559440	416520	241920	91140	179748	103780	59640	38670	9660	76920	44088	25120
(3^221^4)	6652800	2973600	1327200	776160	591660	344960	129360	263520	153220	88560	56990	14000	117300	68040	39244
(32^31^3)	9979200	4536000	2061360	1224720	936540	556290	214200	425376	252600	149940	97200	24570	193140	114657	68040
(2^51^2)	14968800	6917400	3200400	1927800	1482300	894600	352800	687240	415500	252000	164700	42840	318930	193140	117300
(51^7)	3991680	1693440	705600	423360	287280	170520	70560	113400	66360	38640	26880	8232	42840	24570	14000
(421^6)	9979200	4460400	1980720	1194480	872910	523320	214200	381240	226950	134400	91725	26880	164700	97200	56990
(3^21^6)	13305600	6048000	2741760	1653120	1239840	744240	302400	559440	334320	198400	134400	38640	252000	149940	88560
(32^21^5)	19958400	9223200	4257120	2600640	1962540	1197000	495600	903600	550240	334320	226950	66360	415500	252600	153220
(2^41^4)	29937600	14061600	6607440	4082400	3106080	1920240	808920	1460736	903600	559440	381240	113400	687240	425376	263520
(41^8)	19958400	9072000	4092480	2540160	1829520	1128960	493920	808920	495600	302400	214200	70560	352800	214200	129360
(321^7)	39916800	18748800	8789760	5503680	4112640	2569560	1128960	1920240	1197000	744240	523320	170520	894600	556290	344960
(2^31^6)	59875200	28576800	13638240	8618400	6508620	4112640	1829520	3106080	1962540	1239840	872910	287280	1482300	936540	591660
(31^9)	79833600	38102400	18144000	11612160	8618400	5503680	2540160	4082400	2600640	1653120	1194480	423360	1927800	1224720	776160
(2^21^8)	119750400	58060800	28143360	18144000	13638240	8789760	4092480	6607440	4257120	2741760	1980720	705600	3200400	2061360	1327200
(21^{10})	239500800	117936000	58060800	38102400	28576800	18748800	9072000	14061600	9223200	6048000	4460400	1693440	6917400	4536000	2973600
(1^{12})	479001600	239500800	119750400	79833600	59875200	39916800	19958400	29937600	19958400	13305600	9979200	3991680	14968800	9979200	6652800

TABLE I. (lower half continued): Coefficients in Terms of Roots.

	$[42^21^4]$	$[431^5]$	$[521^5]$	$[61^6]$	$[2^6]$	$[32^41]$	$[3^22^21^2]$	$[3^31^3]$	$[42^31^2]$	$[4321^3]$	$[4^21^4]$	$[52^21^3]$	$[531^4]$	$[621^4]$	$[71^5]$
(541^3)	106	30	10		450	184	66	18	36	9	0	3	0		
(5321^2)	327	145	35		1020	476	216	96	110	47	8	11	4		
(52^31)	618	321	75		1620	793	390	189	207	103	30	24	13		
(4^221^2)	488	210	50		1710	817	376	162	189	78	12	18	6		
(43^21^2)	781	360	80		2580	1296	644	318	314	147	24	31	12		
(432^21)	1466	755	170		4260	2210	1140	582	599	305	80	68	34		
(42^4)	2694	1500	360		7020	3763	2026	1080	1116	610	201	150	88		
(3^321)	2328	1230	270		6660	3564	1914	1038	1008	540	144	117	60		
(3^22^3)	4248	2390	570		11100	6114	3386	1884	1875	1056	344	258	150		
(71^5)	20	10	5	1											
(621^4)	245	120	50	6	360	168	78	36	48	22	6	9	4	1	
(531^4)	872	390	155	15	2160	1048	488	216	291	124	24	48	16	4	
(52^21^3)	1593	828	306	30	3420	1752	886	444	528	264	78	96	48	9	
(4^21^4)	1294	560	220	20	3690	1812	844	360	498	204	36	78	24	6	
(4321^3)	3711	1905	675	60	9180	4900	2578	1332	1506	763	204	264	124	22	
(42^31^2)	6678	3705	1305	120	15120	8352	4592	2505	2754	1506	498	528	291	48	
(3^31^3)	5820	3060	1050	90	14400	7920	4332	2364	2505	1332	360	444	216	36	
$(3^22^21^2)$	10428	5840	2020	180	24060	13592	7672	4332	4592	2578	844	886	488	78	
(32^41)	18504	10860	3840	360	40320	23376	13592	7920	8352	4900	1812	1752	1048	168	
(2^6)	32580	19800	7200	720	67950	40320	24060	14400	15120	9180	3690	3420	2160	360	
(61^6)	630	335	171	37	720	360	180	90	120	60	20	30	15	6	1
(521^5)	3970	2135	985	171	7200	3840	2020	1050	1305	675	220	306	155	50	5
(431^5)	9140	4800	2135	335	19800	10860	5840	3060	3705	1905	560	828	390	120	10
(42^21^4)	16201	9140	3970	630	32580	18504	10428	5820	6678	3711	1294	1593	872	245	20
(3^221^4)	25120	14250	6070	930	52200	30220	17404	9972	11085	6284	2160	2640	1442	392	30
(32^31^3)	44088	26130	11130	1740	87660	52020	30858	18300	19986	11841	4524	5037	2976	786	60
(2^51^2)	76920	47100	20220	3240	148140	49350	54640	33300	35940	22020	9060	9540	5940	1560	120
(51^7)	9660	5460	2856	672	15120	8400	4620	2520	3150	1715	630	882	476	196	36
(421^6)	38670	22425	11085	2355	70200	40980	23720	13590	15975	9110	3420	4446	2489	910	135
(3^21^6)	59640	34600	16800	3440	113400	67200	39520	23040	26460	15270	5620	7308	4056	1440	200
(32^21^5)	103780	62600	30130	6210	190800	115800	70120	42360	47385	28530	11440	13656	8132	2780	380
(2^41^4)	179748	111600	530640	11160	323460	200328	124176	77040	84780	52668	22464	25380	15840	5316	720
(41^8)	91140	54600	29400	7560	151200	90720	54040	31920	37800	22260	9030	11844	6888	2940	600
(321^7)	241920	149100	78120	19320	415800	257880	159460	98280	111510	68495	28980	35658	21644	8680	1620
(2^31^6)	416520	263070	137070	33930	707400	446940	282360	178380	198765	125550	55800	65412	41310	16200	2970
(31^9)	559440	352800	196560	55440	907200	574560	362880	228480	260820	168800	73080	90720	56448	25200	5760
(2^21^8)	958440	616560	341040	95760	1549800	997920	642320	413280	463680	298200	138180	164808	105840	46200	10320
(21^{10})	2192400	1436400	831600	257040	3402000	2230200	1461600	957600	1077300	705600	340200	408240	267120	126000	32400
(1^{12})	4989600	3326400	1995840	665280	7484400	4989600	3326400	2217600	2494800	1663200	831600	997920	665280	332640	95040

TABLE I. (lower half continued): Coefficients in Terms of Roots.

	$[3^22^3]$	$[3^321]$	$[42^4]$	$[432^21]$	$[43^21^2]$	$[4^221^2]$	$[52^31]$	$[5321^2]$	$[541^3]$	$[62^21^2]$	$[631^3]$	$[721^3]$	$[81^4]$	$[3^4]$	$[43^22]$	$[4^22^2]$
(541^3)	24	6	13	3	0	0	1									
(5321^2)	68	27	34	13	5	2	3	1								
(52^31)	117	51	60	27	12	7	6	3	1							
(4^221^2)	153	66	76	31	10	4	7	2	0					12	5	2
(43^21^2)	264	132	126	60	29	10	12	5	0					28	12	4
(432^21)	467	237	244	123	60	31	27	13	3					48	24	12
(42^4)	828	438	456	244	126	76	60	34	13					90	48	28
(3^321)	828	451	438	237	132	66	51	27	6					108	58	30
(3^22^3)	1486	828	828	467	264	153	117	68	24					204	117	68
(71^5)																
(621^4)																
(531^4)	150	60	88	34	12	6	13	4	0	1						
(52^21^3)	258	117	150	68	31	18	24	11	3	2	1					
(4^21^4)	344	144	201	80	24	12	30	8	0	2	0			24	12	6
(4321^3)	1056	540	610	305	147	78	103	47	9	8	3			108	58	30
(42^31^2)	1875	1008	1116	599	314	189	207	110	36	18	10			204	117	68
(3^31^3)	1884	1038	1080	582	318	162	189	96	18	15	6			256	141	72
$(3^22^21^2)$	3386	1914	2026	1140	644	370	390	216	66	34	18			480	286	168
(32^41)	6114	3564	3768	2210	1296	817	798	476	184	78	48			936	584	368
(2^6)	11100	6660	7020	4260	2580	1710	1620	1020	450	180	120			1860	1200	795
(61^6)																
(521^5)	570	270	360	170	80	50	75	35	10	11	5	1				
(431^5)	2390	1230	1500	755	360	210	321	145	30	43	15	3		240	140	80
(42^21^4)	4248	2328	2694	1466	781	488	618	327	106	90	47	7		468	284	172
(3^221^4)	7728	4428	4900	2779	1570	954	1153	632	192	167	84	12		1128	699	420
(32^31^3)	13986	8289	9054	5361	3174	2046	2277	1344	507	354	208	27		2220	1434	924
(2^51^2)	25470	15540	16800	10300	6320	4245	4500	2800	1200	750	480	60		4440	2960	1990
(51^7)	1260	630	840	420	210	140	210	105	35	42	21	7	1			
(421^6)	9630	5400	6420	3575	1950	1280	1713	926	320	331	174	46	4	1080	690	440
(3^21^6)	17660	10260	11760	6760	3840	2460	3192	1762	570	608	306	78	6	2640	1710	1090
(32^21^5)	32040	19320	21600	12980	7780	5180	6177	3670	1420	1241	722	166	12	5280	3525	2340
(2^41^4)	58536	36336	40002	24864	15468	10620	12012	7500	3240	2532	1596	348	24	10656	7398	5022
(41^8)	21840	12600	15120	8680	4900	3360	4536	2548	966	1064	588	204	33	2520	1680	1120
(321^7)	73500	45150	51240	31360	19110	13160	16212	9793	3983	3878	2289	703	94	12600	8680	5950
(2^31^6)	134760	85140	94860	59925	37860	26640	31221	19722	8766	7737	4890	1422	180	25680	18070	12720
(31^9)	168840	105840	120960	75600	47040	33390	41580	25704	11088	11340	6888	2520	468	30240	21420	15120
(2^21^8)	310800	199920	224280	144200	92680	66780	79632	51128	23604	22288	14280	4968	870	62160	44800	32270
(21^{10})	718200	470400	529200	346500	226800	166950	200340	131040	63000	61740	40320	15840	3420	151200	111300	81900
(1^{12})	1663200	1108800	1247400	831600	554400	415800	498960	332640	166320	166320	110880	47520	11880	369600	277200	207900

TABLE I. (lower half continued): Coefficients in Terms of Roots.

	$[4^231]$	$[532^2]$	$[53^21]$	$[5421]$	$[5^21^2]$	$[62^3]$	$[6321]$	$[641^2]$	$[72^21]$	$[731^2]$	$[821^2]$	$[91^3]$	$[4^3]$	$[543]$	$[5^22]$	$[63^2]$
(541^3)																
(5321^2)																
(52^31)																
(4^221^2)	0	1														
(43^21^2)	2	2	1													
(432^21)	5	5	2	1												
(42^4)	12	12	6	4	1											
(3^321)	18	12	7	3	0								3	1		
(3^22^3)	39	31	18	11	2								6	3	1	
(71^5)																
(621^4)																
(531^4)																
(52^21^3)																
(4^21^4)	0	4	0	0	0	1										
(4321^3)	12	18	7	3	0	3	1									
(42^31^2)	31	39	18	11	2	6	3	1								
(3^31^3)	42	42	24	9	0	6	3	0					6	3	0	1
$(3^22^21^2)$	96	96	54	30	4	15	8	2					15	8	2	2
(32^41)	216	216	128	83	20	36	22	9					36	22	9	6
(2^6)	480	480	300	210	66	90	60	30					90	60	30	20
(61^6)																
(521^5)																
(431^5)	30	55	20	10	0	16	5	0	1							
(42^21^4)	80	114	53	32	6	30	14	4	2	1						
(3^221^4)	238	279	154	86	12	72	37	8	5	2			36	22	6	9
(32^31^3)	546	606	357	228	54	159	93	34	12	7			93	60	24	24
(2^51^2)	1225	1320	820	565	172	360	230	105	30	20			240	165	81	70
(51^7)																
(421^6)	210	315	150	95	20	108	51	15	13	6	1					
(3^21^6)	600	776	420	250	40	262	132	30	32	12	2		90	60	20	30
(32^21^5)	1390	1652	976	635	160	552	320	115	72	40	5		240	165	70	81
(2^41^4)	3132	3552	2220	1536	480	1206	760	336	168	108	12		639	456	228	228
(41^8)	560	840	420	280	70	336	168	56	56	28	8	1				
(321^7)	3570	4410	2632	1771	490	1722	1008	378	302	168	38	8	630	455	210	252
(2^31^6)	8040	9417	5952	4188	1380	3696	2337	1041	631	432	87	6	1710	1266	660	707
(31^9)	9240	11592	7056	4914	1512	5040	3024	1218	1030	624	189	28	1680	1260	630	756
(2^21^8)	20720	24696	15848	11896	4004	10752	6888	3164	2392	1528	418	54	4620	3528	1932	2128
(21^{10})	53550	64260	42000	30870	11592	30240	19740	9450	7740	5040	1665	300	12600	9870	5670	6300
(1^{12})	138600	166320	110880	83160	33264	83160	55440	27720	23760	15840	5940	1320	34650	27720	16632	18480

TABLE I. (lower half continued): Coefficients in Terms of Roots.

	[642]	[651]	[732]	[741]	[82²]	[831]	[921]	[$\overline{10}$-1²]	[6²]	[75]	[84]	[93]	[$\overline{10}$-2]	[$\overline{11}$-1]	[$\overline{12}$]
(541³)															
(5321²)															
(52³1)															
(4²21²)															
(43²1²)															
(432³1)															
(42⁴)															
(3³21)															
(3²2³)															
(71⁵)															
(621⁴)															
(531⁴)															
(52²1³)															
(4²1⁴)															
(4321³)															
(42³1²)															
(3³1³)															
(3²2²1²)	1														
(32⁴1)	4	1													
(2⁶)	15	6							1						
(61⁶)															
(521⁵)															
(431⁵)															
(42²1⁴)															
(3²21⁴)	4	0	1												
(32³1³)	15	3	3	1											
(2⁵1²)	50	17	10	5					2	1					
(51⁷)															
(421⁶)															
(3²1⁶)	15	0	6	0	1										
(32²1⁵)	50	10	17	5	2	1									
(2⁴1⁴)	160	52	52	24	6	4			6	4	1				
(41⁸)															
(321⁷)	161	35	70	21	15	7	1								
(2³1⁶)	498	165	207	93	42	27	3		20	15	6	1			
(31⁹)	504	126	252	84	72	36	9	1							
(2²1⁸)	1526	532	736	336	201	128	26	2	70	56	28	8	1		
(21¹⁰)	4620	1722	2460	1170	810	525	145	21	252	210	120	45	10	1	
(1¹²)	13860	5544	7920	3960	2970	1980	660	132	924	792	495	220	66	12	1

TABLE II. (upper half): Roots in Terms of Coefficients.

	(12)	(11-1)	(10-2)	(93)	(84)	(75)	(6^2)	(101^2)	(921)	(831)	(82^2)	(741)	(732)	(651)	(642)
[1^{12}]	1														
[21^{10}]	− 12	1													
[2^21^8]	+ 54	− 10	1												
[31^9]	+ 12	− 1	− 2					1							
[2^31^6]	− 112	+ 35	− 8	1											
[321^7]	− 96	+ 19	+ 16	− 3				− 9	1						
[41^8]	− 12	+ 1	+ 2	+ 3				− 1	− 8						
[2^41^4]	+ 105	− 50	+ 20	− 6	1										
[32^21^5]	+ 252	− 87	− 32	+ 18	− 4			+ 27	− 7	1					
[3^21^6]	+ 42	− 9	− 17	+ 3	+ 2			+ 9	− 1	− 2	1				
[421]	+ 84	− 18	− 14	− 21	+ 4			+ 8	+ 23	− 1	− 2				
[51^7]	+ 12	− 1	− 2	− 3	− 4			+ 1	+ 3	+ 4	+ 2				
[2^51^2]	− 36	+ 25	− 16	+ 9	− 4	1									
[32^31^3]	− 240	+ 130	0	− 21	+ 16	− 5		− 30	− 14	− 5	0	1			
[3^221^4]	− 180	+ 70	+ 70	− 36	− 4	+ 5		− 45	− 13	+ 10	− 6	− 3	1		
[42^21^4]	− 180	+ 70	+ 20	+ 45	− 20	+ 5		− 20	− 54	+ 5	+ 12	− 1	− 2		
[431^5]	− 72	+ 17	+ 32	+ 18	− 8	− 5		− 17	− 22	+ 5	0	+ 5	− 1		
[521^5]	− 72	+ 17	+ 12	+ 18	+ 24	− 5		− 7	− 20	− 27	− 12	+ 1	+ 5		
[61^6]	− 12	+ 1	+ 2	+ 3	+ 4	+ 5		− 1	− 3	− 4	− 2	− 5	− 5		
[2^6]	+ 2	− 2	+ 2	− 2	+ 2	− 2	1								
[32^41]	+ 60	− 49	+ 20	− 6	− 4	+ 10	− 6	+ 9	− 7	+ 5	0	− 3	0	1	
[$3^22^21^2$]	+ 180	− 114	− 45	+ 63	− 28	− 5	+ 9	+ 54	− 30	− 5	+ 9	− 9	− 4	− 4	1
[3^31^3]	+ 40	− 18	− 30	+ 17	+ 8	− 5	− 2	+ 18	− 6	− 11	+ 6	− 3	− 1	+ 2	− 2
[42^31^2]	+ 120	− 76	+ 10	− 39	+ 24	− 15	+ 6	+ 16	+ 39	− 5	− 18	− 3	+ 8	− 1	− 2
[4321^3]	+ 240	− 108	− 100	− 33	+ 48	+ 5	− 12	+ 68	+ 73	− 27	− 12	− 17	+ 7	+ 7	+ 4
[4^21^4]	+ 30	− 8	− 15	− 21	+ 6	+ 5	+ 3	+ 8	+ 23	− 3	− 1	− 5	+ 1	− 3	− 3
[52^21^3]	+ 120	− 54	− 10	− 30	− 40	+ 20	− 6	+ 14	+ 37	+ 52	+ 18	− 4	− 23	+ 1	+ 2
[531^4]	+ 60	− 16	− 30	− 15	− 20	+ 10	+ 6	+ 16	+ 19	+ 23	+ 14	− 6	− 4	− 6	+ 2
[621^4]	+ 60	− 16	− 10	− 15	− 20	− 25	+ 6	+ 6	+ 17	+ 23	+ 10	+ 29	+ 25	− 1	− 6
[71^5]	+ 12	− 1	− 2	− 3	− 4	− 5	− 6	+ 1	+ 3	+ 4	+ 2	+ 5	+ 5	+ 6	+ 6
[3^22^3]	− 24	+ 24	− 6	− 3	+ 8	− 11	+ 6	− 9	+ 9	− 5	− 2	+ 3	+ 2	− 1	− 2
[3^321]	− 48	+ 37	+ 28	− 42	+ 16	+ 13	− 12	− 27	+ 19	+ 13	− 12	− 18	+ 7	+ 7	+ 4
[42^4]	− 12	+ 12	− 8	− 12	− 12	+ 12	− 6	− 2	− 4	0	+ 4	0	− 4	0	+ 4
[432^21]	− 144	+ 111	+ 24	− 9	− 16	+ 4	0	− 51	− 36	+ 16	+ 24	+ 3	− 19	− 1	− 8
[43^21^2]	− 72	+ 30	+ 62	− 9	− 40	+ 2	+ 18	− 39	− 20	+ 34	− 6	+ 15	− 4	− 20	+ 2
[4^221^2]	− 72	+ 39	+ 32	+ 45	− 40	+ 2	0	− 24	− 62	+ 16	+ 16	+ 15	− 7	+ 1	+ 8
[52^31]	− 48	+ 37	− 12	− 21	+ 16	− 22	+ 12	− 7	− 16	− 26	− 4	+ 3	+ 19	− 1	− 4

TABLE II. (upper half continued): Roots in Terms of Coefficients.

	(63^2)	(5^22)	(543)	(91^3)	(821^2)	(731^2)	(72^21)	(641^2)	(6321)	(62^3)	(5^21^2)	(5421)	(53^21)	(532^2)
$[1^{12}]$														
$[21^{10}]$														
$[2^21^8]$														
$[31^9]$														
$[2^31^6]$														
$[321^7]$														
$[41^8]$														
$[2^41^4]$														
$[32^21^5]$														
$[3^21^6]$														
$[421^6]$				−8	1									
$[51^7]$				−1	−4									
$[2^51^2]$														
$[32^31^3]$														
$[3^221^4]$														
$[42^21^4]$				+20	−6	1								
$[431^5]$				+8	−1	−2	1							
$[521^5]$				+7	+27	−1	−3							
$[61^6]$				+1	+4	+5	+5							
$[2^6]$														
$[32^41]$														
$[3^22^21^2]$														
$[3^31^3]$	1													
$[42^31^2]$	0			−16	+9	−4	0	1						
$[4321^3]$	−3			−32	+11	+8	−5	−3	1					
$[4^21^4]$	+3			−8	+1	+2	−1	+3	−3	1				
$[52^21^3]$	+3			−14	−51	+4	+15	−1	−3	0				
$[531^4]$	−3			−7	−27	+3	+2	+2	+4	−2				
$[621^4]$	−3			−6	−23	−29	−27	+1	+7	+2				
$[71^5]$	+3			−1	−4	−5	−5	−6	−12	−2				
$[3^22^3]$	0	1												
$[3^321]$	−3	−3	1											
$[42^4]$	0	−2	0	+2	−2	+2	0	−2	0	0	1			
$[432^21]$	+9	+6	−3	+24	−17	0	+5	+6	−3	0	−4	1		
$[43^21^2]$	0	+3	−1	+12	−5	−9	+5	+3	−1	0	+2	−2	1	
$[4^221^2]$	−9	−7	+5	+24	−10	−6	+4	−9	+11	−4	+4	−1	−2	1
$[52^31]$	−9	+2	+3	+7	+23	−2	−15	+2	+9	0	−1	−3	0	0

TABLE II. (upper half continued): Roots in Terms of Coefficients.

	(81^4)	(721^3)	(631^3)	(62^21^2)	(541^3)	(71^5)	(621^4)	(52^21^3)	(4^21^4)	(61^6)	(521^5)	(431^5)	(42^21^4)
$[1^{12}]$													
$[21^{10}]$													
$[2^21^8]$													
$[31^9]$													
$[2^31^6]$													
$[321^7]$													
$[41^8]$													
$[2^41^4]$													
$[32^21^5]$													
$[3^21^6]$													
$[421^6]$													
$[51^7]$	1												
$[2^51^2]$													
$[32^31^3]$													
$[3^221^4]$													
$[42^21^4]$													
$[431^5]$													
$[521^5]$	− 7	1											
$[61^6]$	− 1	− 5				1							
$[2^6]$													
$[32^41]$													
$[3^22^21^2]$													
$[3^31^3]$													
$[42^31^2]$													
$[4321^3]$													
$[4^21^4]$													
$[52^21^3]$	+ 14	− 5	1										
$[531^4]$	+ 7	− 1	− 2	1									
$[621^4]$	+ 6	+ 29	− 1	− 4		− 6	1						
$[71^5]$	+ 1	+ 5	+ 6	+ 9		− 1	− 6			1			
$[3^22^3]$													
$[3^321]$													
$[42^4]$													
$[432^21]$													
$[43^21^2]$													
$[4^221^2]$													
$[52^31]$	− 7	+ 5	− 3	0	1								

TABLE II. (lower half): Roots in Terms of Coefficients.

	(12)	(11·1)	(10·2)	(93)	(84)	(75)	(6^2)	$(10\,1^2)$	(921)	(831)	(82^2)	(741)	(732)	(651)	(642)	(63^2)
$[5321^2]$	−144	+78	+64	+9	+48	−31	0	−48	−43	−63	−32	+21	+39	+7	−16	+9
$[541^3]$	−48	+15	+28	+39	+16	−22	−12	−15	−43	−22	−12	+11	+3	+19	+4	−3
$[62^21^2]$	−72	+39	+2	+18	+24	+37	−18	−9	−23	−33	−10	−42	−27	+2	+22	+9
$[631^3]$	−48	+15	+28	+12	+16	+13	−12	−15	−16	−19	−12	−24	−26	+14	+4	+6
$[721^3]$	−48	+15	+8	+12	+16	+13	+24	−5	−14	−19	−8	−24	−20	−28	−24	−12
$[81^4]$	−12	+1	+2	+3	+4	+12	+6	−1	−3	−4	−2	−5	−5	−13	−6	−3
$[3^4]$	+3	−3	−3	+6	−3	−3	+3	+3	−3	−3	+3	+6	−3	−3	−3	+3
$[43^22]$	+36	−36	−16	+18	−4	−1	0	+26	−2	−14	0	+5	+5	+1	+8	−9
$[4^22^2]$	+18	−18	+2	−18	+22	−18	+9	+8	+16	−4	−14	−4	+16	0	−12	0
$[4^231]$	+36	−25	−36	−9	+44	−1	−18	+25	+34	−34	−4	−26	+10	+26	−8	+9
$[532^2]$	+36	−36	+4	−9	−20	+34	−18	+16	+5	+29	+4	−14	−23	+2	+16	+9
$[53^21]$	+36	−25	−36	+18	−4	−1	0	+25	+7	−1	+20	−6	−17	+5	+4	−9
[5421]	+72	−50	−32	−45	−8	+33	0	+30	+73	+31	0	−19	−28	−25	+16	+9
$[5^21^2]$	+18	−7	−13	−18	−18	+17	+9	+7	+20	+25	+13	−10	−4	−16	−5	0
[62·]	+12	−12	+8	−12	+4	−12	+6	+2	+4	+8	0	+8	+4	0	−12	0
[6321]	+72	−50	−32	+9	−40	−37	+36	+30	+19	+33	+16	+55	+30	−27	−24	−27
$[641^2]$	+36	−14	−26	−36	−4	−1	+18	+14	+40	+18	+10	+11	+27	−27	−18	0
$[72^21]$	+36	−25	+4	−9	−20	−1	−18	+5	+12	+18	+4	+24	+12	+26	+16	+9
$[731^2]$	+36	−14	−26	−9	−4	−1	−18	+14	+13	+15	+10	+11	+21	+15	+30	+9
$[821^2]$	+36	−14	−6	−9	−4	−36	−18	+4	+11	+15	+6	+18	+15	+50	+10	+9
$[91^3]$	+12	−1	−2	−3	−12	−12	−6	+1	+3	+4	+2	+13	+5	+13	+14	+3
$[4^3]$	−4	+4	+4	+4	−12	+4	+2	−4	−8	+8	+4	+8	−8	−8	+8	−4
[543]	−24	+24	+24	−3	−8	−11	+12	−24	−21	+11	−8	+19	+14	−13	−16	+3
$[5^22]$	−12	+12	+2	+12	+12	−23	+6	−7	−14	−24	−2	+11	+21	+11	−14	−12
$[63^2]$	−12	+12	+12	−15	+12	+12	−12	−12	+3	+3	−12	−24	+3	+12	+12	+6
[642]	−24	+24	+4	+24	−8	+24	−24	−14	−28	−16	+12	−16	−28	+24	+28	+12
[651]	−24	+13	+24	+24	+24	−11	−24	−13	−37	−37	−24	−2	−13	+40	+24	+12
[732]	−24	+24	+4	−3	+24	−11	+12	−14	−1	−21	−4	−13	−8	−13	−28	+3
[741]	−24	+13	+24	+24	−8	−11	+12	−13	−37	−5	−8	+2	−13	−2	−16	−24
$[82^2]$	−12	+12	−8	+12	−4	+12	+6	−2	−4	−8	0	−8	−4	−24	+12	−12
[831]	−24	+13	+24	−3	−8	+24	+12	−13	−10	−2	−8	−5	−21	−37	−16	+3
[921]	−24	+13	+4	−3	+24	+24	+12	−3	−8	−10	−4	−37	−1	−37	−28	+3
$[101^2]$	−12	+1	+2	+12	+12	+12	+6	−1	−3	−13	−2	−13	−14	−13	−14	−12
$[6^2]$	+6	−6	−6	−6	−6	−6	+15	+6	+12	+12	+6	+12	+12	−24	−24	−12
[75]	+12	−12	−12	−12	−12	+23	−6	+12	+24	+24	+12	−11	−11	−11	+24	+12
[84]	+12	−12	−12	−12	+20	−12	−6	+12	+24	−8	−4	−8	+24	+24	−8	+12
[93]	+12	−12	−12	+15	−12	−12	−6	+12	−3	−3	+12	+24	−3	+24	+24	−15
[102]	+12	−12	+8	−12	−12	−12	−6	+2	+4	+24	−8	+24	+4	+24	+4	+12
[111]	+12	−1	−12	−12	−12	−12	−6	+1	+13	+13	+12	+13	+24	+13	+24	+12
[12]	−12	+12	+12	+12	+12	+12	+6	−12	−24	−24	−12	−24	−24	−24	−24	−12

TABLE II. (lower half continued): Roots in Terms of Coefficients.

	(5²2)	(543)	(4³)	(91³)	(821²)	(731²)	(72²1)	(641²)	(6321)	(62³)	(5²1²)	(5421)	(53²1)	(532²)	(4²31)	(4²2²)	(43²2)
[5321²]	− 4	− 2		+ 21	+ 75	− 15	− 23	− 3	− 11	− 8	+ 3	+ 8	− 1	− 2			
[541³]	+ 7	− 5		+ 15	+ 26	− 5	− 1	− 8	+ 2	0	− 7	− 2	+ 5	− 1			
[62²1²]	− 2	− 7		+ 9	+ 32	+ 42	+ 33	− 3	− 25	− 8	+ 1	+ 3	+ 4	+ 2			
[631³]	− 3	+ 7		+ 6	+ 23	+ 27	+ 28	− 3	− 11	0	− 2	− 4	− 7	+ 3			
[721³]	+ 7	+ 7		+ 5	+ 19	+ 24	+ 22	+ 29	+ 53	+ 8	− 1	− 8	− 4	− 7			
[81⁴]	− 7	− 7		+ 1	+ 4	+ 5	+ 5	+ 6	+ 12	+ 2	+ 7	+ 14	+ 7	+ 7			
[3⁴]	+ 3	− 3	1														
[43²2]	− 9	+ 11	− 4	− 8	+ 8	+ 2	− 5	− 7	+ 5	0	+ 4	− 1	− 2	0	1		
[4²2²]	+ 8	− 4	+ 2	− 8	+ 8	− 4	− 2	+ 8	− 4	+ 2	− 4	0	+ 4	− 2	− 2	1	
[4²31]	+ 1	− 10	+ 4	− 16	+ 9	+ 16	− 9	+ 1	− 10	+ 4	− 8	+ 10	0	− 1	− 1	− 2	1
[532²]	− 4	− 11	+ 4	− 7	− 21	+ 4	+ 19	− 2	− 11	− 4	+ 1	+ 3	− 2	+ 4	− 1	− 2	0
[53²1]	+ 1	+ 11	− 4	− 7	− 24	+ 11	+ 8	+ 2	+ 14	− 8	− 5	− 10	0	+ 2	0	+ 4	− 2
[5421]	− 3	+ 8	− 8	− 30	− 39	+ 16	+ 18	+ 7	− 10	0	+ 10	− 10	− 10	+ 3	+ 10	0	− 1
[5²1²]	− 7	+ 1	+ 6	− 7	− 27	+ 3	+ 2	+ 9	+ 1	− 1	+ 7	+ 10	− 5	+ 1	− 8	− 4	+ 4
[62³]	+ 2	+ 8	− 4	− 2	− 6	− 10	− 4	+ 2	+ 8	+ 4	− 1	0	− 8	− 4	+ 4	+ 2	0
[6321]	+ 17	− 4	+ 8	− 12	− 41	− 49	− 42	+ 5	+ 56	+ 8	+ 1	− 10	+ 14	− 11	− 10	− 4	+ 5
[641²]	− 4	+ 5	− 4	− 14	− 22	− 25	− 29	+ 17	+ 5	+ 2	+ 9	+ 7	+ 2	− 2	+ 1	+ 8	− 7
[72²1]	− 19	− 6	+ 4	− 5	− 17	− 23	− 16	− 29	− 42	− 4	+ 2	+ 18	+ 8	+ 19	− 9	− 2	− 5
[731²]	− 4	− 22	− 4	− 5	− 19	− 22	− 23	− 25	− 49	− 10	+ 3	+ 16	+ 11	+ 4	+ 16	− 4	+ 2
[821²]	+ 21	+ 13	− 4	− 4	− 15	− 19	− 17	− 22	− 41	− 6	− 27	− 39	− 24	− 21	+ 9	+ 8	+ 8
[91³]	+ 7	+ 15	+ 4	− 1	− 4	− 5	− 5	− 14	− 12	− 2	− 7	− 30	− 7	− 7	− 16	− 8	− 8
[4³]	− 4	+ 8	− 4	+ 4	− 4	− 4	+ 4	− 4	+ 8	− 4	+ 6	− 8	− 4	+ 4	+ 4	+ 2	− 4
[543]	+ 11	− 14	+ 8	+ 15	+ 13	− 22	− 6	+ 5	− 4	+ 8	+ 1	+ 8	+ 11	− 11	− 10	− 4	+ 11
[5²2]	+ 3	+ 11	− 4	+ 7	+ 21	− 4	− 19	− 4	+ 17	+ 2	− 7	− 3	+ 1	− 4	+ 1	+ 8	− 9
[63²]	− 12	+ 3	− 4	+ 3	+ 9	+ 9	+ 9	0	− 27	0	0	+ 9	− 9	+ 9	+ 9	0	− 9
[642]	− 14	− 16	+ 8	+ 14	+ 10	+ 30	+ 16	− 18	− 24	− 12	− 5	+ 16	+ 4	+ 16	− 8	− 12	+ 8
[651]	+ 11	− 13	− 8	+ 13	+ 50	+ 15	+ 26	− 27	− 27	0	− 16	− 25	+ 5	+ 2	+ 26	0	+ 1
[732]	+ 21	+ 14	− 8	+ 5	+ 15	+ 21	+ 12	+ 27	+ 30	+ 4	− 4	− 28	− 17	− 23	+ 10	+ 16	+ 5
[741]	+ 11	+ 19	+ 8	+ 13	+ 18	+ 11	+ 24	+ 11	+ 55	+ 8	− 10	− 19	− 6	− 14	− 26	− 4	+ 5
[82²]	− 2	− 8	+ 4	+ 2	+ 6	+ 10	+ 4	+ 10	+ 16	0	+ 13	0	+ 20	+ 4	− 4	− 14	0
[831]	− 24	+ 11	+ 8	+ 4	+ 15	+ 15	+ 18	+ 18	+ 33	+ 8	+ 25	+ 31	− 1	+ 29	− 34	− 4	− 14
[921]	− 14	− 21	− 8	+ 3	+ 11	+ 13	+ 12	+ 40	+ 19	+ 4	+ 20	+ 73	+ 7	+ 5	+ 34	+ 16	− 2
[101²]	− 7	− 24	− 4	+ 1	+ 4	+ 14	+ 5	+ 14	+ 30	+ 2	+ 7	+ 30	+ 25	+ 16	+ 25	+ 8	+ 26
[6²]	+ 6	+ 12	+ 2	− 6	− 18	− 18	− 18	+ 18	+ 36	+ 6	+ 9	0	0	− 19	− 18	+ 9	0
[75]	− 23	− 11	+ 4	− 12	− 26	− 1	− 1	− 1	− 37	− 12	+ 17	+ 33	− 1	+ 34	− 1	− 18	− 1
[84]	+ 12	− 8	− 12	− 12	− 4	− 4	− 20	− 4	− 40	+ 4	− 18	− 8	− 4	− 20	+ 44	+ 22	− 4
[93]	+ 12	− 3	+ 4	− 3	− 9	− 9	− 9	− 36	+ 9	− 12	− 18	− 45	+ 18	− 9	− 9	− 18	+ 18
[102]	+ 2	+ 24	+ 4	− 2	− 6	− 26	+ 4	− 26	− 32	+ 8	− 13	− 32	− 36	+ 4	− 36	+ 2	− 16
[111]	+ 12	+ 24	+ 4	− 1	− 14	− 14	− 25	− 14	− 50	− 12	− 7	− 50	− 25	− 36	− 25	− 18	− 36
[12]	− 12	− 24	− 4	+ 12	+ 36	+ 36	+ 36	+ 36	+ 72	+ 12	+ 18	+ 72	+ 36	+ 36	+ 36	+ 18	+ 36

TABLE II. (lower half continued): Roots in Terms of Coefficients.

	(3^4)	(81^4)	(721^3)	(631^3)	(62^21^2)	(541^3)	(5321^2)	(52^31)	(4^221^2)	(43^21^2)	(432^21)	(42^4)	(3^321)	(3^22^3)	(71^5)
$[5321^2]$		− 21	+ 9	+ 6	− 4	− 3	1								
$[541^3]$		− 7	+ 1	+ 2	− 1	+ 3	− 3	1							
$[62^21^2]$		− 9	− 41	+ 3	+ 16	− 1	− 4	0							+ 9
$[631^3]$		− 6	− 29	+ 3	+ 3	+ 2	+ 6	− 3							+ 6
$[721^3]$		− 5	− 24	− 29	− 41	+ 1	+ 9	+ 5							+ 5
$[81^4]$		− 1	− 5	− 6	− 9	− 7	− 21	− 7							+ 1
$[3^4]$															
$[43^22]$															
$[4^22^2]$															
$[4^231]$															
$[532^2]$		+ 7	− 7	+ 3	+ 2	− 1	− 2	0	1						
$[53^21]$		+ 7	− 4	− 7	+ 4	+ 5	− 1	0	− 2	1					
$[5421]$		+ 14	− 8	− 4	+ 3	− 2	+ 8	− 3	− 1	− 2	1				
$[5^21^2]$		+ 7	− 1	− 2	+ 1	− 7	+ 3	− 1	+ 4	+ 2	− 4	1			
$[62^3]$		+ 2	+ 8	0	− 8	0	+ 8	0	− 4	0	0	0			− 2
$[6321]$		+ 12	+ 53	− 11	− 25	+ 2	− 11	+ 9	+ 11	− 1	− 3	0			− 12
$[641^2]$		+ 6	+ 29	− 3	− 3	− 8	− 3	+ 2	− 9	+ 3	+ 6	− 2			− 6
$[72^21]$		+ 5	+ 22	+ 28	+ 33	− 1	− 23	− 15	+ 4	+ 5	+ 5	0			− 5
$[731^2]$		+ 5	+ 24	+ 27	+ 42	− 5	− 15	− 2	− 6	− 9	0	+ 2			− 5
$[821^2]$		+ 4	+ 19	+ 23	+ 32	+ 26	+ 75	+ 23	− 10	− 5	− 17	− 2			− 4
$[91^3]$		+ 1	+ 5	+ 6	+ 9	+ 15	+ 21	+ 7	+ 24	+ 12	+ 24	+ 2			− 1
$[4^3]$	1														
$[543]$	− 3	− 7	+ 7	+ 7	− 7	− 5	− 2	+ 3	+ 5	− 1	− 3	0	1		
$[5^22]$	+ 3	− 7	+ 7	− 3	− 2	+ 7	− 4	+ 2	− 7	+ 3	+ 6	− 2	− 3	1	
$[63^2]$	+ 3	− 3	− 12	+ 6	+ 9	− 3	+ 9	− 9	− 9	0	+ 9	0	− 3	0	+ 3
$[642]$	− 3	− 6	− 24	+ 4	+ 22	+ 4	− 16	− 4	+ 8	+ 2	− 8	+ 4	+ 4	− 2	+ 6
$[651]$	− 3	− 13	− 28	+ 14	+ 2	+ 19	+ 7	− 1	+ 1	− 20	− 1	0	+ 7	− 1	+ 6
$[732]$	− 3	− 5	− 20	− 26	− 27	+ 3	+ 39	+ 19	− 7	− 4	− 19	− 4	+ 7	+ 2	+ 5
$[741]$	+ 6	− 5	− 24	− 24	− 42	+ 11	+ 21	+ 3	+ 15	+ 15	+ 3	0	− 18	+ 3	+ 5
$[82^2]$	+ 3	− 2	− 8	− 12	− 10	− 12	− 32	− 4	+ 16	− 6	+ 24	+ 4	− 12	− 2	+ 2
$[831]$	− 3	− 4	− 19	− 19	− 33	− 22	− 63	− 26	+ 16	+ 34	+ 16	0	+ 13	− 5	+ 4
$[921]$	− 3	− 3	− 14	− 16	− 23	− 43	− 43	− 16	− 62	− 20	− 36	− 4	+ 19	+ 9	+ 3
$[101^2]$	+ 3	− 1	− 5	− 15	− 9	− 15	− 48	− 7	− 24	− 39	− 51	− 2	− 27	− 9	+ 1
$[6^2]$	+ 3	+ 6	+ 24	− 12	− 18	− 12	0	+ 12	0	+ 18	0	− 6	− 12	+ 6	− 6
$[75]$	− 3	+ 12	+ 13	+ 13	+ 37	− 22	− 31	− 22	+ 2	+ 2	+ 4	+ 12	+ 13	− 11	− 5
$[84]$	− 3	+ 4	+ 16	+ 16	+ 24	+ 16	+ 48	+ 16	− 40	− 40	− 16	− 12	+ 16	+ 8	− 4
$[93]$	+ 6	+ 3	+ 12	+ 12	+ 18	+ 39	+ 9	+ 21	+ 45	− 9	+ 9	+ 12	− 42	− 3	− 3
$[102]$	− 3	+ 2	+ 8	+ 28	+ 2	+ 28	+ 64	− 12	+ 32	+ 62	+ 24	− 8	+ 28	− 6	− 2
$[111]$	− 3	+ 1	+ 15	+ 15	+ 39	+ 15	+ 78	+ 37	+ 39	+ 39	+ 111	+ 12	+ 37	+ 24	− 1
$[12]$	+ 3	− 12	− 48	− 48	− 72	− 48	− 144	− 48	− 72	− 72	− 144	− 12	− 48	− 24	+ 12

TABLE II. (lower half continued): Roots in Terms of Coefficients.

	(621^4)	(531^4)	(52^21^3)	(4^21^4)	(4321^3)	(42^31^2)	(3^31^3)	$(3^22^21^2)$	(32^41)	(2^6)	(61^6)	(521^5)	(431^5)	(42^21^4)	(3^221^4)
$[5321^2]$															
$[541^3]$															
$[62^21^2]$	− 4	1													
$[631^3]$	− 1	− 2	1												
$[721^3]$	+ 29	− 1	− 5								− 5	1			
$[81^4]$	+ 6	+ 7	+ 14								− 1	− 7			
$[3^4]$															
$[43^22]$															
$[4^22^2]$															
$[4^231]$															
$[532^2]$															
$[53^21]$															
$[5421]$															
$[5^21^2]$															
$[62^3]$	+ 2	− 2	0	1											
$[6321]$	+ 7	+ 4	− 3	− 3	1										
$[641^2]$	+ 1	+ 2	− 1	+ 3	− 3	1									
$[72^21]$	− 27	+ 2	+ 15	− 1	− 5	0					+ 5	− 3	1		
$[731^2]$	− 29	+ 3	+ 4	+ 2	+ 8	− 4					+ 5	− 1	− 2	1	
$[821^2]$	− 23	− 27	− 51	+ 1	+ 11	+ 9					+ 4	− 27	− 1	− 6	
$[91^3]$	− 6	− 7	− 14	− 8	− 32	− 16					+ 1	− 7	+ 8	+ 20	
$[4^3]$															
$[543]$															
$[5^22]$															
$[63^2]$	− 3	− 3	+ 3	+ 3	− 3	0	1								
$[642]$	− 6	+ 2	+ 2	− 3	+ 4	− 2	− 2	1							
$[651]$	− 1	− 6	+ 1	− 3	+ 7	− 1	+ 2	− 4	1						
$[732]$	+ 25	− 4	− 23	+ 1	+ 7	+ 8	− 1	− 4	0		− 5	+ 5	− 1	− 2	1
$[741]$	+ 29	− 6	− 4	− 5	− 17	+ 3	+ 3	+ 9	− 3		− 5	+ 1	+ 5	− 1	− 3
$[82^2]$	+ 10	+ 14	+ 18	− 1	− 12	− 18	+ 6	+ 9	0		− 2	− 12	0	+ 12	− 6
$[831]$	+ 23	+ 23	+ 52	− 3	− 27	− 5	− 11	− 5	+ 5		− 4	− 27	+ 5	+ 5	+ 10
$[921]$	+ 17	+ 19	+ 37	+ 23	+ 73	+ 39	− 6	− 30	− 7		− 3	− 20	− 22	− 54	+ 13
$[101^2]$	+ 6	+ 16	+ 14	+ 8	+ 68	+ 16	+ 18	+ 54	+ 9		− 1	− 7	− 17	− 20	− 45
$[6^2]$	+ 6	+ 6	− 6	+ 3	− 12	+ 6	− 2	+ 9	− 6	1					
$[75]$	− 25	+ 10	+ 20	+ 5	+ 5	− 15	− 5	− 5	+ 10	− 2	+ 5	− 5	− 5	+ 5	+ 5
$[84]$	− 20	− 20	− 40	+ 6	+ 48	+ 24	+ 8	− 28	− 4	+ 2	+ 4	+ 24	− 8	− 20	− 4
$[93]$	− 15	− 15	− 30	− 21	− 33	− 39	+ 17	+ 63	− 6	− 2	+ 3	+ 18	+ 18	+ 45	− 36
$[102]$	− 10	− 30	− 10	− 15	− 100	+ 10	− 30	− 45	+ 20	+ 2	+ 2	+ 12	+ 32	+ 20	+ 70
$[111]$	− 16	− 16	− 54	− 8	− 108	− 76	− 18	− 114	− 49	− 2	+ 1	+ 17	+ 17	+ 70	+ 70
$[12]$	+ 60	+ 60	+ 120	+ 30	+ 240	+ 120	+ 40	+ 180	+ 60	+ 2	− 12	72	− 72	− 180	− 180

TABLE II. (lower half continued): Roots in Terms of Coefficients.

	(32^31^3)	(2^51^2)	(51^7)	(421^6)	(3^21^6)	(32^21^5)	(2^41^4)	(41^8)	(321^7)	(2^31^6)	(31^9)	(2^21^8)	(21^{10})	(1^{12})
$[5321^2]$														
$[541^3]$														
$[62^21^2]$														
$[631^3]$														
$[721^3]$														
$[81^4]$			1											
$[3^4]$														
$[43^22]$														
$[4^22^2]$														
$[4^231]$														
$[532^2]$														
$[53^21]$														
$[5421]$														
$[5^21^2]$														
$[62^3]$														
$[6321]$														
$[641^2]$														
$[72^21]$														
$[731^2]$														
$[821^2]$			− 4	1										
$[91\ ]$			− 1	− 8				1						
$[4^3]$														
$[543]$														
$[5^22]$														
$[63^2]$														
$[642]$														
$[651]$														
$[732]$														
$[741]$	1													
$[82^2]$	0		+ 2	− 2	1									
$[831]$	− 5		+ 4	− 1	− 2	1								
$[921]$	+ 14		+ 3	+ 23	− 1	− 7		− 3	1					
$[101^2]$	− 30		+ 1	+ 8	+ 9	+ 27		− 1	− 9		1			
$[6^2]$														
$[75]$	− 5	1												
$[84]$	+ 16	− 4	− 4	+ 4	+ 2	− 4	1							
$[93]$	− 21	+ 9	− 3	− 21	+ 3	+ 18	− 6	+ 3	− 3	1				
$[102]$	0	− 16	− 2	− 14	− 17	− 32	+ 20	+ 2	+ 16	− 8	− 2	1		
$[111]$	+ 130	+ 25	− 1	− 18	− 9	− 87	− 50	+ 1	+ 19	+ 35	− 1	− 10	1	
$[12]$	− 240	− 36	+ 12	+ 84	+ 42	+ 252	+ 105	− 12	− 96	− 112	+ 12	+ 54	− 12	1

Some Elliptic Function Formulae.

By Thomas Craig, *Johns Hopkins University.*

On page 102 of Prof. Cayley's Treatise on Elliptic Functions occur the formulae for the differentiation of $\operatorname{sn} u$, $\operatorname{cn} u$, $\operatorname{dn} u$ with respect to the modulus k; these are

$$\frac{d \operatorname{sn} u}{dk} = -\frac{k}{k'^2} \operatorname{cn} u \operatorname{dn} u \int_0 \operatorname{cn}^2 u\, du + \frac{k}{k'^2} \operatorname{sn} u \operatorname{cn}^2 u,$$

$$\frac{d \operatorname{cn} u}{dk} = \frac{k}{k'^2} \operatorname{sn} u \operatorname{dn} u \int_0 \operatorname{cn}^2 u\, du - \frac{k}{k'^2} \operatorname{sn}^2 u \operatorname{cn} u,$$

$$\frac{d \operatorname{dn} u}{dk} = \frac{k^3}{k'^2} \operatorname{sn} u \operatorname{cn} u \int_0 \operatorname{cn}^2 u\, du - \frac{k}{k'^2} \operatorname{sn}^2 u \operatorname{dn} u.$$

The following three forms for these differential coefficients may be obtained from these equations, or they may be obtained directly in a slightly different manner. The formulae may have been given before, but I do not remember to have seen them; they are, however, sufficiently interesting to make a note of them here. Starting with the relation

$$\frac{d}{du} \log \operatorname{dn} u = -k^2 \frac{\operatorname{sn} u \operatorname{cn} u}{\operatorname{dn} u},$$

drop the factor $-k^2$ and differentiate this a second time; we have thus on reduction,

$$\frac{d}{du} \frac{\operatorname{sn} u \operatorname{cn} u}{\operatorname{dn} u} = -\operatorname{sn}^2 u + \frac{\operatorname{cn}^2 u}{\operatorname{dn}^2 u}.$$

Subtract this from $\operatorname{cn}^2 u$ and we have

$$\operatorname{cn}^2 u - \frac{d}{du} \frac{\operatorname{sn} u \operatorname{cn} u}{\operatorname{dn} u} = 1 - \frac{\operatorname{cn}^2 u}{\operatorname{dn}^2 u},$$

or

$$\operatorname{cn}^2 u - \frac{d}{du} \frac{\operatorname{sn} u \operatorname{cn} u}{\operatorname{dn} u} = \frac{k'^2 \operatorname{sn}^2 u}{\operatorname{dn}^2 u};$$

and on integration

A $$k'^2\int_0\frac{\operatorname{sn}^2 u}{\operatorname{dn}^2 u}=\int_0\operatorname{cn}^2 u\, du-\frac{\operatorname{sn} u \operatorname{cn} u}{\operatorname{dn} u}.$$

But

$$\int_0\operatorname{cn}^2 u\, du = u-\int_0\operatorname{sn}^2 u\, du =$$

$$u-\frac{1}{k^2}\left\{\frac{\Theta'(u)}{\Theta(u)}-u\left(1-\frac{E}{K}\right)\right\}=\frac{1}{k^2}\left\{\left(\frac{E}{K}-k'^2\right)u+\frac{\Theta'(u)}{\Theta(u)}\right\}.$$

Assume now $\phi = \operatorname{am} u$ and write

$$u=\int_0\frac{d\varphi}{\sqrt{1-k^2\operatorname{sn}^2\varphi}};$$

then

$$\frac{du}{dk}=\int_0\frac{k\operatorname{sn}^2\varphi d\varphi}{(1-k^2\operatorname{sn}^2\varphi)^{\frac{3}{2}}},$$

or

$$\frac{du}{dk}=\int_0\frac{k\operatorname{sn}^2 u\, du}{\operatorname{dn}^2 u}.$$

Equation A now becomes

A′ $$kk'^2\frac{du}{dk}=\left\{\left(\frac{E}{K}-k'^2\right)u+\frac{\Theta'(u)}{\Theta(u)}\right\}-k^2\frac{\operatorname{sn} u \operatorname{cn} u}{\operatorname{dn} u}.$$

Now

$$\frac{d\operatorname{sn} u}{dk}=-\operatorname{cn} u \operatorname{dn} u\frac{du}{dk},$$

$$\frac{d\operatorname{cn} u}{dk}=\operatorname{sn} u \operatorname{dn} u\frac{du}{dk},$$

$$\frac{d\operatorname{dn} u}{dk}=k^2\operatorname{sn} u \operatorname{cn} u\frac{du}{dk};$$

substituting from A′ for $\frac{du}{dk}$ we have the following formulae:

$$\frac{d\operatorname{sn} u}{dk}=\frac{-1}{kk'^2}\operatorname{cn} u \operatorname{dn} u\left\{\left(\frac{E}{K}-k'^2\right)u+\frac{\Theta'(u)}{\Theta(u)}\right\}+\frac{k}{k'^2}\operatorname{sn} u \operatorname{cn}^2 u,$$

B $$\frac{d\operatorname{cn} u}{dk}=\frac{1}{kk'^2}\operatorname{sn} u \operatorname{dn} u\left\{\left(\frac{E}{K}-k'^2\right)u+\frac{\Theta'(u)}{\Theta(u)}\right\}-\frac{k}{k'^2}\operatorname{sn}^2 u \operatorname{cn} u,$$

$$\frac{d\operatorname{dn} u}{dk}=\frac{k}{k'^2}\operatorname{sn} u \operatorname{cn} u\left\{\left(\frac{E}{K}-k'^2\right)u+\frac{\Theta'(u)}{\Theta(u)}\right\}+\frac{k^3}{k'^2}\frac{\operatorname{sn}^2 u \operatorname{cn}^2 u}{\operatorname{dn} u}.$$

By means of the formulae

$$\operatorname{sn} u = e^{-\frac{\pi}{4K}(K'-2iu)}\frac{-i}{\sqrt{k}}\Theta(u-iK') \div \Theta(u)$$

$$\operatorname{cn} u = e^{-\frac{\pi}{4K}(K'-2iu)}\sqrt{\frac{k'}{k}}\,\Theta(u+K+iK') \div \Theta(u)$$

$$\operatorname{dn} u = \sqrt{k'}\,\Theta(u+K) \div \Theta(u),$$

these may be written in the forms

$$\frac{d\operatorname{sn} u}{dk} = -e^{\frac{*}{4}}\frac{1}{k'k\sqrt{k}}\frac{\theta(u+K+iK')\,\theta(u+K)}{\theta^2(u)}\left\{\left(\frac{E}{K}-k'^2\right)u+\frac{\theta'(u)}{\theta(u)}\right\}$$
$$-e^{\frac{3*}{4}}\frac{i}{k\sqrt{kk'}}\frac{\vartheta(u+iK')\,\theta^2(u+k+iK')}{\theta^3(u)}$$

B'. $$\frac{d\operatorname{cn} u}{dk} = -e^{\frac{*}{4}}\frac{i}{kk'\sqrt{kk'}}\frac{\theta(u+K)\,\theta(u+iK')}{\theta^2(u)}\left\{\left(\frac{E}{K}-k'^2\right)u+\frac{\theta'(u)}{\theta(u)}\right\}$$
$$-\epsilon^{\frac{3*}{4}}\frac{\sqrt{k}}{k'}\frac{\theta^2(u+iK')\,\theta(u+K+iK')}{\theta^3(u)}$$

$$\frac{d\operatorname{dn} u}{dk} = -e^{\frac{*}{2}}\frac{i}{k'\sqrt{k'}}\frac{\theta(u+iK')\,\theta(u+K+iK')}{\theta^2(u)}\left\{\left(\frac{E}{K}-k'^2\right)u+\frac{\theta'(u)}{\theta(u)}\right\}$$
$$-e^{*}\frac{k^2}{k'\sqrt{k'}}\frac{\theta^2(u+iK')\,\theta^2(u+K+iK')}{\theta(u+K)\,\theta^3(u)}$$

where for brevity * is written to denote $-\frac{\pi}{K}(K'-2iu)$.

Substituting in equation A the value of $\int \operatorname{cn}^2 u\,du$ we have

$$\int\frac{\operatorname{sn}^2 u}{\operatorname{dn}^2 u}du = \frac{1}{k^2k'^2}\left\{\left(\frac{E}{K}-k'^2\right)u+\frac{\theta'(u)}{\theta(u)}\right\}-\frac{1}{k'^2}\frac{\operatorname{sn} u\operatorname{cn} u}{\operatorname{dn} u}.$$

We can easily find eight other relations of this character, *i. e.* we can get the group of nine integrals,

$$\int\frac{1}{\operatorname{sn}^2 u}du,\quad \int\frac{\operatorname{cn}^2 u}{\operatorname{sn}^2 u}du,\quad \int\frac{\operatorname{dn}^2 u}{\operatorname{sn}^2 u}du,$$
$$\int\frac{\operatorname{sn}^2 u}{\operatorname{cn}^2 u}du,\quad \int\frac{1}{\operatorname{cn}^2 u}du,\quad \int\frac{\operatorname{dn}^2 u}{\operatorname{cn}^2 u}du,$$
$$\int\frac{\operatorname{sn}^2 u}{\operatorname{dn}^2 u}du,\quad \int\frac{\operatorname{cn}^2 u}{\operatorname{dn}^2 u}du,\quad \int\frac{1}{\operatorname{dn}^2 u}.$$

These together with

$$\int \mathrm{sn}^2 u\, du\,, \int \mathrm{cn}^2 u\, du\,, \int \mathrm{dn}^2 u\, du$$

make a series of twelve integrals of which the differential coefficients relatively to u are the squares of the twelve quantities

$$\begin{array}{ccc} \mathrm{sn}\, u, & \mathrm{cn}\, u, & \mathrm{dn}\, u \\ \dfrac{1}{\mathrm{sn}\, u}, & \dfrac{\mathrm{cn}\, u}{\mathrm{sn}\, u}, & \dfrac{\mathrm{dn}\, u}{\mathrm{sn}\, u} \\ \dfrac{\mathrm{sn}\, u}{\mathrm{cn}\, u}, & \dfrac{1}{\mathrm{cn}\, u}, & \dfrac{\mathrm{dn}\, u}{\mathrm{cn}\, u} \\ \dfrac{\mathrm{sn}\, u}{\mathrm{dn}\, u}, & \dfrac{\mathrm{cn}\, u}{\mathrm{dn}\, u}, & \dfrac{1}{\mathrm{dn}\, u} \end{array}$$

whose integrals with respect to u have been given by Mr. J. W. L. Glaisher in the B. A. Report for 1881. We easily find the following values:

$$\frac{1}{\mathrm{sn}^2 u} = k^2\, \mathrm{sn}^2 u - \frac{d}{du}\frac{\mathrm{cn}\, u\, \mathrm{dn}\, u}{\mathrm{sn}\, u},$$

$$\frac{\mathrm{cn}^2 u}{\mathrm{sn}^2 u} = -\,\mathrm{dn}^2 u - \frac{d}{du}\frac{\mathrm{cn}\, u\, \mathrm{dn}\, u}{\mathrm{sn}\, u},$$

$$\frac{\mathrm{dn}^2 u}{\mathrm{sn}^2 u} = k^2\, \mathrm{sn}^2 u - k^2 - \frac{d}{du}\frac{\mathrm{cn}\, u\, \mathrm{dn}\, u}{\mathrm{sn}\, u}.$$

$$\frac{\mathrm{sn}^2 u}{\mathrm{cn}^2 u} = \frac{1}{1+k^2}\left\{ \mathrm{dn}^2 u + \frac{d}{du}\frac{\mathrm{sn}\, u\, \mathrm{dn}\, u}{\mathrm{cn}\, u} \right\},$$

$$\frac{1}{\mathrm{cn}^2 u} = \frac{k^2}{1+k^2} + \frac{k^2}{1+k^2}\mathrm{sn}^2 u + \frac{1+2k^2}{1+k^2}\frac{d}{du}\frac{\mathrm{sn}\, u\, \mathrm{dn}\, u}{\mathrm{cn}\, u},$$

$$\frac{\mathrm{dn}^2 u}{\mathrm{cn}^2 u} = k^2\, \mathrm{sn}^2 u + \frac{d}{du}\frac{\mathrm{sn}\, u\, \mathrm{dn}\, u}{\mathrm{cn}\, u},$$

$$\frac{\mathrm{sn}^2 u}{\mathrm{dn}^2 u} = \frac{1}{k'^2}\mathrm{cn}^2 u\, \mathrm{dn} - \frac{1}{k'^2}\frac{d}{du}\frac{\mathrm{sn}\, u\, \mathrm{cn}\, u}{\mathrm{dn}\, u}$$

$$\frac{\mathrm{cn}^2 u}{\mathrm{dn}^2 u} = \mathrm{sn}^2 u + \frac{d}{du}\frac{\mathrm{sn}\, u\, \mathrm{cn}\, u}{\mathrm{dn}\, u}$$

$$\frac{1}{\mathrm{dn}^2 u} = \frac{1}{k'^2} - \frac{k^2}{k'^2}\mathrm{sn}^2 u - \frac{k^2}{k'^2}\frac{d}{du}\frac{\mathrm{sn}\, u\, \mathrm{cn}\, u}{\mathrm{dn}\, u}.$$

The integrals of these quantities are dependent then upon the integral of $\mathrm{sn}^2 u$. Taking no account of signs or numerical factors we see that in the first group of three equations, which has as the denominator on the left-hand side the

quantity $\operatorname{sn}^2 u$, the exact differential is $\frac{d^2}{du^2}\log \operatorname{sn} u$; in the second group where the corresponding denominator is $\operatorname{cn}^2 u$ the exact differential is $\frac{d^2}{du^2}\log \operatorname{cn} u$; and also in the third group with denominator $\operatorname{dn}^2 u$ the differential is $\frac{d^2}{du^2}\log \operatorname{dn} u$. The complete set of integrals is

$$\int \frac{du}{\operatorname{sn}^2 u} = \left\{\left(1 - \frac{E}{K}\right)u - \frac{\Theta'(u)}{\Theta(u)}\right\} - \frac{d}{du}\log \operatorname{sn} u,$$

$$\int \frac{\operatorname{cn}^2 u}{\operatorname{sn}^2 u}du = -\left\{\frac{E}{K}u + \frac{\Theta'(u)}{\Theta(u)}\right\} - \frac{d}{du}\log \operatorname{sn} u,$$

$$\int \frac{\operatorname{dn}^2 u}{\operatorname{sn}^2 u}du = \left\{\left(k'^2 - \frac{E}{K}\right)u - \frac{\Theta'(u)}{\Theta(u)}\right\} - \frac{d}{du}\log \operatorname{sn} u.$$

$$\int \frac{\operatorname{sn}^2 u}{\operatorname{cn}^2 u}du = \frac{1}{1+k^2}\left\{\frac{E}{K}u + \frac{\Theta'(u)}{\Theta(u)}\right\} - \frac{1}{1+k^2}\frac{d}{du}\log \operatorname{cn} u,$$

$$\int \frac{du}{\operatorname{cn}^2 u} = \frac{k^2}{1+k'^2}\left\{u + \frac{u}{k^2}\left(1 - \frac{E}{K}\right) - \frac{1}{k^2}\frac{\Theta(u)}{\Theta(u)}\right\} - \frac{1+2k^2}{1+k^2}\frac{d}{du}\log \operatorname{cn} u,$$

$$\int \frac{\operatorname{dn}^2 u}{\operatorname{cn}^2 u}du = \left\{\left(1 - \frac{E}{K}\right)u - \frac{\Theta'(u)}{\Theta(u)}\right\} - \frac{d}{du}\log \operatorname{cn} u,$$

$$\int \frac{\operatorname{sn}^2 u}{\operatorname{dn}^2 u}du = \frac{1}{k'^2}u - \frac{1}{k^2 k'^2}\left\{\left(\frac{E}{K} - 1\right)u + \frac{\Theta'(u)}{\Theta(u)}\right\} + \frac{1}{k^2 k'^2}\frac{d}{du}\log \operatorname{dn} u,$$

$$\int \frac{\operatorname{cn}^2 u}{\operatorname{dn}^2 u}du = \frac{1}{k^2}\left\{\left(1 - \frac{E}{K}\right)u - \frac{\Theta'(u)}{\Theta(u)}\right\} - \frac{1}{k^2}\frac{d}{du}\log \operatorname{dn} u,$$

$$\int \frac{1}{\operatorname{dn}^2 u}du = \frac{1}{k'^2}\left\{\frac{E}{K}u + \frac{\Theta'(u)}{\Theta(u)}\right\} + \frac{1}{k^2}\frac{d}{du}\log \operatorname{dn} u.$$

Of course the whole set of integrals might be given in terms of the Θ-functions only, by substituting for $\frac{d}{du}\log \operatorname{sn} u$, $\frac{d}{du}\log \operatorname{cn} u$, $\frac{d}{du}\log \operatorname{dn} u$ their values, but it is not worth while to write them down. It is not difficult to obtain the integrals of the quantities

$$\operatorname{sn}^2 u \operatorname{cn}^2 u,\ \operatorname{cn}^2 u \operatorname{dn}^2 u,\ \operatorname{dn}^2 u \operatorname{sn}^2 u;$$

in fact it is obvious that they will all depend upon the integral

$$\int_0 \operatorname{sn}^4 u\, du,$$

and this can be shown to depend upon the integral of $\operatorname{sn}^2 u$.

The above reference to Mr. Glaisher's paper was made from memory, but I find on referring to it that he remarks that the integral of $\frac{1}{\operatorname{sn}^2 u}$ may be deduced from that of $\operatorname{sn}^2 u$ by the formula

$$\frac{d^2}{du^2}\log \operatorname{sn} u = k^2 \operatorname{sn}^2 u - \frac{1}{\operatorname{sn}^2 u};$$

he further shows in general how the integral of $\operatorname{sn}^n u$ may be obtained. I may perhaps be permitted to quote the last few lines of Mr. Glaisher's paper, which bear directly upon this question. In connection with the formulae of reduction for $\int \operatorname{sn}^n u\, du$, Mr. Glaisher says: "We have

1. $\frac{d^2}{du^2}\operatorname{sn}^n u = u(u-1)\operatorname{sn}^{n-2} u - u^2(1+k^2)\operatorname{sn}^n u + u(u+1)k^2 \operatorname{sn}^{n+2} u \ldots$

and by means of this formula the integral of $\operatorname{sn}^n u$ may be reduced to depend upon the integrals of $\operatorname{sn} u$ $\operatorname{sn}^2 u$, $\frac{1}{\operatorname{sn} u}$, $\frac{1}{\operatorname{sn}^2 u}$ according as u is positive and uneven, positive and even, negative and uneven, negative and even.

The integral of $\operatorname{sn}^2 u$ involves the function $Z(u)$" $\left(i.\ e.\ \text{as above} \frac{\Theta'(u)}{\Theta(u)}\right)$; "and the integral of $\frac{1}{\operatorname{sn}^2 u}$ may be deduced from that of $\operatorname{sn}^2 u$ by the formula

2. $$\frac{d^2}{du^2}\log \operatorname{sn} u = k^2 \operatorname{sn}^2 u - \frac{1}{\operatorname{sn}^2 u}.$$

Corresponding to (1) and (2) there are eleven other pairs of formulae which involve the other eleven functions in place of $\operatorname{sn} u$, and differ from one another only in the k-coefficients. *It can thus be shown, that the integrals of the* n^{th} *powers of the twelve functions are all finitely expressible in terms of elliptic functions if u is uneven, and in terms of elliptic functions, and of the Zeta function if u is even; and that the twelve formulae of reduction are similar in form and deducible from any one of them.*"

The italics are my own. Mr. Glaisher has given in his paper the values of the integrals of the group of twelve primary functions marked (C) in the above.

It may be as well to work out the integral of $\operatorname{sn}^4 u$, as by means of it we can find at once the integrals of

$$\operatorname{cn}^4 u,\ \operatorname{dn}^4 u,\ \operatorname{sn}^2 u,\ \operatorname{cn}^2 u,\ \operatorname{cn}^2 u \operatorname{dn}^2 u,\ \operatorname{dn}^2 u \operatorname{sn}^2 u,\ \&c.$$

Applying Mr. Glaisher's general formula for the second differential coefficient of $\operatorname{sn}^n u$ with respect to u we have

$$\frac{d^2}{du^2}\operatorname{sn}^2 u = 2 - 4(1+k^2)\operatorname{sn}^2 u + 6k^2 \operatorname{sn}^4 u .$$

From this on integration and reduction follows

$$\int_0 \operatorname{sn}^4 u\, du = \frac{1}{3k^2}\left\{\left[1+(1+k^2)\left(1-\frac{2E}{K}\right)\right]u - \frac{2(1+k^2)}{k^2}\frac{\theta' u}{\theta u} - \operatorname{sn} u \operatorname{cn} u \operatorname{dn} u\right\}.$$

This taken with the formula for $\int \operatorname{sn}^2 u\, du$ serves at once for the reduction of $\int \operatorname{cn}^4 u\, du$, $\int \operatorname{dn}^4 u\, du$, $\int \operatorname{sn}^2 u \operatorname{cn}^2 u\, du$, &c.

For the integral of $\dfrac{1}{\operatorname{sn}^2 u \operatorname{cn}^2 u}$ it is not necessary to use the above formula, for multiplying this by $\operatorname{sn}^2 u + \operatorname{cn}^2 u = 1$ it reduces to the sum of the two integrals

$$\int_0 \frac{du}{\operatorname{cn}^2 u} + \int_0 \frac{du}{\operatorname{sn}^2 u}$$

the values of which are given above; on substituting these values we have

$$\int_0 \frac{du}{\operatorname{sn}^2 u \operatorname{cn}^2 u} = \left[2 - \frac{E}{K}\left(\frac{2+k^2}{1+k^2}\right)\right]u - \frac{\theta'(u)}{\theta(u)}\left(\frac{2+k^2}{1+k^2}\right)$$

$$- \frac{k^2}{1+k^2}\frac{d}{du}\log \operatorname{cn} u - \frac{d}{du}\log \operatorname{sn} u \operatorname{cn} u$$

or

$$= \left[2 - \frac{E}{K}\left(\frac{2+k^2}{1+k^2}\right)\right]u - \frac{\theta'(u)}{\theta(u)}\left(\frac{2+k^2}{1+k^2}\right)$$

$$- \frac{d}{du}\log \operatorname{sn} u\,(\operatorname{cn} u)^{\frac{1+2k^2}{1+k^2}}.$$

In like manner since

$$\frac{1}{\operatorname{sn}^2 u \operatorname{dn}^2 u} = \frac{1}{\operatorname{sn}^2 u} + \frac{k^2}{\operatorname{dn}^2 u}$$

we have

$$\int_0 \frac{du}{\operatorname{sn}^2 u \operatorname{dn}^2 u} = \left[1 - \frac{1-2k^2}{k'^2}\frac{E}{K}\right]u - \frac{1-2k^2}{k'^2}\frac{\theta'(u)}{\theta(u)}$$

$$+ \frac{k^2}{k'^2}\frac{d}{du}\log \operatorname{dn} u - \frac{d}{du}\log \operatorname{sn} u$$

or

$$= \left[1 - \frac{1-2k^2}{k'^2}\frac{E}{K}\right]u - \frac{1-2k^2}{k'^2}\frac{\Theta'(u)}{\Theta(u)}$$

$$-\frac{d}{du}\log \operatorname{sn} u\,(\operatorname{dn} u)^{\frac{k^2}{k'^2}}$$

and also since

$$\frac{1}{\operatorname{cn}^2 u\operatorname{dn}^2 u} = \frac{1}{k'^2}\left(\frac{1}{\operatorname{cn}^2 u} - \frac{k^2}{\operatorname{dn}^2 u}\right)$$

we have

$$\int_0 \frac{du}{\operatorname{cn}^2 u\operatorname{dn}^2 u} = \frac{u}{k'^4(1+k^2)}\left[1 - k^4 - (1+k^4)\frac{E}{K}\right] - \frac{1}{k'^4}\frac{1+k^4}{1+k^2}\frac{\Theta'(u)}{\Theta(u)}$$

$$-\frac{1}{k'^2}\frac{1+2k^2}{1+k^2}\frac{d}{du}\log\operatorname{cn} u + \frac{1}{k'^2}\frac{d}{du}\log\operatorname{dn} u$$

or

$$= \frac{u}{k'^4(1+k^2)}\left[1 - k^4 - (1+k^4)\frac{E}{K}\right] - \frac{1}{k'^4}\frac{1+k^4}{1+k^2}\frac{\Theta'(u)}{\Theta(u)}$$

$$-\frac{1}{k'^2}\frac{d}{du}\log\frac{(\operatorname{cn} u)^{\frac{1+2k^2}{1+k^2}}}{\operatorname{dn} u}.$$

A further set not mentioned in Mr. Glaisher's paper, nor in Prof. Cayley's Treatise, are the following three integrals

$$\int_0 \frac{\operatorname{sn} u}{\operatorname{cn} u\operatorname{dn} u}du,\quad \int_0 \frac{\operatorname{cn} u}{\operatorname{sn} u\operatorname{dn} u}du,\quad \int_0 \frac{\operatorname{dn} u}{\operatorname{sn} u\operatorname{cn} u}du$$

or

$$\int_0 \frac{du}{\frac{d}{du}\log\operatorname{sn} u},\quad -\int_0 \frac{du}{\frac{d}{du}\log\operatorname{cn} u},\quad -k^2\int_0 \frac{du}{\frac{d}{du}\log\operatorname{dn} u}.$$

These are very easily found; take for example the first one $\int \frac{\operatorname{sn} u}{\operatorname{cn} u\operatorname{dn} u}du$: write $\operatorname{sn} u = x$, then the integral is

$$\int \frac{x\,dx}{1-x^2.1-k^2x^2}.$$

Make

$$\frac{x}{1-x^2.1-k^2x^2} = \frac{A}{1-x} + \frac{B}{1+x} + \frac{C}{1-kx} + \frac{D}{1+kx}$$

and we find at once

$$A,\ B,\ C,\ D = \frac{1}{2k'^2},\ -\frac{1}{2k'^2},\ \frac{-k}{2k'^2},\ \frac{k}{2k'^2}.$$

Substituting these values we have

$$\int \frac{x\,dx}{1-x^2 . 1-k^2x^2} = \frac{1}{2k'^2} \log \frac{1-k^2x^2}{1-x^2}.$$

Replacing x by its value gives

$$\int_0 \frac{\operatorname{sn} u}{\operatorname{cn} u \operatorname{dn} u} du = \frac{1}{k'^2} \log \frac{\operatorname{dn} u}{\operatorname{cn} u};$$

and similarly

$$\int \frac{\operatorname{cn} u}{\operatorname{sn} u \operatorname{dn} u} du = \log \frac{\operatorname{sn} u}{\operatorname{dn} u},$$

$$\int_0 \frac{\operatorname{dn} u}{\operatorname{sn} u \operatorname{cn} u} du = -\log \frac{\operatorname{cn} u}{\operatorname{sn} u}.$$

From any two of these we obtain the third by addition. The similar integrals involving the squares of $\operatorname{sn} u$, $\operatorname{cn} u$, $\operatorname{dn} u$ will involve the Θ-functions. We have, viz:—

$$\int_0 \frac{\operatorname{sn}^2 u}{\operatorname{cn}^2 u \operatorname{dn}^2 u} du = \frac{1}{k'^2} \int \left(\frac{1}{\operatorname{cn}^2 u} - \frac{1}{\operatorname{dn}^2 u}\right) du$$

$$\int_0 \frac{\operatorname{cn}^2 u}{\operatorname{sn}^2 u \operatorname{dn}^2 u} du = \int_0 \left(\frac{1}{\operatorname{sn}^2 u} - \frac{k'^2}{\operatorname{dn}^2 u}\right) du$$

$$\int_0 \frac{\operatorname{dn}^2 u}{\operatorname{sn}^2 u \operatorname{cn}^2 u} du = \int_0 \left(\frac{1}{\operatorname{sn}^2 u} + \frac{k'^2}{\operatorname{cn}^2 u}\right) du.$$

Substituting from the above values of these quantities, we find readily

$$\int_0 \frac{\operatorname{sn}^2 u}{\operatorname{cn}^2 u \operatorname{dn}^2 u} du = -\frac{1}{(1+k^2)(1-k^2)^2}\left[1+k^4+2\left(1-\frac{E}{K}\right)u\right] +$$

$$\frac{2}{(1+k^2)(1-k^2)^2} \frac{\Theta'(u)}{\Theta(u)} - \frac{1}{1-k^2} \frac{d}{du} \log\left[\operatorname{cn} u^{\frac{1+2k^2}{1+k^2}} \operatorname{dn} u^{\frac{1}{k'^2}}\right]$$

$$\int_0 \frac{\operatorname{cn}^2 u}{\operatorname{sn}^2 u \operatorname{dn}^2 u} du = -\left[\frac{E}{K} u + \frac{\Theta'(u)}{\Theta(u)}\right] + \frac{d}{du} \log \frac{\operatorname{sn} u}{\operatorname{dn} u}$$

$$\int_0 \frac{\operatorname{dn}^2 u}{\operatorname{sn}^2 u \operatorname{cn}^2 u} du = \frac{1}{1+k^2}\left\{(2+5k^2+k^4) - 2(1+k^2)^3 \frac{E}{K}\right\} u$$

$$-2(1+k^2)\frac{\Theta'(u)}{\Theta(u)} - \frac{d}{du} \log \frac{\operatorname{sn} u}{(\operatorname{cn} u)^{\frac{1+k^2-2k^4}{1+k^2}}}.$$

Of course, one could go on and extend these integrals indefinitely, but all subsequent forms would only depend upon those given in the above and in

Mr. Glaisher's paper. It may be worth while, however, to give the formulae for the integration of the quantities

$$\operatorname{sn}^\alpha u \operatorname{cn}^\beta u$$
$$\operatorname{cn}^\alpha u \operatorname{dn}^\beta u$$
$$\operatorname{dn}^\alpha u \operatorname{sn}^\beta u$$

in the several cases where α and β are either both even, one even and the other odd, or both odd, *i. e.* four cases in all for each of the three products. Suppose α and β both even, say $\alpha = 2n$, $\beta = 2m$:

Then

$$\operatorname{sn}^{2n} u \operatorname{cn}^{2n} u = \operatorname{sn}^{2n} u\,(1 - \operatorname{sn}^2 u)^m .$$

Again make $\alpha = 2n + 1$, $\beta = 2m$: then

$$\operatorname{sn}^{2n+1} u \operatorname{cn}^{2m} u = \operatorname{sn}^{2n+1} u\,(1 - \operatorname{sn}^2 u)^m.$$

Again make $\alpha = 2n$, $\beta = 2m + 1$: then

$$\operatorname{sn}^{2n} u \operatorname{cn}^{2m+1} u = \operatorname{cn}^{2m+1} u\,(1 - \operatorname{cn}^2 u)^n.$$

For these three cases we have

1. $$\int_0 \operatorname{sn}^{2n} u \operatorname{cn}^{2m} u\, du = \sum_{h=0}^{h=m} (-)^h \frac{m(m-1)\ldots(m-h+1)}{h!} \int_0 \operatorname{sn}^{2(h+n)} u\, du$$

2. $$\int_0 \operatorname{sn}^{2n+1} u \operatorname{cn}^{2m} u\, du = \sum_{h=0}^{h=m} (-)^h \frac{m(m-1)\ldots(m-h+1)}{h!} \int_0 \operatorname{sn}^{2(h+n+} u\, du$$

3. $$\int_0 \operatorname{sn}^{2n} u \operatorname{cn}^{2m+1} u\, du = \sum_{h=0}^{h=n} (-)^h \frac{n(n-1)\ldots(n-h+1)}{h!} \int_0 \operatorname{cn}^{2(h+m+\frac{1}{2})} u\, du$$

and for the case of both odd we have

4. $$\int_0 \operatorname{sn}^{2n+1} u \operatorname{cn}^{2m+1} u\, du = \sum_{h=0}^{h=m} (-)^h \frac{m(m-1)\ldots(m-h+1)}{h!}$$
$$\int_0 \operatorname{sn}^{2(h+n+\frac{1}{2})} u \operatorname{cn} u\, du .$$

Each term in (4) is of the form

$$\int_0 \operatorname{sn}^{2j+1} u \operatorname{cn} u\, du ;$$

this is

$$-\frac{1}{k^2}\int_0 \mathrm{sn}^{2j} u \, d(\mathrm{dn}\, u) = -\frac{1}{k^{2(j+1)}}\int_0 (1-\mathrm{dn}^2 u)^j \, d\, \mathrm{dn}\, u$$

$$= -\frac{1}{k^{2(j+1)}}\sum_{l=0}^{l=j}(-)^l \frac{j(j-1)\ldots(j-l+1)}{l!}\int \mathrm{dn}^{2l} u \, d(\mathrm{dn}\, u)$$

$$= -\frac{1}{k^{2(j+1)}}\sum_{l=0}^{l=j}(-)^l \frac{j(j-1)\ldots(j-l+1)}{l!\,(2l+1)}\,\mathrm{dn}^{2l+1} u .$$

Giving j the value that it has in (4), viz.

$$j = 2h + 2n + 1$$

we have for the integral in the general term of (4)

$$\int_0 \mathrm{sn}^{2(h+n+1)} u \, du = -\frac{1}{k^{2(j+1)}}\sum_{l=0}^{l=2(h+n+1)}(-)^l \frac{(2h+2n+1)(2h+2n)\ldots(2h+2n-l+2)}{l!\,(2l+1)}\, du^{2l+1} u .$$

and so

4. $$\int_0 \mathrm{sn}^{2\ +1} u \, \mathrm{cn}^{2m+1} u \, du$$

$$= -\frac{1}{k^{2(j+1)}}\sum_{h=0}^{h=m}(-)^h \frac{m(m-1)\ldots(m-h+1)}{h!}$$

$$\sum_{l=0}^{l=2h+2n+1}(-)^l \frac{(2h+2n+1)(2h+2n)\ldots(2h+2n-l+2)}{l!\,(2l+1)}\,\mathrm{dn}^{2l+1} u .$$

By aid of the relations

$$k'^2(1+\tau^2 \mathrm{cn}^2 u) = \mathrm{dn}^2 u$$

$$-\frac{1}{\tau^2}\left(1-\frac{\mathrm{dn}^2 u}{k'^2}\right) = \mathrm{cn}^2 u \qquad \tau^2 = \frac{k^2}{k'^2}$$

we readily find the integrals of

$$\mathrm{cn}^{\alpha} u \ \mathrm{dn}^{\beta} u$$

for the various odd and even values of α and β. These are

1'. $$\int_0 \mathrm{cn}^{2n} u \ \mathrm{dn}^{2m} u \, du = k'^{2m}\sum_{h=0}^{h=m}\frac{m(n-1)\ldots(m-h+1)}{h!}\,\tau^{2h}\int_0 \mathrm{cn}^{2(h+n)} u \, du ,$$

2'. $$\int_0 \mathrm{cn}^{2n+1} u \ \mathrm{dn}^{2m} u \, du = k'^{2m}\sum_{h}\frac{m(m-1)\ldots(m-h+1)}{h!}\,\tau^{2h}\int \mathrm{cn}^{2(h+n)} u \, du ,$$

$$3'.\quad \int_0 \mathrm{cn}^{2n} u\, \mathrm{dn}^{2m+1} u\, du = (-)^n \frac{1}{\tau^{2n}} \sum_{h=0}^{h=n} (-)^n \frac{n(n-1)\ldots(n-h+1)}{h!\, k'^{2h}} \int_0 \mathrm{dn}^{2h} u\, du,$$

$$4'.\quad \int_0 \mathrm{cn}^{2n+1} u\, \mathrm{dn}^{2m+1} u\, du =$$

$$-k'^2 \sum_{h=0}^{h=n} \frac{m(m-1)\ldots(m-h+1)}{h!} \tau^{2h} \sum_{l=0}^{l=2h+2n+1} (-)^l \frac{(2h+2n+1)\ldots(2h+2n-l+2)}{l!\,(2l+1)} \mathrm{sn}^{2l+1} u$$

Again since

$$\mathrm{sn}^2 u = \frac{1}{k^2}(1 - \mathrm{dn}^2 u),$$

we have

$$1''.\quad \int_0 \mathrm{dn}^{2n} u\, \mathrm{sn}^{2m} u\, du = \frac{1}{k^{2m}} \sum_{h=0}^{h=m} (-)^h \frac{m(m-1)\ldots(m-h+1)}{h!} \int_0 \mathrm{dn}^{2(h+n)} u\, du,$$

$$2''.\quad \int_0 \mathrm{dn}^{2n+1} u\, \mathrm{sn}^{2m} u\, du = \frac{1}{k^{2m}} \sum_{h=0}^{h=m} (-)^h \frac{m(m-1)\ldots(m-h+1)}{h!} \int_0 \mathrm{dn}^{2(h+n+\frac{1}{2})} u\, du,$$

$$3''.\quad \int_0 \mathrm{dn}^{2n} u\, \mathrm{sn}^{2m+1} u\, du = \sum_{h=0}^{h=n} (-)^h \frac{n(n-1)\ldots(n-h+1)}{h!} k'^{2h} \int_0 \mathrm{sn}^{2(h+m+\frac{1}{2})} u\, du,$$

$$4''.\quad \int_0 \mathrm{dn}^{2n+1} u\, \mathrm{sn}^{2m+1} u\, du = \frac{1}{k^{2m}} \sum_{h=0}^{h=m} (-)^h \frac{m(m-1)\ldots(m-h+1)}{h!} (-k'^{2h+2n+1})$$

$$\sum_{l=0}^{l=2h+2n+1} \frac{(2h+2n+1)\ldots(2h+2n-l+2)}{l!\,(2l+1)} \tau^{2l}\, \mathrm{cn}^{2l+1} u.$$

The formulae 2, 3, 2′, 3′, 2″, 3″ might have been given in forms similar to 4, 4′, 4″. For the integrations in these sets of equations we need only to employ the formulae

$$\frac{d^2}{du^2} \mathrm{sn}^n u = n(n-1)\mathrm{sn}^{n-2} u - n^2(1+k^2)\,\mathrm{sn}^n u + n(n+1)k^2\,\mathrm{sn}^{n+2} u,$$

$$\frac{d^2}{du^2} \mathrm{cn}^n u = n(n-1)k'^2\,\mathrm{cn}^{n-2} u + n^2(k^2 - k'^2)\,\mathrm{cn}^n u - n(n+1)k^2\,\mathrm{cn}^{n+2} u,$$

$$\frac{d^2}{du^2} \mathrm{dn}^n u = -n(n-1)k'^2\,\mathrm{dn}^{n-2} u + n^2(1+k'^2)\,\mathrm{dn}^n u - n(n+1)\,\mathrm{dn}^{n+2} u.$$

I have tried to find reasonably simple and symmetrical forms for the expansions of the right-hand members of the above equations, by grouping together all terms containing only elliptic functions, and all containing the Θ-function, but, though I think such forms should exist, I have not been able to hit upon them. Another set of formulae might be found for the integrals of

$$\frac{1}{\operatorname{sn}^\alpha u \operatorname{cn}^\beta u}, \quad \frac{1}{\operatorname{cn}^\alpha u \operatorname{dn}^\beta u}, \quad \frac{1}{\operatorname{dn}^\alpha u \operatorname{sn}^\beta u},$$

$$\frac{\operatorname{sn}^\alpha u \operatorname{cn}^\beta u}{\operatorname{dn} u}, \quad \frac{\operatorname{cn}^\alpha u \operatorname{dn}^\beta u}{\operatorname{sn}^\gamma u}, \quad \frac{\operatorname{dn}^\alpha u \operatorname{sn}^\beta u}{\operatorname{cn}^\gamma u},$$

$$\frac{\operatorname{dn}^\gamma u}{\operatorname{sn}^\alpha u \operatorname{cn}^\beta u}, \quad \frac{\operatorname{sn}^\gamma u}{\operatorname{cn}^\alpha u \operatorname{dn}^\beta u}, \quad \frac{\operatorname{cn} u}{\operatorname{dn}^\alpha u \operatorname{sn}^\beta u},$$

but they are easily deducible from the preceding and following forms.

The formulae of integration for the several cases involved in the expression

$$\int_0 \operatorname{sn}^{\alpha'} u \operatorname{cn}^{\beta'} u \operatorname{dn}^{\gamma'} u \, du$$

are very easily found; there are in all 8 cases, viz.

	α',	β',	γ',		α',	β',	γ',
1.	2α,	2β,	2γ,	5.	$2\alpha+1$,	$2\beta+1$,	2γ,
2.	$2\alpha+1$,	2β,	2γ,	6.	2α,	$2\beta+1$,	$2\gamma+1$,
3.	2α,	$2\beta+1$,	2γ,	7.	$2\alpha+1$,	2β,	$2\gamma+1$,
4.	2α,	2β,	$2\gamma+1$,	8.	$2\alpha+1$,	$2\beta+1$,	$2\gamma+1$.

For brevity write

$$[\zeta, \lambda] = \frac{\zeta(\zeta-1)\ldots(\zeta-\lambda+1)}{\lambda!}.$$

It is very easy now to find the integral of $\operatorname{sn}^{\alpha'} u \operatorname{cn}^{\beta'} u \operatorname{dn}^{\gamma'} u$ for the above 8 cases by aid of the relations

$$1-\operatorname{sn}^2 u = \operatorname{cn}^2 u, \quad 1-k^2 \operatorname{sn}^2 u = \operatorname{dn}^2 u, \quad 1-\operatorname{cn}^2 u = \operatorname{sn}^2 u$$

$$-\frac{1}{\tau^2}\left(1-\frac{\operatorname{dn}^2 u}{k'^2}\right) = \operatorname{cn}^2 u, \; k'^2(1+\tau^2 \operatorname{cn}^2 u) = \operatorname{dn}^2 u, \; \frac{1}{k^2}(1-\operatorname{dn}^2 u) = \operatorname{sn}^2 u$$

where as above $\tau^2 = \frac{k^2}{k'^2}$. The integrals are

$$1. \quad \int_0 \operatorname{sn}^{2\alpha} u \operatorname{cn}^{2\beta} u \operatorname{dn}^{2\gamma} u \, du = \sum_{h=0}^{h=\beta} \sum_{l=0}^{l=\beta} (-)^{h+l}[\beta, h][\gamma, l] k^{2l} \int \operatorname{sn}^{2(h+l+\alpha)} u \, du$$

2. $$\int_0 \mathrm{sn}^{2a+1}u\ \mathrm{cn}^{2\beta}u\ \mathrm{dn}^{2\gamma}u = \sum_{h=0}^{h=\beta}\sum_{l=0}^{l=\gamma}[\beta,\ h][\gamma,\ l]\,k^{2l}\int \mathrm{sn}^{2(h+l+a+\frac{1}{2})}u\,du$$

3. $$\int_0 \mathrm{sn}^{2a}u\ \mathrm{cn}^{2\beta+1}u\ \mathrm{dn}^{2\gamma}u\,du = k'^{2\gamma}\sum_{h=0}^{h=a}\sum_{l=0}^{l=\gamma}(-)^{h+l}[a,\ h][\gamma,\ l]\tau^{2l} \int_0 \mathrm{cn}^{2(h+l+\beta+\frac{1}{2})}u\,du$$

4. $$\int_0 \mathrm{sn}^{2a}u\,\mathrm{cn}^{2\beta}u\ \mathrm{dn}^{2\gamma+1}u\,du = (-)^{\beta}\frac{1}{\tau^{2\beta}k^{2a}}\sum_{h=0}^{h=a}\sum_{l=0}^{l=\beta}\frac{[a,\ h][\beta,\ l]}{k'^{2l}}\int_0 \mathrm{dn}^{2(h+l+\gamma+\frac{1}{2})}u\,du$$

5. $$\int_0 \mathrm{sn}^{2a+1}u\ \mathrm{cn}^{2\beta+1}u\ \mathrm{dn}^{2\gamma}u\,du$$
$$= (-)^{\beta+1}\frac{1}{\tau^{2\beta}k^{2(a+1)}}\sum_{h=0}^{h=a}\sum_{l=0}^{l=\beta}(-)^{h+l}\frac{[a,\ h][\beta,\ l]}{k'^{2l}}\cdot\frac{\mathrm{dn}^{2(h+l+\gamma+\frac{1}{2})}u}{2(h+l+\gamma+\frac{1}{2})}$$

6. $$\int_0 \mathrm{sn}^{2a}u\ \mathrm{cn}^{2\beta+1}u\ \mathrm{dn}^{2\gamma+1}u\,du = \sum_{h=0}^{h=\beta}\sum_{l=0}^{l=\gamma}[\beta,\ h][\gamma,\ l]\,k^{2l}\frac{\mathrm{sn}^{2(h+l+a+\frac{1}{2})}u}{2(h+l+a+\frac{1}{2})}$$

7. $$\int_0 \mathrm{sn}^{2a+1}u\ \mathrm{cn}^{2\beta}u\,\mathrm{dn}^{2\gamma+1}u\,du = -k'^{2\gamma}\sum_{h=0}^{h=a}\sum_{l=0}^{l=}(-)^{h}[a,\ h][\gamma,\ l]\tau^{2l}\frac{\mathrm{cn}^{2(h+l+\beta+\frac{1}{2})}u}{2(h+l+\beta+\frac{1}{2})}$$

8. $$\int_0 \mathrm{sn}^{2a+1}u\ \mathrm{cn}^{2\beta+1}u\ \mathrm{dn}^{2\gamma+1}u\,du = \sum_{h=0}^{h=\beta}\sum_{l=0}^{l=\gamma}(-)^{h+l}[\beta,\ h][\gamma,\ l]k^{2l}\frac{\mathrm{sn}^{2(h+l+a+1)}u}{2(h+l+a+1)}.$$

Note on the Counter-Pedal Surface of an Ellipsoid.

By Thomas Craig, *Johns Hopkins University.*

In Vol. IV of this Journal I defined the counter-pedal surface of a given surface, and worked out some of its properties for the case of the ellipsoid. The definition given in the paper referred to is: (for the ellipsoid) the counter-pedal surface is the locus of the intersections of the normals to the ellipsoid with the diametral planes parallel to the tangent planes at each point. This surface may also be defined as follows: the counter-pedal surface, when the origin is taken as the pole, is the locus of the intersections of the normals at corresponding points of the ellipsoid and its first pedal.

The ellipsoid being given by

$$\frac{\xi'^2}{a^2}+\frac{\eta'^2}{b^2}+\frac{\zeta'^2}{c^2}=1,$$

the first pedal is

$$\phi\equiv(a^2x'^2+b^2y'^2+c^2z'^2)-(x'^2+y'^2+z'^2)^2=0.$$

Denoting by P the central perpendicular upon the tangent plane to the ellipsoid, we have for the coördinates x', y', z' of the point corresponding to ξ', η', ζ',

$$x',\ y',\ z'=\frac{P^2\xi'}{a^2},\quad \frac{P^2\eta'}{b^2},\quad \frac{P^2\zeta'}{c^2}.$$

The direction-cosines of the normal to the ellipsoid at a given point ξ', η', ζ' are

$$\frac{P^2\xi'}{a^2},\quad \frac{P^2\eta'}{b^2},\quad \frac{P^2\zeta'}{c^2};$$

the direction-cosines of the normal at the corresponding point x', y', z' on the pedal are proportional to

$$\frac{d\varphi}{dx},\quad \frac{d\varphi}{dy},\quad \frac{d\varphi}{dz}$$

$$=2x'(a^2-2r^2),\ 2y'(b^2-2r^2),\ 2z'(c^2-2r^2),$$

where $r^2 = x'^2 + y'^2 + z'^2$. Substituting for x', y', z' the above values and we have as the equations of the two normals,

$$\left.\begin{aligned} x &= \frac{c^2\xi'}{a^2\zeta'}z - \frac{\beta\xi'}{a^2} \\ y &= \frac{c^2\eta'}{b^2\zeta'}z + \frac{\alpha\eta'}{b^2} \end{aligned}\right\} \text{Ellipsoid.}$$

$$\left.\begin{aligned} x &= \frac{c^2\xi'}{a^2\zeta'}\cdot\frac{a^2-2P^2}{c^2-2P^2}z + \frac{P^2\xi'}{a^2}\cdot\frac{a^2-2P^2}{c^2-2P^2} \\ y &= \frac{c^2\eta'}{b^2\zeta'}\cdot\frac{b^2-2P^2}{c^2-2P^2} - \frac{P^2\eta'}{b^2}\cdot\frac{\alpha}{c^2-2P^2} \end{aligned}\right\} \text{Pedal.}$$

In these x, y, z are the current coördinates of points in the normals, and α, β, $\gamma = b^2 - c^2$, $c^2 - a^2$, $a^2 - b^2$. The values of z obtained from the first and third of these equations, and from the second and fourth, are easily seen to be the same, and therefore the normals at corresponding points on the ellipsoid and its first pedal will intersect. The coördinates of the points of intersection are readily found to be

$$x = \xi'\left(1 - \frac{P^2}{a^2}\right),$$

$$y = \eta'\left(1 - \frac{P^2}{b^2}\right),$$

$$z = \zeta'\left(1 - \frac{P^2}{c^2}\right);$$

but these are the coördinates of a point on the counter-pedal surface, and therefore this latter is the locus of the intersections of normals at corresponding points on the ellipsoid and its first pedal.

This, of course, affords a construction for the normals and tangent planes to the first pedal, two points of the normal being given by the above equations.

The general definition of the counter-pedal, to any surface, as given in the above-mentioned paper, was that it is the locus of the points of intersection of the normals to a surface with planes through a fixed point (the pole) parallel to the tangent planes to the given surface. For convenience the pole may be taken at the origin. It is easy to show now that the above method of generating the counter-pedal in the case of the ellipsoid is applicable to the cases of any surface. A very simple geometrical proof of this is as follows.

Denote by O the pole, and by SQS' the given surface. PQ is the tangent plane at the point Q, and OP the perpendicular upon the tangent plane. Now it is known (*vide* Salmon's Geom. of Three Dimen.) that the sphere described upon OQ as a diameter touches the locus of P, *i. e.* touches the pedal; then if M is the middle point of OQ, the line PM is the normal to the pedal of the surface SS'; but QN the normal to SS' is parallel to OP, and therefore the normal PM passes through the point N; but N is the point on the counter-pedal corresponding to Q on the given surface, hence it follows that the counter-pedal surface to any given surface is the locus of the intersections of the normals at corresponding points on the given surface and its first pedal.

There is another class of surfaces of some interest, which may be called counter-centro-surfaces; these are found by drawing lines through the pole parallel to the normals at each point of a given surface, and laying off upon these lines lengths equal to the radii of curvature at the corresponding points on the surface. The investigation of these surfaces is reserved for the present. It may be remarked, however, that the equation of the counter-centro-surface of the ellipsoid is

$$\frac{a^2x^2}{Q-a^2}+\frac{b^2y^2}{Q-b^2}+\frac{c^2z^2}{Q-c^2}=0,$$

where

$$Q=(a^2x^2+b^2y^2+c^2z^2)^{\frac{1}{2}}.$$

The equation when rationalized is of the eighth degree; the surface has a conjugate quadruple point at the origin, and its principal sections each consist of a conic and a sextic, viz. an ellipse and what may appropriately be called the counter-evolute to the corresponding principal section of the ellipsoid.

It is of course clear that we might have spoken of a quadric in general instead of confining attention merely to the ellipsoid.

On Subinvariants, i. e. Semi-Invariants to Binary Quantics of an Unlimited Order.

By J. J. Sylvester.

Er macht kein System, sondern es wird, es concrescirt in ihm, wie das Kind im Mutterleibe. (*Schopenhauer*) Deutsche Rundschau, July, 1882, p. 69.

§ 1. Proem.

Any rational integer function ϕ of the letters $a, b, c, \ldots$ indefinitely continued, which satisfies the partial differential equation $(a\delta_b + 2b\delta_c + 3c\delta_d \ldots)\phi = 0$ may be termed as subinvariant in respect to the elements $a, b, c, \ldots$ or simply a subinvariant to or *quâ* those elements. It follows from this definition that any rational integer function of one or more subinvariants is itself one.

The same function of the letters $a, b, c, \ldots$ which, when regarded as the coefficient of the highest power of the first variable x in a covariant to the quantic $(a, b, c, \ldots \between x, y)^i$ or the polynomial $(a, b, c, \ldots \between x, 1)^i$ is termed a differenciant of the quantic or polynomial, when regarded as an individual of the infinite scale to which ϕ belongs, assumes the name of a subinvariant in respect to the letters $a, b, c, \ldots$

Of course a differenciant derives its name from reference to the fact that, when multiplied by a suitable power of a it may be regarded as a function of the differences of the roots of any one of the infinite series of polynomials, of some covariant of each of which it is the principal coefficient.

It follows also from the definition that if any composite function is a subinvariant, each of its factors must be so too. For if the function be $P^\alpha . Q^\beta . R^\gamma \ldots$ writing $a\delta_b + 2b\delta_c + \ldots = E$, we must have $\alpha \dfrac{EP}{P} + \beta \dfrac{EQ}{Q} + \gamma \dfrac{ER}{R} + \ldots = 0$, which for denominators $P, Q, R, \ldots$ relatively prime to each other is obviously impossible unless $EP = 0$, $EQ = 0$, $ER = 0 \ldots$, *i. e.* $P, Q, R \ldots$ are subinvariants.

Again, suppose U, V, Ω to be three subinvariants so related that the equation $XU + VU = \Omega$ is capable of being satisfied at all. I say that it must be capable of being satisfied by subinvariantive values of X, Y.*

For from the equation it follows that $EX.U + EY.V = 0$, of which the most general solution is $EX = K\left(\frac{V}{\Delta}\right)$, $EY = -K\left(\frac{U}{\Delta}\right)$. Hence $X = \left(\frac{V}{\Delta}\right)E^{-1}K + U_1$, $Y = -\left(\frac{U}{\Delta}\right)E^{-1}K + V_1$, where U_1, V_1 are subinvariants. Substituting these values of X, Y in the original equation, there results $U_1U + V_1V = \Omega$, as was to be shown possible. The same or a similar manner of proof will serve to show that if for three functions U, V, W, $XU + YV + ZW = 0$, X, Y, Z, are, or may be replaced by subinvariants. I do not know for certain, but think that the proposition may be extended to any number of given functions $U, V, W, \ldots$

It is scarcely necessary to add the fundamental theorem that if for the elements $a, b, c, \ldots$ be substituted the elements a, $a\lambda + b$, $a\lambda^2 + 2\lambda b + c, \ldots$ where λ is arbitrary, any subinvariant remains unchanged; the proof being that if such a change be made in the elements of any function F, ΔF (the change in F) is expressible by $(e^E - 1)F$, which, when F is a subinvariant, so that $EF = 0$, vanishes identically. Hence it is that subinvariants become differenciants.†

It may be worth while here to notice that if in place of the operator on ϕ in the above equation *any* numerical linear function of $a\delta_b$, $b\delta_c$, $c\delta_d \ldots$ be substituted,‡ the value of ϕ which satisfies the transformed equation will be a subinvariant *quâ* the elements $a, b, c, \ldots$ divided respectively by appropriate numbers; viz., if the linear function be $pa\delta_b + qb\delta_c + rc\delta_d$, these numbers will be 1, p, $\frac{p.q}{1.2}$, $\frac{p.q.r}{1.2.3}$, as will be evident by making $a = a$, $p\beta = b$, $\frac{p.q}{1.2}\gamma = c$, $\frac{p.q.r}{1.2.3}\delta = d, \ldots$ which being done the operator last above written may be changed into $a\delta_\beta + 2\beta\delta_\gamma + 3\gamma\delta_d \ldots$

* For instance, in the above equation, U, V may be supposed to be two subinvariants of equal extent, exceeding by a unit that of Ω, their resultant in respect to their final letter. We know, by a principle demonstrated further on in the text, that Ω must be a subinvariant. The present theorem shows that X and Y also are (or may be replaced by) subinvariants.

† Or more simply for any number of letters $a_1, a_2, \ldots a_i$, not fewer than the number of ratios between $a, b, c, \ldots$, if $a\Sigma a_1 = ib$, $a\Sigma a_1 a_2 = \frac{i(i-1)}{2}c$, $a\Sigma a_1 a_2 a_3 = \frac{i(i-1)(i-2)}{2.3}d \ldots$ then $a\delta_b + 2b\delta_c + 3c\delta_d \ldots = \Sigma \frac{d}{da}$, because $\Sigma\frac{db}{da} = a$, $\Sigma\frac{dc}{da} = 2b$, $\Sigma\frac{dd}{da} = 3c \ldots$ Hence any subinvariant to the letters $a, b, c, \ldots$ is a function of the differences of $a_1, a_2, \ldots a_i$.

‡ So *ex gr.* $(a\delta_b + b\delta_c + c\delta_d \ldots)^{-1}0$ is a subinvariant *quâ* the elements $a, b, \frac{c}{1.2}, \frac{d}{1.2.3} \ldots$

As a consequence of this it will readily be seen that if $\phi\,(a,\, b,\, c,\, d, \ldots)$ be a subinvariant to the elements $a,\, b,\, c,\, d \ldots$

$\phi\,(0,\, b,\, c,\, d, \ldots),\ \phi\,(0,\, 0,\, c,\, d, \ldots),\ \phi\,(0,\, 0,\, 0,\, d, \ldots)$

will respectively be subinvariants *quâ* the elements

$$b,\ \frac{c}{2},\ \frac{d}{3},\ \frac{e}{4},\ \ldots$$
$$c,\ \frac{d}{3},\ \frac{e}{6},\ \ldots$$
$$d,\ \frac{e}{4},\ \ldots$$

and so on, the denominators following the law of figurate numbers.

This theorem, although foreign to the original and primary object of the present paper, as given in § 4, is of some considerable importance to the method of deduction. I mean the method (theoretically perfect but practically very difficult of application for quantics beyond the 4th order) according to which all the groundforms of a quantic, or which is the same thing, their ground-differenciants,* may be deduced by an exhaustive algebraical process in successive strata or categories from one another beginning with the known forms a, $ac - b^2$, $a^2d - 3abc + 2b^3, \ldots$ as the first category. See § 3.

It follows from the definition above given that a subinvariant may contain any given number of letters, and the number which it actually contains, less one (*i. e.* the weight of the most advanced letter which appears in it), may be called its *extent.* Any subinvariant will then be a differenciant to a quantic whose order is not less than such extent.

Of course the definition of subinvariant may be extended to sets of letters $a,\, b,\, c \ldots;\ a',\, b',\, c' \ldots;\ a'',\, b'',\, c'' \ldots.$ Any function ϕ of these sets of letters may be called a subinvariant, or when necessary, by way of distinction, a pluri-subinvariant, which satisfies the equality $(a\delta_b + 2b\delta_c + \ldots + a'\delta_{b'} + 2b'\delta_{c'} + \ldots + a''\delta_{b''} + 2b''\delta_{c''} \ldots)\,\phi = 0$. But for greater simplicity, except when a necessity arises for enlarging the horizon, I shall, in what follows, confine myself to the case of a single set of letters, *i. e.* of uni-subinvariants.†

* I shall frequently use the term groundform to signify the leading coefficient of what is ordinarily so termed.

† Eventually I am inclined to substitute the word binariant for subinvariants, and to speak of simple, double, treble or multiple binariants. The functions similarly related to ternary forms will then be styled simple or multiple ternariants, and so in general.

By an irreducible subinvariant is of course to be understood one which cannot be expressed as a rational integer function of any others. A differenciant to an irreducible quantic is of necessity a subinvariant, but not necessarily or even generally an irreducible subinvariant in the absolute sense in which the word is employed above; it will, however, be inexpressible as a rational integer function of any other subinvariants whose extent does not exceed the order of the quantic concerned, and may thus be said to be *relatively* irreducible. Thus *ex gr.* the subinvariant $a^2d^2 + 4ac^3 + 4db^3 - 3b^2c^2 - 6abcd$ is irreducible, relatively to the extent 3 or *quâ* the letters a, b, c, d, that is to say, cannot be expressed as a rational integer function of subinvariants whose elements are limited to a, b, c, d, but it is not an irreducible subinvariant in the absolute sense of the term, because it can be represented by a combination of the subinvariants a, $ac - b^2$, $ae - 4bd + 3c^2$, $(ac - b^2)e + 2bcd - ad^2 - c^3$, the letter e being eliminated by the process of taking the difference between the product of the 2d and 3d and that of the 1st and 4th of the preceding groundforms.*

Here I may take occasion to state a theorem of wide generality suggested by the above decomposition. It is well known that if ϕ be a subinvariant extending to the letter l as the highest letter which it contains, all the successive derivatives ϕ in respect to l will also be subinvariants, as is evident from the fact that if $(a\delta_b + 2b\delta_c + \ldots + ik\delta_l)\phi$ is zero, the same must be true of $(\delta_l)(a\delta_l + 2b\delta_c + \ldots ik\delta_l)\phi$, or what is the same thing, of $(a\delta_b + 2b\delta_c + \ldots + ik\delta_l)\delta_l\phi$.

Suppose then that $\phi, \psi, \omega, \ldots$ are any number of subinvariants limited to l as their highest letter, and regarded, each of them, as a homogeneous function of l and 1, then I say that any differenciant in respect to l of this system of quantics will be a subinvariant *quâ* the elements $a, b, c, \ldots k$. For we know that any differenciant of the system $\phi(x)$, $\psi(x)$, ... say $(\alpha, \beta, \gamma \ldots \lambda \between x, 1)^i$; $(\alpha', \beta', \gamma' \ldots \lambda' \between x, 1)^{i'}$, ... remains unaltered when α, $\alpha + \beta x$, $\alpha + 2\beta x + \gamma x^2 \ldots \phi$, respectively, are substituted for $\alpha, \beta, \gamma \ldots \lambda$, and at the same time α', $\alpha' + \beta' x$, ... ψ, for $\alpha', \beta' \ldots, \lambda'$, respectively, and so on; that is to say, any subinvariant of the equation above written may be regarded as a function of $\phi x, \phi' x, \phi'' x, \ldots; \psi x, \psi' x, \psi'' x, \ldots; \ldots$ Hence in regard of the system of subinvariants any of its differenciants is a function of the members of the

* So it may be shown that the subinvariants of deg-orders 5.7, 5.1, 5.5 to the Quintic (which are perfectly determinate), may be regarded as the resultants in respect to g of the sextic groundforms 2.0 and 4.6, 2.0 and 4.0, 2.0 and 4.4 respectively, all four of which are linear in g. See Sextic Germ Table, § 2.

system, and the successive derivatives in respect to l of each member, all of which are subinvariants. Hence the differenciant in question may be regarded as a function exclusively of subinvariants, and is therefore a subinvariant of the letters $a, b, c, \ldots k$. As a particular application of the theorem we see that the resultant in regard to their last letter of two subinvariants of like extent and the discriminant of any subinvariant in regard to its last letter are subinvariants. Thus *ex gr.* if the discriminant of a cubic be exhibited as a quadratic function of d, viz. under the form $a^2d^2 + (4b^3 - 6abc)d + (4ac^3 - 3b^2c^2)$, its discriminant, viz. $(2b^3 - 3abc)^2 - a^2(4ac^3 - 3b^2c^2)$, *i. e.* $4(b^3 - 3ab^3c + 3a^2b^2c^2 - a^3x^3)$ is as it ought to be a subinvariant, viz. it is $4(b^2 - ac)^3$. So more generally, if we regard any number of pluri-subinvariants (all of the same extent in each set of letters) as a system of multi-partite polynomials in the extreme letter of each set, any differenciant of such system will be a subinvariant (of course with diminished extent in each set) in regard to the original letters. The simple instance already given will serve as a diagram to make the reason self-evident. The invariant in respect to d of the discriminant of the cubic is the same as in respect to x of $a^2(x + d)^2 + (4b^3 - babc)(x + d) + (4ac^3 - 3b^2c^2)$, *i.e.* of $a^2x^2 + 2(a^2d - 3abc + 2b^3)x + (a^2d^2 + 4b^3d - babcd + 4ac^3 - 3b^2c^2)$, hence being a function of the three coefficients, which are all of them subinvariants, it is itself a subinvariant.*

It has been shown above that the same form which regarded as a differenciant is irreducible, *i. e.* is incapable of being decomposed into products of other differenciants of no higher extent than its own, when regarded as a subinvariant may be, and as a matter of fact, far oftener than not will be decomposable into products of subinvariants of higher extent. Thus the irreducible differenciants to any quantic naturally resolve themselves into two classes, those which are absolutely irreducible and those which are only relatively so; and it would seem that in any natural method of proof of Gordan's theorem these would, it is likely, have to be considered separately. There is comparatively little difficulty in proving that the first class are finite in number; the proof of the second class being likewise finite, must depend upon the fact that they are the resultants of a finite number of functions.

I use the word resultant in the above paragraph in an enlarged sense. If $U, V, W, \ldots$ are any given polynomials in $x, y, \ldots z, t, \ldots u$, I call any

* The method of proof here employed, it will be seen, is the same in kind as that employed in the ordinary proof of Taylor's theorem.

quantity not containing $x, y, \ldots z$ capable of being exhibited under the form of the syzygetic function $U_1 U + V_1 V + W_1 W \ldots$ a resultant of the given polynomials in respect to $x, y, \ldots z$. For resultants thus defined, the following important proposition admits of easy proof, viz. Every such resultant is capable of being represented as a sum of products $U_1 U$, $V_1 V, \ldots$ of which the orders in $x, y, \ldots z$ are *limited* in extent, and consequently the most general representation of such resultant can contain only a finite number of arbitrary parameters. When the number of the eliminables $x, y, \ldots$ is one less than the number of the given functions which contain them, we fall back upon the ordinary kind of resultant, having only one arbitrary parameter. When there is but one eliminable x; and any number of polynomials $U, V, W, \ldots$ of orders $\alpha, \beta, \gamma, \ldots$ in x, the order in x of each syzygetic product U_1U, $V_1V, \ldots$ in a syzygetic function of $U, V, W, \ldots$ which is competent to represent any resultant of the system, is (if I mistake not) at most one unit less than the sum of the two highest (or of the two as high as any) of the numbers $\alpha, \beta, \gamma \ldots$

The orders of the syzygetic multipliers being once determined, the number of indeterminate constants is known, and these will be subject to satisfy a known number of *linear* equations, viz. a number greater by unity than the order of the $U_1U + V_1V \ldots$ polynomial, and thus the problem of finding the complete system of resultants of the original system of polynomials in one variable is brought to depend upon the problem of finding the complete system of resultants of a system of homogeneous *linear* functions of several variables, a problem of which the solution and the number of arbitrary parameters which at most can appear in it are perfectly well known and need not be here set forth.

The syzygetic products U_1U, $V_1V, \ldots$ whose sum is competent to express every resultant of $U, V, \ldots$, I have said, need none of them be taken of an order so high as the sum of the two greatest of the quantities $\alpha, \beta, \gamma \ldots$ Thus for instance in the case of $U, V, W, \ldots$ being linear functions, the syzygetic multipliers, as is well known, need only to be taken as constants; or again when $\alpha, \beta, \gamma, \ldots$ form a descending series, the syzygetic products need only to be all of them made of the same order as the highest of the given functions. Take, to fix the ideas, three functions, U, V, W, all of them quadratics in x. The syzygetic multipliers may be taken all linear functions in x: there will thus arise six disposable constants subject to three conditions, inasmuch as the coefficients of x^3, x^2, x, must vanish in the sum of the products: if two of the

multipliers, say of U, V, were made quadratic functions, there would be eight disposable constants subject to four conditions, since an additional coefficient, viz. of x^4, would have to vanish in the sum of the products; there would therefore be one additional arbitrary parameter, viz. $8-4$ instead of $6-3$, but the form of the *resultant* would be not more general than on the preceding supposition, because if to U_1, V_1 (the most general values of the linear multipliers of U, V), λV, $-\lambda U$ respectively be added, there will then be four arbitrary parameters, and consequently the solution must be the same as on the second supposition, but the value of the resultant remains unaltered by the change made in U_1, V_1.

Or again if U, V, W were the two first quadratics and the second a linear function in x, their syzygetic multipliers might be taken two constants and a linear function respectively: by raising the orders of any two of these multipliers by a unit, an additional arbitrary constant would be gained, but the sum of the products resulting therefrom would not thereby gain in generality, as may be shown by the same method as in the preceding example.

It might probably not be difficult to give a universal rule for determining the lowest orders of the syzygetic multipliers required for expressing the resultant in its most general form, of functions of one or even of several variables, but this is an inquiry which it is necessary to postpone, as it might lead to too long a deviation from the immediate purpose in view, and there are some difficulties attending the subject more than present themselves at first sight.

It is enough to know, and that only for the case of a single eliminable, the existence of a limit to the orders of the multipliers, which it is quite easy to demonstrate. That being premised, it will follow as an easy consequence, that any combination *inter se* of subinvariants of any given extent and each containing the highest letter corresponding thereto can only give rise to a limited number of subinvariants of lower extent, and from that it is easy by repeated applications of the same *principle of the limit* to infer that only a finite number of *relatively* irreducible subinvariants of any given extent (*i. e.* irreducible into combinations of subinvariants of the same or lower extent) can arise from the combinations of a finite number of subinvariants of any given higher extent; but it will appear in the sequel that the degree and consequently that the number of irreducible subinvariants of any given extent is subject to a limit; consequently if the number of relatively irreducible subinvariants of any given

extent (or which is the same thing, if the covariants of a quantic of any given order were unlimited in number), this could only be in consequence of there being no extent so large but that subinvariants of that extent and containing the most advanced letter corresponding thereto, would be needed in order to exhibit the composition of the relatively irreducible, but in an absolute sense, reducible subinvariants referred to.

In § 4 I propose to show how to obtain the types (*i. e.* deg-weights) of the absolutely irreducible subinvariants of the first few degrees. Besides the intrinsic interest of the inquiry, the result obtained without going beyond subinvariants of the 7th degree will serve to show conclusively that *it is not true* "that syzygants and groundforms of the same degree and order cannot appertain to the same binary quantic," but that when the order of the quantic is sufficiently elevated there *must* appertain to it, syzygants (*compound* ones) and groundforms of the same degree and order.

Let it be observed that the proposition here about to be disproved is not coëxtensive with the law of parsimony, but goes considerably beyond it—*i. e.* implies much more than that law gives warrant for.

Let us for the moment call the number of linearly independent forms of the deg-order (j, ω) to a given quantic given by Cayley's rule, the denumerator to the type (j, ω), and the number of forms of such type that can be obtained by compounding together groundforms of lower types, the aggregator to the same type. Let us further suppose that the duad (j, ω) may be compounded of (j', ω'), (j'', ω'').*

Suppose further that the aggregator to the type (j', ω') exceeds its denumerator, and also that there exists one or more, say Δ' linearly independent invariantive forms of the deg-order (ω'', j''), but that (*if possible*) the aggregator to the type (j, ω) is equal to or less than its denumerator, the difference being Δ. Obviously if such a case can occur, the law of parsimony (*i. e.* the Newtonian rule of not assuming more causes to exist than are necessary to the explanation of a phenomenon or set of phenomena) will, on such a supposition, lead to the conclusion, not that there are Δ groundforms and *no* syzygies, but $\Delta + \Delta'$ groundforms and Δ' syzygies. Such a case does not present itself for quantics of the lower orders; it seems natural and logical therefore to seek for it in the case of a quantic of an infinite order, *i. e.* in the case of subinvariants unlimited in

*I mean that $j = j' + j''$ $\omega = \omega' + \omega''$.

extent. If it can be shown (as in §4 it will be shown) that with an unlimited number of letters, an irreducible subinvariant and a compound syzygy of subinvariants co-exist for a given degree and for the weight ω, it will follow from the nature of the process employed in what follows, that the same conclusion must hold when the *extent* of the subinvariants is limited, provided (at the very worst) that the limit is not less than ω, for it will be seen that no letter of higher weight than ω enters into the process which leads to the result under consideration. It is in all human probability true that the proposition holds good in the form in which it was originally presented, viz. that *irreducible* syzygants and irreducible invariantive derivatives of the same type, to the same quantic cannot coëxist; but whether the proposition so limited is sufficient to support the substitution of the process of tamisage performed upon the numerator of the representative generating fraction, in lieu of tamisage performed upon the development of that fraction in an infinite series, or how the method of substitutive tamisage, if at present inexact, may be modified *pari passu* with the needful modification in brute tamisage so as to recover its validity, is a matter which must be reserved for future consideration.

§2. Germs.

Before proceeding to the more immediate object of this paper I think it will be profitable to insert the following table of the multipliers of the highest letter or power of the highest letter f in the relatively irreducible subinvariants of the extent 5 (*i. e.* the leading coefficients in the groundforms of the quintic), and a similar table for the groundforms of the sextic arranged according to the powers of g.* For many purposes these tables will be found as serviceable as the entire function of the letters or even as the entire covariant written out at length. Those relating to the quintic may be verified by comparison with the tables (as far as they extend) contained in the *Formes Binaires* of M. Faà de Bruno, but the order of arrangement of the terms in those tables is not what my method of representation points out as the most natural, and proceeds upon some principle not easy to divine. It is also necessary to state that there are very many errors and misprints in those tables. With regard to the particular choice of the groundforms of any deg-order I believe that in all cases but one the tables of M. de Bruno are in accordance with those

*Any such multiplier I call the *germ* of the form to which it appertains.

employed by myself; and which are on the face of them the *simplest* that can be employed, with one exception, viz. in the expression for the covariant of deg-order 9.3 the multiplier of the power of f, or *germ* as it may well be styled, is $(ac-b^2)^3$, whereas in the extended tables of M. de Bruno the germ will be found to be some numerical linear function (its exact value I have forgotten) of $(ac-b^2)^3$, $a^2(ac-e^2)(ae-4bd+3c^2)$, and $a^3\begin{vmatrix} a & b & c \\ b & c & d \\ c & d & e \end{vmatrix}$ or which comes to the same thing, of the two former and $a^2d^2+4ac^3+4db^3-3b^2c^2-6abcd$; the covariant thus given of deg-order 9.3 is accordingly more complicated than it need have been.

It may be well to notice that whenever two consecutive terms in either table occur with the same germ but different powers of the last letter, the complete subinvariant of the antecedent is (to a numerical factor *près*) the differential derivative of the consequent in respect to that letter; thus *ex gr.* the leading coefficient in the covariant to the quintic of the deg-order 7.5 will be found by simply differentiating the invariant of the degree 8 and dividing the result by the number 3.

In the table immediately following (c), (d), (e), $(e)'$, Δ stand for a, $ac-b^2$, $a^2d-3abc+2b^3$, $ae-4bd+3c^2$, $ace-ad^2+2bcd-c^3-ad^2$ and $a^2d^2+4ac^3+4db^3-3b^2c^2-6abcd$ respectively. The quantities which appear in the outside vertical column are the germs; the double figures which fill the occupied spaces are the deg-orders. Thus *ex gr.* 7.5 being opposite to the germ $(c)(d)$ and in the column headed by f^2, indicates that the covariant to the quintic of degree 7 and order 5 has for its differenciant a quantity of the form $(ac-b^2)(a^2d-3abc+2b^3)^2+$ a linear function of f, and so in general.

GERM TABLE TO THE QUINTIC.

	1	f	f^2	f^3	f^4	f^7
a	1.5					
(c)	2.6					
(d)	3.9	4.4				
(e)	2.2					
(e')	3.3					
a^2		3.5	4.0			
$a(c)$		4.6	5.1			
$a(d)$			6.4			
$(c)^2$		5.7	6.2			
$(c)(d)$			7.5	8.0		
$3a(e') - 2(c)(e)$		5.3				
$a^2(c)$				7.1		
$a(c)^2$				8.2		
$(c)^3$				9.3		
$(c)^2(d)$					11.1	
$(c)^2$					12.0	
$(c)^3(e')$					13.1	
$a(c)^5$						18.0

In the annexed table (c) (d) (e) (e') (f) (Δ) retain their previous significations. The additional symbols (cf) (c^2f), (df); (cef) represent respectively the differenciants to the quintic of the deg-orders 4.6, 5.7, 4.4, 5.3, all of which are linear functions of f (see preceding table).

Germ Table to the Sextic.

	1	g	g^2	g^3	g^5	g^6
a	1.6	2.0				
(c)	2.8	3.2				
(d)	3.12	4.6				
(Δ)			6.0			
(e)	2.4					
(e')	3.6	4.0				
(f)	3.8					
$a(c)$			5.2			
$a(d)$			6.6			
$a(e)$		4.4				
$(c)(d)$				8.2.		
$(d)(e)$			7.4			
$(c)(f)$	4.10	5.4				
$(d)^3$						15.0
$a(d)(e)$				9.4		
$(c)^2(f)$		6.6*				
$a(d)(f)$			7.2			
$(c)(c)^2(f)$				10.2		
$(c)(e)(f)$	5.8					
$a^2(c)(d)$					12.2	
a^5					10.0	
$(c)(c^2f)$				10.2		

§ 3. Groundforms.

Quantitative Deduction of their Categories.

I will now proceed to explain what I mean by the exhaustive or quantitative method of deducing the ground differenciants to a given quantic, referred to in the course of the preceding observations.

The well-known functions of alternately the second and third degrees $ac - b^2$, $a^2d - 3abc + 2b^3$, $ae - 4bd + 3c^2$, ... limited in extent to the order of the quantic under consideration, may be called the protomorphs or primaries.

Suppose then the groundforms to the cubic are to be deduced. The primaries or protomorphs, omitting a, are $ac - b^2$, $a^2d - 3abc + 2b^3$, and the residues (meaning thereby the remainders when these quantities are divided by a) are $-b^2$, $2b^3$. Hence $(a^2d - 3abc + 2b^3)^2 + 4(ac - b^2)^3$ will divide out by a (as it happens by a^2) and give the new groundform $a^2d^2 + 4ac^3 + 6abcd + 4b^3d - 3b^2c^2$.

Between its residue $4b^3d - 3b^2$, and the two former, it is obvious that no new relation can arise. Hence the four forms a, $ac - b^2$, $a^2d - 3abc + 2b^3$, $a^2d^2 + 4ac^3 - 6abcd + 4b^3d + 3b^2c^2$ constitute the complete system of ground differenciants, and the corresponding co- and-invariants comprehend the complete system of such for the cubic.

Proceeding to the quartic, a new protomorph or base-form comes into view, viz. $ae - 4bd + 3c^2$, whose residue is $-4bd + 3c^2$ in addition to the antecedent ones $4b^3d - 3b^2c^2$, $2b^3$, $-b^2$, and since the second of these is the product of the first and last it follows that $-(a^2d^2 + \ldots) + (ac - b^2)(ae - 4bd + 3c^2)$ must contain the factor a, and on performing the division there emerges the new groundform

$$\begin{vmatrix} a & b & c \\ b & c & d \\ c & d & e \end{vmatrix}$$

so that $(a^2d^2 + \ldots)$ being equal to this multiplied by a less the product of two other groundforms, ceases itself to be one, and the groundforms now subsisting are the one last named in addition to the base-forms a, $ac - b^2$, $a^2d - 3abc + 2b^3$, $ae - 4bd + 3c^2$, which, since the new one is the only one of the *five* containing the letter e, can enter into no combination with them of which the residue is zero, and consequently the *deduction* is at an end and the five named constitute the complete system of groundforms.

Beyond this point the method of deduction has not hitherto been pushed, nor could it have been, without the use of the theorem concerning the subinvariantive character of the residues, in consequence of their enormous complexity when regarded as simple functions of the letters. In what follows the deduction is extended to the case of the quintic.*

Algebraical Deduction of the Groundforms of the Quintic.†

The complete system of groundforms to be deduced may be denoted by the deg-order or the deg-weight: viewed as subinvariants, the latter is the

* In Salmon's Modern Algebra, 3d Ed., pp. 170-71, 195-6, the base-forms employed in the deduction of the quartic groundforms are not identical with those employed above, the third one being of the fourth instead of the second degree in the letters, and consequently not a groundform, whereby the deduction is rendered somewhat longer than that given in the text. The most eligible base-forms to employ in any case are alternately of the second and third degree, whereas those given by Prof. Cayley, the author of this important method, are of degrees continually increasing by a unit.

† By algebraical, I mean in this connection, that which deals only with the ordinary algebraical processes of addition, multiplication and division, as contradistinguished from transcendental processes involving differential operation, or which is substantially the same thing, symbolical resolution.

The preceding deduction for the Cubic and the Quartic is by far the simplest mode of obtaining the complete systems of groundforms for these quantics, and proving their completeness, which, at an earlier period of the theory, was regarded as a problem of some little difficulty. See Faà de Bruno's *Formes Binaires*, Chapter 7, pp. 260-263, where the same results are obtained through the medium of "*Formes Associées.*" I cannot but think that sooner or later this method, first discovered by the eagle-gaze of Cayley, will lead to the object which I presume he had in view when he originated it, viz. a proof of Gordan's theorem by ordinary algebra.

I think I see looming in the not far distance such a proof, depending ultimately upon the fact of a certain succession of increasing integer multiplets, subject to stated laws of limitation, not being capable of being indefinitely produced. To render sensible the sort of arithmetical theorem which I have in view, I subjoin a theorem *ejusdem generis* concerning singlets (simple integers), which, as far as I know, is new, and admits of easy proof.

A succession of integers of which no one is a multiple of one nor the sum of the multiples of two others cannot be continued ad infinitum.

To prove this we may begin with the case where one of the integers written down is a prime number, for which case the proof is immediate. Then it is easy from this to show that if the theorem is true for the case where one of the integers is a product of only i-primes, it must be true for the case where one of the integers is a product of only $(i+1)$ primes; for this case, by virtue of the supposition made, may easily be reduced to the case where one of the numbers is a relative prime to all the others, for which case the theorem is true, for the same reason as the number in question were an absolute prime. Consequently the theorem is true universally.

By the quotient of a duad (in what follows) is to be understood the quotient of the second element by the first; by the sum of two duads, the duad whose elements are the sums of the corresponding elements of the two, and by a multiple of a duad the duad whose elements are the elements of that duad multiplied each by the same *integer*. The foregoing theorem may then be extended as follows:

A succession of duads, the quotients of all which but two are intermediate to the quotients of those two, and such that no duad is a multiple of any one or the sum of the multiples of any two or three of the others, cannot be indefinitely continued.

Again, one couple of quantities may be said to be *intermediate* to three others when the point representing the first is situated within the triangle whose apices represent the other three; a point

more natural mode of designation: if j and ω are the degree and weight, the order ε will be $5j - 2\omega$. For greater facility of reference to the known list of groundforms, it will be convenient to set out the order as well as the degree; the complete system of the designating $j; \varepsilon . \omega$, of the twenty-three groundforms, *i. e.* of the twenty-three relatively irreducible subinvariants of *extent* not exceeding five, will then be as follows: 1;5.0, 2;2.4, 2;6.2, 3;5.5, 3;9.3, 3;3.6, 4;0.10, 4;4.8, 4;6.7, 5;1.12, 5;3.11, 5;7.9, 6;2.14, 6;4.13, 7;1.17, 7;5.15, 8;0.20, 8;2.19, 9;3.21, 11;1.27, 12;0.30, 13;1.32, 18;0.45. The protomorphs or base-forms are the five first of these, viz. 1;5.0 is a, 2;6.2 is $ac - b^2$, 3;9.3 is $a^2d - 3abc + 2b^3$, 2;2.4 is $ae - 4bd + 3c^2$, 3;5.5 is $a^2f - 5abe + 2acd + 8b^2d - 6bc^2$.

Again, 3;3.6 is the determinant

$$\begin{vmatrix} a & b & c \\ b & c & d \\ c & d & e \end{vmatrix}$$

being said to represent the two quantities which are equal to its two coördinates in respect to any two given axes. So a triplet of quantities, by aid of an analogous representation in space, may be said to be intermediate to four others when its representative point lies inside the pyramid whose apices represent those four.

It will readily be understood that these definitions may be translated into conditions of inequality between determinants, and thus translated may be extended so as to yield a definition of one pollad of $n-1$ elements being *intermediate* to n, or indeed to any number of other such pollads. Also the quotient-system of an n-ad will be understood to mean the system of $(n-1)$ quotients got by dividing the first element of the n-ad into the $n-1$ others. The following general theorem may then be enunciated:

A succession of n*-ads such that the quotient-systems of all but* n *of them are intermediate to the quotient-systems of those* n cannot be indefinitely continued, if every n-ad which is either a multiple of some one or a sum of multiples of 2, 3, ... n or $n+1$ of the others, is excluded from the succession.

More generally, and with a less stringent negative condition, *a succession of* n*-ads such that the quotient-systems of all but ν given ones (ν being any number) are intermediate to the quotient-systems of those ν, cannot be indefinitely continued, if every* n*-ad which is a multiple or a sum of multiples of any or all of the n-ads of a group of $\nu+1$ others (whereof ν are the given ones) is excluded from the succession.*

The hypothetical ground of connection between this theorem and Gordan's algebraical one is as follows: It may be shown to be implied in the method of deduction, that if the number of groundforms to the quintic were infinite, then there must exist a certain infinite succession of products, some of the form $b^x Q^y R^z S^t$, the others of the form $b^\xi Q^\eta R^\zeta S^\tau T$, such that neither any product $b^x Q^y R^z S^t$ nor any product $b^\xi Q^\eta R^\zeta S^\tau$ could be (a power of one or) a product of powers of any number of the products not involving T. If then it could be shown that there exists a set of quadruplets of the kind x, y, z, t such that every other one of that kind and also every one of the kind ξ, η, ζ, τ is *intermediate* to that set, the existence of such a succession would be impossible by virtue of the arithmetical theorem, and the possibility of the existence of an infinite number of groundforms would consequently be disproved. A similar kind of proof could conceivably, but with more difficulty, be extended to quantics of any order.

This, not involving the letter f, has been previously deduced, and it has been shown that its integrating factor (*i. e.* the power of a by which it must be multiplied to give a rational integer function of the base-forms) is a^3; it has, in fact, been shown (dropping the second integer and dealing only with deg-weights) that $(1.0)^3\,(3.6) = (1.0)^2\,(2.2)\,(2.4) - 4\,(2.2)^3 + (3.3)^2$.

I shall denote the residue of any form ϕ by the symbol $\mathfrak{R}\phi$; each such residue is a function of the *five* letters b, c, d, e, f, being in fact a subinvariant in regard to the letters $b, \frac{c}{2}, \frac{d}{3}, \frac{e}{4}, \frac{f}{5}$, and therefore of the four ground-forms proper to the diminished extent 4, *i. e.* of the five following functions

$$b,\ \frac{bd}{3} - \frac{c^2}{4},\ \frac{b^2e}{4} - 3\frac{bcd}{6} + 2\frac{c^3}{8},\ \frac{bf}{5} - 4\frac{ce}{8} + 3\frac{d^2}{9},$$

$$\begin{vmatrix} b, & \frac{c}{2}, & \frac{d}{3} \\ \frac{c}{2}, & \frac{d}{3}, & \frac{e}{4} \\ \frac{d}{3}, & \frac{e}{4}, & \frac{f}{5} \end{vmatrix}$$

or (getting rid of the denominators) of b, $4bd - 3c^2$, $2b^2e - bcd + 2c^3$, $6bf - 15ce + 10d^2$,

$$\begin{vmatrix} 3b, & 3c, & 2d \\ 3c, & 4d, & 3e \\ 10d, & 15e, & 12f \end{vmatrix}$$

of which the deg-weights are 1.1, 2.4, 3.6, 2.6, 3.9 respectively; the first of these is b, the others I shall call Q, T, R, S respectively. In all that follows I shall denote a numerical linear function of two or more quantities by enclosing them in brackets with commas interposed*; thus *ex gr.* (ϕ, ψ, θ) will mean $\lambda\phi + \mu\psi + \nu\theta$, where λ, μ, ν are certain determinate (but unexpressed) numbers.

We know from the theory of the groundforms of extent 4 (*i. e.* differenciants of a quartic) that the above five quantities are not algebraically independent, but are connected by an equation of the form

$$T^2 = (Q^3, b^3S, b^2QR).$$

*The brackets will sometimes for convenience be omitted.

We have also the following expressions for the residues of the groundforms denoted by their deg-orders, and their first deduct, viz.

$$\mathfrak{R}(2;6)=b^2,\ \mathfrak{R}(3;9)=b^3,\ \mathfrak{R}(2,2)=Q,\ \mathfrak{R}(3;5)=bQ,\ \mathfrak{R}(3;3)=T,$$

or, using deg-weights instead of deg-orders,

$$\mathfrak{R}(2.2)=b^2,\ \mathfrak{R}(3.3)=b^3,\ \mathfrak{R}(2.4)=Q,\ \mathfrak{R}(3.5)=bQ,\ \mathfrak{R}(3.6)=T.$$

Since $b^3.Q=b^2.bQ$, *i. e.* $\mathfrak{R}(3,9)\,\mathfrak{R}(2,2)-\mathfrak{R}(2.6).\mathfrak{R}(3,5)=0$, it follows that $((3;9)(2;2),\ (2;6)(3;5))$ must contain a.

Also it is obvious that the effect of throwing out a from a differenciant to the quintic which contains it, is to diminish the degree by one unit, leaving the weight unaltered, and therefore diminishes the order by five units.

Hence $$\frac{1}{a}((3;9)(2;2),\ (2,6)(3;5))=4;6.$$

It will be more convenient here and hereafter to use exclusively deg-weights instead of deg-orders to denote the forms; the above equation thus expressed becomes

$$\frac{1}{a}((3.3)(2.4),\ (2.2)(3.5))=4.7.$$

Turning now to the deg-weights of the residues, it will be seen that 4.7 can only be composed of 1.1 and 3.6.

Hence $\mathfrak{R}(4.7)=bT$, which is not a product of residues; so 4.7 must be a new groundform. Again, (adhering to the use of deg-weights) we have $(\mathfrak{R}(3.5))^2=b^2Q^2=(\mathfrak{R}\,2.2)(\mathfrak{R}(2.4))^2$.

Hence $$\frac{1}{a}((3.5)^2,\ (2.2).(2.4)^2)=5.10.$$

The only mode of resolving 5.10 into sums of the duads 1.1, 2.4, 3.6, 2.6, 3.9, is by the addition of 2.4 and 3.6.

Hence $\mathfrak{R}(5.10)$ is a numerical multiple of QT, *i. e.* of $\mathfrak{R}(2.4)$ and $\mathfrak{R}(3.6)$. Hence $((5.10),\ (2.4)(3.6))$ contains a; consequently 5.10 is not a groundform, but we shall have $\frac{1}{a}((5.10),\ (2.4)(3.6))=4.10$, and 4.10 can be resolved into $1.1+3.9$ and $2.4+3.6$. Hence $\mathfrak{R}(4.10)=(bS,\ QR)$ and 4.10* will be a new groundform.

So again $(3.3)(3.5)=b^3.bQ$, and $(2.2)^2(2.4)=(b^2)^2Q$. Hence $\frac{1}{a}\{(3.3)(3.5),\ (2.2)^2(2.4)\}=5.8$, which can be resolved in only one way into a sum

*4.10 which is the same (using deg-orders) as 4.0 obviously cannot undergo further depression, and is consequently a groundform.

of the duads 1.1, 2.4, 3.6, 2.6, 3.9, viz. into $1.1 + 1.1 + 3.6$. Hence $\mathfrak{R}(5.8) = \mathfrak{R}(2.2)\,\mathfrak{R}(3.6)$, and consequently 5.8 is not a groundform, but $\frac{1}{a}\{5.8,\ (2.2)(3.6)\} = [4.8]$, which, in respect to the duads above mentioned, is resoluble (and only resoluble) into $2.4 + 2.4$ and $1.1 + 1.1 + 2.6$.

Hence $\mathfrak{R}(4.8) = (Q^2,\ b^2R)$, and since $Q = \mathfrak{R}(2.4)$ we have $\mathfrak{R}((4.8),\ (2.4)^2) = b^2R$; and since $([4.8], (2.4)^2)$ is of the deg-weight 4.8, we see that there is a form 4.8 such that $\mathfrak{R}(4.8) = b^2R$, and is consequently a groundform, since b^2R is not a rational integer function of any of the previous residues. Thus, then, from the base-forms 2.2, 3.3, 2.4, 3.5, besides the groundform not containing f, viz. 3.6, we have derived the three additional groundforms 4.7, 4.10, 4.8 Of these 4.7 and 4.8 belong to the same category as 3.6, being like it derived immediately from the base-forms. Whereas, in obtaining 4.10 it has been necessary to employ 3.6, so that it belongs to a more distant category. If we call the base-forms primaries, 3.6, 4.7, 4.8 will be secondaries, and 4.10 a tertiary. So again we shall find $\mathfrak{R}(3.3)\,\mathfrak{R}(3.6) = b^3.T$, and $\mathfrak{R}(2.2)\,\mathfrak{R}(4.7) = b^2.bT$. Hence $\frac{1}{a}\{(3.3)(3.6),\ (2.2)(4.7)\} = 5.9$, and $\mathfrak{R}(5.9) = b^3.R$, which cannot be compounded out of the preceding residues, so that (5.9) is another tertiary.

Again $\mathfrak{R}(4.7)\,\mathfrak{R}(2.4) = Q.bT$, and $\mathfrak{R}(3.5)\,\mathfrak{R}(3.6) = bQ.T$. Hence $\frac{1}{a}((4.7)(2.4),\ (3.5)(3.6) = 5.11$, and $\mathfrak{R}(5.11) = (b^2S,\ bQR))$, for 5.11, in regard to the oft-quoted duads, is resoluble only into $1.1 + 1.1 + 3.6$ and $1.1 + 2.4 + 2.6$. Hence 5.11 is also a tertiary groundform.

Again $\mathfrak{R}(2.2)\,\mathfrak{R}(2.4)\,\mathfrak{R}(3.6) = b^2.Q.T$, and $\mathfrak{R}\,4.7\,\mathfrak{R}\,3.5 = bT.bQ$. Hence $\frac{1}{a}((2.2)(2.4)(3.6),\ (4.7),\ (3.5)) = [6.12]$, and the duad 6.12 is resoluble into $3.9 + (1.1)^3$* $(2.6) + (2.4) + (1.1)^2(3.6)^2$ and $(2.4)^3$, corresponding to b^3S, b^2QR, Q^3, T^2. Now Q, T, b^2R are all residues, as already shown, and since b^2 and $(bS,\ QR)$ are residues $(b^3S,\ b^2R.\,Q)$, and therefore b^3S is a residue.

Hence a form denotable by 6.12 which shall be a linear function of [6.12] and of the combinations of inferior groundforms, will have a residue zero, and consequently [6.12] will not be a groundform, but the 6.12 last spoken of

* It will be often found convenient to use $(p.q)^i$ to mean the sum of i duads $p.q$.

will be divisible by a, and the quotient will give a groundform 5.12, whose residue corresponding to the composition $3.6 + 2.6$ is RT. We shall thus have obtained for our tertiary or third batch of groundforms (descendants, *i. e.* in the second degree from the base-forms) the subinvariants denoted by 4.10, 5.9, 5.11, 5.12.

Again $\mathfrak{R}(3.3)\mathfrak{R}(4.10) = b^3(bS,\ QR)$; $\mathfrak{R}(2.2)\mathfrak{R}(5.11) = b^2(b^2S,\ bQR)$; $\mathfrak{R}(2.4)\mathfrak{R}(5.9) = Q(b^3R)$. Hence between these three equations the two arguments b^4S, b^3QR may be eliminated, and there results $\frac{1}{a}\{(3.3)(4.10),$ $(2.2)(5.11),\ (2.4)(5.9)\} = 6.13$, and 6.13 will be resoluble only into $3.6 + 2.6 + 1.1$, so that $\mathfrak{R}6.13 = bRT$.

Again $\mathfrak{R}3.5\mathfrak{R}5.12 = bQ.RT$; $\mathfrak{R}3.6\mathfrak{R}5.11 = T\,(b^2S,\ bQR)$; $(\mathfrak{R}4.7)$ $.(\mathfrak{R}4.10) = bT\,(bS.\ QR)$, on the right-hand side of which three equations $bQRT$, b^2ST are the only two arguments appearing, so that ($\mathfrak{R}3.5$, $\mathfrak{R}5.12$, $\mathfrak{R}3.6$, $\mathfrak{R}5.11$, $\mathfrak{R}4.7$, $\mathfrak{R}4.10$) may be made equal to zero. Hence we have a new deduct 7.17, and $\mathfrak{R}7.17$ will be found $= (Q^2S,\ b^2RS,\ bQR)$, and 7.17 will be a groundform, as is apparent at once from the fact that it is the same (using a deg-order instead of deg-weight) as 7;1 which is obviously indecomposable into any inferior forms.

But it may be objected that conceivably there might exist a syzygy between (3.5)(5.12), (3.6)(5.11), (4.7)(4.10), so that the forms 7.17 obtained by dividing a linear combination of the three products by a may really be a null quantity. But not to mention the unlikelihood that a syzygy should occur between so low a number as only three products of groundforms of elevated degrees, the existence of such a syzygy may be directly disproved as follows: (3.6)(6.11) will contain only the first power of f, and writing

$$5.12 = Lf^2 + 2Mf + N, \quad 4.10 = Pf^2 + 2Qf + R,$$

we shall have $\quad 4.7 = Lf + M, \qquad 3.5 = Pf + Q,$

so that if the supposed syzygy exists we must have $LQ - MP = 0$, but $L = -a^2$, $M = 5abc - 2acd + 8b^2d + 6bc^2$, $P = (a^2c - ab^2)$, $Q = \ldots$ Hence since M does not contain a as a factor, MP cannot equal LQ, so that the conceivable syzygy does not exist, and the groundform 7.17 is correctly deduced.*

*I shall eventually supersede this proof of the non-existence of the syzygy under discussion by a method involving no algebraical computation. It is a remarkable feature in this deduction that although it is in its nature quantitative, no algebraical computations whatever need to nor will be employed in working it out and establishing its validity at each stage, thanks to the use made of the factors of integration, as will presently appear.

Again $\mathfrak{R}\,5.9\,\mathfrak{R}\,3.5 = b^3R.bQ$, $\mathfrak{R}\,3.3\,\mathfrak{R}\,5.11 = b^3(b^2S,\ bQR)$, $(\mathfrak{R}\,2.2)^2.(\mathfrak{R}\,4.10) = b^4(bS,\ QR)$, between which equations b^5S, b^4QR can be eliminated; thus there will be a form [7.14] deduced from $\frac{1}{a}((5.9)(3.5),\ (3.3)(5.11),\ (2.2)^2(4.10))$.

Also the sole components of $\mathfrak{R}\,7.14$ will be easily seen to be $(3.6\times(2.4)^2,\ 3.6\times 2.6\times(1.1)^2)$.

Hence $\mathfrak{R}[(7.14)] = (Q^2T,\ b^2RT)$, in which each of the two arguments is a residue.* Hence we may find a 7.14 which will be divisible by a and thus obtain a form 6.14, which (since 5.14 is necessarily non-existent) cannot be further depressed.

That this is not a *null* form will presently be demonstrated. It results that 6.14 is a new groundform, and we have now completed a new (quatertiary) group, *i. e.* the third in order of descent from the primaries, viz. the group 5.12, 6.13, 7.17, 6.14.

Here, having reached the middle of this long deduction, it will be expedient to pause for a while and take stock of the relations so far established between the base-forms and their deducts.

I enclose, in what follows, the deg-weight numbers within square brackets, in order to indicate that the forms which they represent are not necessarily identical with the simplified forms represented by the same numbers, but are the immediate quotients which present themselves after dividing out by a or a power of a in the course of the deduction. We have thus

$$a^3[3.6] - a^3[2.2].[2.4] = (2.2)^3, (3.3)^2 \qquad (3)$$
$$a\,[4.7] = [2.2]\,[3.5], [2.4]\,[3.3] \qquad (1)$$
$$a^2[4.8] + a(?) = [3.3]\,[3.5], [2.2]^2\,[2.4] \qquad (2)$$
$$a^2[4.10] + a(?) = [3.5]^2, [2.4]^2\,[2.2] \qquad (2)$$
$$a\,[5.9] = [4.7]\,[2.2], [3.3]\,[3.6] \qquad (4)$$
$$a\,[5.11] = [4.7]\,[2.4], [3.5]\,[3.6] \qquad (4)$$
$$a^2[5.12] + a(?) = [3.6]\,[2.4]\,[2.2], [4.7]\,[3.5] \qquad (5)$$
$$a\,[7.17] = [5.12]\,[3.5], [3.6]\,[5.11], [4.10]\,[4.7] \qquad (6)$$
$$a\,[6.13] = [4.10]\,[3.3], [2.2]\,[5.11], [2.4]\,[5.9] \qquad (5)$$
$$a^2[6.14] = [5.9]\,[3.5], [5.11]\,[3.3], [2.2]^2\,[4.10] \qquad (6)$$

---

* For Q^2, b^2R, T are each of them residues.

In the above table the quantities connected by one or more commas represent a linear function of themselves, and the sign of interrogation means "some known rational integral function of the base-forms." The numerals to the right (beginning with (3) and ending with (6)) indicate the power of (a) by which each corresponding deduct has to be multiplied in order to become an integral function of the base-forms, and which may be called its integrating factor. Thus *ex gr.* the integrating factor of $[5.9]$ is a^4, because the integrating factors of the two arguments in the linear function expressing $a[5.9]$ are a, a^3 respectively; so again a^5 is the integrating factor of $[5.12]$, because the integrating factors of the arguments of the linear function which expresses $a^2[5.12]+a(?)$ are a^3, a respectively. So again the arguments corresponding to $a[7.17]$ having the integrating factors a^5, a^4, a^3 respectively, the integrating factor of $[7.17]$ will be $1+5$ (the dominant of the numbers $3, 4, 5$), *i. e.* 6. This will be sufficient to show how the integrating factors are to be successively obtained, it being of course borne in mind that the integrating factor of a product of deducts is the product of the integrating factors of the deducts taken separately. With the aid of this table we may see *à priori* that the linear forms representing $[7.17]$, $[6.13]$, $[6.14]$ cannot be identically *nulls*. In the preceding cases no proof is required because we know subinvariants can only be decomposed in one way into factors.

Thus 1^0 for $[7.17]$ the integrating factors of the three arguments being a^5, a^4, a^3; for if a syzygy existed between them we should have $B_1+aB_2+a^2B_3=0$, where each B is a rational integer function of the base-forms not containing a as a factor.

2^0 for $[6.13]$ the separate integrating factors being a^2, a^4, a^4 respectively, did a syzygy exist, we must have $a^2B+B_1+B_2=0$, and consequently $[2.2][5.11]$ would be in syzygy with $[2.4][5.9]$, which is impossible.

3^0 for $[6.14]$ the separate integrating factors being a^4, a^4, a^2, the syzygy is impossible, for the same reason as in the preceding case.

I pass on now to the fifth group, *i. e.* to the deducts four degrees of succession removed from the base-forms.

$\mathfrak{R}2.2\ \mathfrak{R}6.13=b^2.bRT$, $\mathfrak{R}3.6\ \mathfrak{R}5.9=T.b^3R$. Hence there is a deduct $[7.15]$. Its integrating factor will be a into the dominant of the integrating factors of 6.13, 5.9, which are a^4, a^5, *i. e.* it is a^6. Also in regard to the duads 1.1, 2.4, 2.6, 3.9, 3.6, the compositions of 7.15 are $(1.1)^3+(2.6)^2$,

$(1.1)^2 + (2.4) + (3.9)$, $(1.1) + (2.4)^2 + (2.6)$, or b^3R^2, b^2QS, bQ^2R, and the two latter being residues we may write $\mathfrak{R}\,7.15 = b^3R^2$. Its integrating factor is a into the dominant of the integrating factors of 6.13, 5.9 (which are a^5, a^4), and is therefore a^6; 7.15 is necessarily a groundform, for b^3R^2 is obviously indecomposable into simpler residues.

Again $\mathfrak{R}\,3.6\ \mathfrak{R}\,6.13 = T.bRT$, and $\mathfrak{R}\,5.12\ \mathfrak{R}\,4.7 = RT.bT$. Hence 8.19 is a deduct, and its decompositions in respect to the customary duads being $3.6 \times 3.9 \times 2.4$, $3.6 \times (2.6)^2 \times 1.1$, we have $\mathfrak{R}\,8.19 = (QST,\ bR^2T)$. Also 8.19 is a groundform, for the existence of such a form as 7.19 is impossible, inasmuch as 5 times 7 is less than the double of 19. Its integrating index will be the dominant of those of $(3.6)(6.13)$ and $(4.7)(5.12)$ [which are $3+5$ and $1+5$ respectively] increased by unity, *i. e.* is 9. I use here and shall in future use the phrase 'index of integration' to signify the index of the power of a which is the integrating factor.

Again, $\mathfrak{R}\,4.7\ \mathfrak{R}\,6.13 = bT.b.RT$, $\mathfrak{R}\,5.12\ \mathfrak{R}\,3.6\ \mathfrak{R}\,2.2 = RT.T.b^2$. Hence there is a deduct [9.20].

The resolutions of the duad 9.20 in respect to 3.6, 3.9, 2.6, 2.4, 1.1 are $3.6 + 3.9 + 2.4 + 1.1$, $3.6 + (2.6)^2 + (1.1)^2$, $3.6 + (2.4)^2 + 2.6$, corresponding to $bQST$, b^2R^2T, Q^2RT. Now Q^2, b^2R, RT are already known to be residues, and $\mathfrak{R}\,2.2\ \mathfrak{R}\,3.3\ \mathfrak{R}(4.0) = (bQST,\ Q^2RT)$. Hence b^2R^2T, Q^2RT, $bQST$ are all residues. Hence there exists a deduct 9.20 such that $\mathfrak{R}\,9.20 = 0$, and consequently there is a deduct 8.20 which must be a groundform,* since 7.20 is *à priori* known to be impossible. Its resolutions (regarded as a duad) in respect to the customary duads are $(1.1)^2 + (3.9)^2$, $(1.1) + (3.9) + (2.4) + (2.6)$, $(2.4)^2 + (2.6)^2$, $(1.1)^2 + (2.6)^3$, so that $\mathfrak{R}\,8.20 = b^2S^2$, $bQRS$, Q^2R^2, b^2R^3. The index of integration to $(4.7)(6.13)$ is $1 + 5 = 6$, and of $(5.12)(3.6)(2.2)$ is $5 + 3 = 8$. Hence the index of integration to 8.20 is $2 + 8$ or 10.

We have now obtained a new group of ground-deducts, fourth in descent from the primaries, viz. 7.15, 8.19, 8.20, whose integrating factors are a^6, a^9, a^{10} respectively.

Again, we have the following group $1^0\ \mathfrak{R}\,2.2\ \mathfrak{R}\,5.12\ \mathfrak{R}\,6.13 = b^2.RT.bRT$, $(\mathfrak{R}\,3.6)^2\ \mathfrak{R}\,7.15 = T^2.b^3R^2$. Hence there is a deduct [12.27].

In writing out the decomposition table (*quâ* 1.2, 2.4, 2.6, 3.9, 3.6 of 12.27), no account need be taken of $(3.6)^2$, inasmuch as T^2 which it repre-

* I have accidentally omitted here (and may possibly have done so in some other cases) the usual proof by means of the indices of integration, that the deduct is not a null.

sents is a rational integral function of b, Q, R, S, consequently $(3.6)^3$ will not appear therein.

The table will thus be $3.6+(3.9)^2+(1.1)^3$, $3.6+3.9+2.6+2.4+(1.1)^2$, $3.6+3.9+(2.4)^3$, $3.6+(2.6)^3+(1.1)^3$. Hence $\mathfrak{R}[12.27]=(b^3S^2T, b^2QRST, Q^3ST, b^3R^3T)$. But b^3R^2, b^2QS, RT have all been seen to be residues, hence b^3R^3T, b^2QRST are residues.

Also $(\mathfrak{R}4.10)^2=(b^2S^2, bQRS, Q^2R^2)$ is a residue, as is also bT. Hence $(b^3S^2T, b^2QS.RT, bQ^2R.RT)$ is a residue, and $bQ(bS, QR)$, $Q(b^2S, bQR)$ being each of them residues, b^2QS, bQ^2R are each of them separately residues. Hence b^3S^2T is a residue. Also $Q^2\mathfrak{R}8.2=(Q^3ST, bQ^2R^2T)$ is a residue, and bQ^2R^2T is a residue, because bQ^2R, RT are residues. Hence Q^3ST is a residue. Hence all the arguments in expression for $\mathfrak{R}[12.27]$, viz. b^3R^3T, b^2QRST, $b^3S^2T^2$, Q^3ST are residues; consequently a deduct 12.27 may be found such that $\mathfrak{R}12.27=0$, and there will be a deduct 11.27 which cannot be still further reducible, because 10.27 is necessarily non-existent. Its index of integration will be two greater than the dominant of those of $(5.12)(6.13)$ and 7.15, which are $5+5$ and 6, *i. e.* it is 12. Its residue $\mathfrak{R}11.27$ will easily be seen to be $(b^3R^4, b^3RS^2, bQ^2R^3, bQ^2S^2, b^2QSR^2, Q^3RS)$.

Again, 2^0
$$\mathfrak{R}5.9\ \mathfrak{R}5.12=RT.b^3R,$$
$$\mathfrak{R}3.6\ \mathfrak{R}7.15=T.b^3R^3.$$

Hence there is a deduct 9.21 which cannot be further depressed, because 8.21 is necessarily non-existent, and it will readily be found that $\mathfrak{R}9.21=(b^3S^2, b^3R^3, b^2QRS.Q^3S)$, and that the index of integration is $1+4+5$, *i. e.* is 10.

Again, 3^0
$$\mathfrak{R}6.13\ \mathfrak{R}7.17=bRT(Q^2S, b^2RS, bQR^2)$$
$$\mathfrak{R}5.11\ \mathfrak{R}8.19=(b^2S, bQR)(QST, bR^2T)$$
$$\mathfrak{R}3.6\ \mathfrak{R}^25.12=T.(RT)^2=R^2T(Q^3, b^2QR, b^3S)$$
$$\mathfrak{R}5.12\ \mathfrak{R}4.8\ \mathfrak{R}4.10=RT.b^2R(bS, QR)$$
$$\mathfrak{R}2.4\ \mathfrak{R}3.6\ \mathfrak{R}^24.10=Q.T.(bS, QR)^2$$
$$\mathfrak{R}2.4\ \mathfrak{R}3.6\ \mathfrak{R}8.20=Q.T(b^2S^2, bQRS, Q^2RQ^2, b^2R^3).$$

Hence it will be seen that the arguments on the right-hand side of the equation are the five following, viz. bQ^2RST, b^3R^2ST, b^2QR^3T, b^2QS^2T, Q^3R^2T, and no others. Hence the six products on the left may be linearly combined so as to give a result zero, and there will consequently be a deduct 12.30.

To prove that this is not a null, take the integrating factors of (6.13)(7.17), (5.11)(8.19), $(3.6)(5.12)^2$, (5.12)(4.8)(4.10), $(2.4)(3.6)(4.10)^2$, (2.4)(3.6)(8.20). These will be found to be

$5+6,\ 4+9,\ 3+5+5,\ 5+2+2,\ 3+2+2,\ 3+10$, or 11, 13, 13, 9, 7, 13.

Hence if there were any syzygy between these products it must be between the 2d, 3d and 6th, which have a common integrating factor a^{13}, but the 3d and 6th products have a common factor 3.6; hence the three cannot be syzygetically connected, and consequently 12.30 is a *bona-fide* existing deduct, and being incapable of further depression, is necessarily a groundform.

The index of integration will be a unit greater than the dominant of the indices last found, *i. e.* it is 14.

Its residue will be found to be of the form

$$(b^3S^3,\ b^3R^3S,\ b^2QR^4,\ b^2QRS^2,\ bQ^2R^2S,\ Q^3R^3,\ Q^3S^2).$$

Again, 4⁰.

$$\begin{aligned}
&\mathfrak{R}6.13\ \mathfrak{R}8.19 = bRT.(QST,\ bR^2T)\\
&\mathfrak{R}^2 5.12\ \mathfrak{R}4.8 = R^2T^2.b^2R\\
&\mathfrak{R}^2 5.12\ \mathfrak{R}^2 2.4 = R^2T^2.Q^2\\
&\mathfrak{R}^2 3.6\ \mathfrak{R}^2 4.10 = T^2.(bS,\ QR)^2\\
&\mathfrak{R}2.4\ \mathfrak{R}3.6\ \mathfrak{R}5.12\ \mathfrak{R}4.10 = QT.RT.(bS,\ QR).
\end{aligned}$$

In these five equations the arguments on the left-hand side are four in number, viz. $b^2R^3T^2$, $b^2S^2T^2$, $bQRST^2$, $Q^2R^2T^2$. Accordingly a linear combination of the five quantities on the right-hand side will be zero, and there is a deduct 13.32 which cannot be further depressed (since 12.32 is necessarily non-existent), and may be easily seen to be an actual quantity and not a null, inasmuch as the indices of integration of the products of which the quantities to the left are the residues (the anti-residues as they may be termed), are $5+9$, $5+5+2$, $5+5$, $3+3+2$, $3+5+2$, *i. e.* 14, 12, 10, 8, 10, of which only *a pair are equal.* Its index of integration is one unit more than the dominant of these numbers, *i. e.* is 15.

Finally $\mathfrak{R}13.32 = (b^2RT,\ b^2RS^2T,\ bQR^2ST,\ Q^2R^3T,\ Q^2S^2T)$. The four last deducts 11.27, 9.21, 12.30, 13.32 form the batch fifth in descent from the primaries, and their indices of integration have been shown to be 12, 10, 14, 15.

We are now within sight of the goal of our wearisome pilgrimage. We may form eight equations leading to 18.45, the skew-invariant, as follows:

(1) $\mathfrak{R}4.10\ \mathfrak{R}7.17\ \mathfrak{R}3.6\ \mathfrak{R}5.12 = (bS,\ QR)(Q^2S,\ b^2RS,\ bQR^2).T.R.T$
(2) $\mathfrak{R}^24.10\ \mathfrak{R}3.6\ \mathfrak{R}8.19 = (bS,\ QR)^2.T.(QST,\ bR^2T)$
(3) $\mathfrak{R}^24.10\ \mathfrak{R}6.13\ \mathfrak{R}5.12 = (bS,\ QR)^2.bRT.RT$
(4) $\mathfrak{R}8.20\ \mathfrak{R}3.6\ \mathfrak{R}8.19 = (b^2S^2,\ b^2R^3,\ bQRS,\ Q^2R^2)\,T(QST.bR^2T)$
(5) $\mathfrak{R}8.20\ \mathfrak{R}6.13\ \mathfrak{R}5.12 = (b^2S^2,\ b^2R^3,\ bQRS,\ Q^2R^2)\,bRT.RT$
(6) $\mathfrak{R}11.27\ \mathfrak{R}3.6\ \mathfrak{R}5.12 = (b^3R^4,\ bQ^2R^3,\ bQ^2S^2,\ b^2QSR^2)\,T.RT$
(7) $\mathfrak{R}6.13\ \mathfrak{R}13.32 = bRT(b^2R^4T,\ b^2RS^2T,\ bQR^2ST,\ Q^2R^3T,\ Q^2S^2T)$
(8) $\mathfrak{R}9.21\ \mathfrak{R}^25.12 = (b^3S^2,\ b^3R^3,\ b^2QRS,\ Q^3S)\,R^2T^2.$

The arguments on the right-hand side of these equations will be seen to be the seven following: $T^2b^3R^5$, $T^2b^3R^2S^2$, $T^2b^2QR^3S$, $T^2b^2QS^3$, $T^2bQ^2R^4$, $T^2bQ^2RS^2$, $T^2Q^3R^2S$. Hence a linear function of the anti-residues to the eight products to the left can be made zero, and the sums of each set of duads being 19.45, there emerges the deduct 18.45 corresponding to the skew-invariant 18;0.

That this is not a null may be shown in the usual manner as follows: The indices of integration of the several anti-residues are $2+6+3+5$, $2+2+3+9$, $2+2+5+5$, $10+3+9$, $10+5+5$, $12+3+5$, $5+15$, $10+5$, *i.e.* 16, 16, 14, 22, 20, 20, 20, 15. The 5th, 6th and 7th indices constitute the only triad of equal indices, but the 5th, 6th and 7th anti-residues cannot be in syzygy, inasmuch as the two first of them have the factor 5.12 in common. Hence the value of 18.45 found as above will not be null.

Its index of integration will be one unit more than the dominant of the above numbers, *i.* e. it is 23, and its residue will be of the form $(b^3R^6T,\ b^3R^3S^2T,\ b^3S^4T,\ b^2QR^4ST,\ b^2QRS^3T,\ bQ^2R^5T,\ bQ^2R^2S^2T,\ Q^3R^3ST,\ Q^3S^3T)$.

We ought now to be able to show that there exists no other deduct of which the residue is not a rational integral function of the 22 residues which have been determined in order to prove that the system of groundforms obtained is complete. But this inquiry is one of considerable difficulty and must be reserved for future consideration.

I will now bring together the several steps of the deduction (several of which, especially in the earlier stages, would admit of abridgment), separating the successive strata from one another and substituting the more familiar designation of deg-orders for the equivalent deg-weights. The single numbers on the left-hand side are the indices of integration to the corresponding deducts.

TABLE OF DEDUCTION FOR THE QUINTIC.

(3) $a^3(3;3)+a^2(?)=(2;6)^3,\ (3;9)^2$

(1) $a(4;6)=(2;6)(3;5),\ (2;2)(3;9)$

(2) $a^2(4;4)+a(?)=(3;9)(3;5);\ (2.6)^2(2.2)$

(2) $a^2(4;0)+a(?)=(3;5)^2,\ (2;2)^2(2;6)$

(4) $a(5;3)=(4;6)(2.2),\ (3;5)(3;3)$

(5) $a^2(5;1)+a(?)=(3;3)(2;2)(2;6),\ (4.6)(3.5)$

(4) $a(5;7)=(4;6)(2;6),\ (3;9)(3;3)$

(5) $(a)(6;4)=(4;0)(3;9),\ (2;6)(5;3),\ (2;2)(5;7)$

(8) $a(7;1)=(3;5)(5;1),\ (3;3)(5;3),\ (4;6)(4;0)$

(6) $a^2(6;2)+a(?)=(5;7)(3;5),\ (3;9)(5;3);\ (2.6)^2(4.0)$

(8) $a(7;5)=(2;6)(6;4),\ (3;3)(5;7)$

(9) $a(8;2)=(3;3)(6;4),\ (5;1)(4;6)$

(10) $a^2(8;0)+a(?)=(4;6)(6;4),\ (5;1)(3;3)(2;6)$

(12) $a^2(11;1)+a(?)=(2;6)(5;1)(6;4),\ (3;3)^2(7;5)$

(10) $a(9;3)=(3;3)(7;5),\ (5;7)(5;1)$

(14) $a(12;0)=(6;4)(7;1), (5;3)(8;2), (3;3)(5;1)^2(5;1)(4.4)(4.0), (2;2)(3;3)(8;0)$

(15) $a(13;1)=(6;4)(8;2), (5;1)^2(6;4), (5;1)^2(2;2)^2, (3;3)^2(4;0)^2, (2;2)(3;3)(5;1)(4;0)$

(23) $18;0=(4;0)(7;1)(3;3)(5;1),\ (4;0)(3;3)(8;2),\ (4;0)^2(6;4)(5;1),$
$(8;0)(6;4)(5;1),\ (8;0)(3;3)(8;2),\ (6;4)(13;1),\ (9;3),\ (5;1)^2$

In addition to the deducts which appear in the above table, the groundform 1.5 and the four protomorphs 2;2 2;6 3;5 3;9 have to be taken into account. Thus the twenty-three groundforms to the quintic will be seen to be distributed among seven batches or categories containing respectively 1, 4, 3, 4, 3, 3, 4, 1 individuals.

It was my intention to have simplified some of the steps of the deduction, and to have supplied the omissions, to show in one or two cases that the deducts as obtained are actual and not null forms,* but unfortunately the proof-sheets

* When the deduct is a zero instead of a possible new groundform, it indicates a syzygy between anterior groundforms.

have been kept back, owing to the necessities of the printing-office, for some weeks, and in the meanwhile my attention has been drawn off to other parts of the subject, and I am unable to give sufficient time to call back to mind the intended ameliorations or rectifications of the text.

§ 4. Perpetuants.

On Absolutely Irreducible Binary Subinvariants.

Any rational integral value of $(\lambda a\delta_b + \mu b\delta_c + \nu c\delta_d \ldots)^{-1}0$ is a binary subinvariant. If none of the numerical coefficients $\lambda, \mu, \nu \ldots$ are zero, the subinvariant is simple. If in the series of coefficients $\lambda, \mu, \nu, \pi, \rho \ldots$, any number i of breaks occur in consequence of i non-contiguous terms $\nu, \rho \ldots$ vanishing, it becomes a multiple subinvariant corresponding to a semi-invariant of i distinct binary quantics. If, however, the subinvariant is to appertain to a system of quantics, all of unlimited order, it would be necessary for the breaks in the series to be each of them at an infinite distance from the initial term and from one another.

In what follows I shall confine my attention to simple binary subinvariants, and investigate the types, *i. e.* the deg-weights (order ceases to be predicable) of those of them which are absolutely indecomposable, *i. e.* incapable of being expressed as rational integral functions of others of lower types of any extent whatever.

It may be convenient to give a name to absolutely indecomposable subinvariants, and I propose, until an apter word presents itself, to call them perpetuants.* The present section then will be occupied with the successive determination of the types of all possible simple binary perpetuants up to a certain limit of degree.

We know, by Cayley's rule, that the number of linearly independent binariants of degree j and weight w is the difference between the number of partitions of w into j parts, and the number of partitions of $w-1$ into such parts, and therefore by Euler's law of reciprocity is the difference between the number of partitions of w into parts none exceeding j, and the number of

* Perhaps *Revenants* would be more expressive to signify the forms (or ghosts of forms, if one pleases to say so) which never die out, but *continually return* as the leading coefficients of irreducible covariants. Such I need not say is not the case with conditionally irreducible integrals of the above partial differential equation (as for instance the discriminants to the cubic), which sooner or later die out and are seen no more as sources of irreducible covariants to quantics of a superior order.

partitions of $w-1$ into such parts; it is therefore the coefficient of x^w in $\left\{\frac{1}{(1-x)(1-x^2)\ldots(1-x^j)}-\frac{x}{(1-x)(1-x^2)\ldots(1-x^j)}\right\}$ or the coefficient of x^w in $\frac{1}{(1-x^2)(1-x^3)\ldots(1-x^j)}$, which I shall call the generating function for the degree j of the linearly independent subinvariants.

Thus for the degree 1 the generating function is simply 1, and there will be one subinvariant (a) of the degree 1 and weight zero.

For the degree 2 the generating function is $\frac{1}{1-x^2}$, which expanded gives the series $1+x^2+x^4+\ldots$; there is consequently one semi-invariant of the degree 2 for every even weight $0, 2, 4, 6\ldots$; but the first of these will be merely the square of the one of degree 0 and weight 1; hence the generating function for the perpetuants of degree 2 is $\frac{1}{1-x^2}-1$ or $\frac{x^2}{1-x^2}$ giving rise to the deg-weights $2.2 \quad 2.4 \quad 2.6\ldots$ corresponding to the well-known series of quadinvariants or quadri-semi-invariants $ac-b^2$, $ac-4bd+3c^2$, ... Again, for $j=3$ the generating function to the linearly independent binariants, or for brevity sake say the *total* generating function is $\frac{1}{(1-x^2)(1-x^3)}$.

To find the irreducible forms, or say the *limited* generating function, we must take away the cube of the one of degree 1 and weight zero, and the product of this one and each indecomposable one of the degree 2, and consequently the limited generating function will be

$$\frac{1}{(1-x^2)(1-x^3)}-\left(\frac{x^2}{1-x^2}+1\right) \textit{ i. e. } \frac{x^3}{(1-x^2)(1-x^3)};$$

thus we obtain perpetuants of the deg-weights $3.i$, where the least value of i is 3 and the number of such for $i=3,4,5,6,7,8;\ 9,10,11,12,13,14;\ 15,16,17,\ldots$ will be $1,0,1,1,1,1;\ 2\ 1\ 2\ 2\ 2\ 2;\ 3,2,3,\ldots$

Again, for $j=4$, the total generating function is $\frac{1}{(1-x^2)(1-x^3)(1-x^4)}$.

To determine the subtrahend consider the total partitions of 4 (the number itself not counting as a partition). These are $1^4, 1^2.2, 1.3, 2^2$. The three former will give rise to the partial subtrahends $1, \frac{x^2}{1-x^2}, \frac{x^3}{(1-x^2)(1-x^3)}$, but for 2^2 *i. e.* 2.2 the case is different.

Taking the development of $\frac{x^2}{1-x^2}$ *i. e.* $x^2+x^4+x^6+x^8+\ldots$ the function corresponding to 2.2 to be subtracted is not $\left(\frac{x^2}{1-x^2}\right)^2$, but the sum of the *homogeneous products* of the second order of the infinite succession $x^2, x^4, x^6, x^8, \ldots$ or calling s_1 the sum of the terms and s_2 the sum of their squares, is $\frac{s_1^2+s_2}{2}$,

i. e. is $\frac{1}{2}\left\{\left(\frac{x^2}{1-x^2}\right)^2+\frac{x^4}{1-x^4}\right\}$ or $\frac{x^4(1+x^2)+x^4(1-x^2)}{(1-x^2)(1-x^4)}$, *i. e.* $\frac{x^4}{(1-x^2)(1-x^4)}$.

Hence the *limited* generating function for the degree 4 is

$$\frac{1}{(1-x^2)(1-x^3)(1-x^4)}-\left(\frac{x^3}{(1-x^2)(1-x^3)}+\frac{x^2}{1-x^2}+1\right)-\frac{x^4}{(1-x^2)(1-x^4)}$$

which is

$$\frac{1}{(1-x^2)(1-x^3)(1-x^4)}\left(1-(1-x^4)-x^4(1-x^3)\right),\ i.\,e.\ \frac{x^7}{(1-x^2)(1-x^3)(1-x^4)}.$$

Let us pause a moment in the deduction to draw an inference from this result. The lowest power of x in the development of the limited generating function for the degree 4 being x^7, we see that an absolutely indecomposable binariant of the 4th degree cannot be of lower weight than 7. Consider any semi-invariant of degree 4 to a quantic of order i. Its weight must be less than $2i$. Hence if it is indecomposable, 7 must be less than $2i$ or i, or i must be at least 4. Thus we see that there can be no absolutely indecomposable binariant of the 4th degree appertaining to a cubic. This shows *à priori* that the discriminant to the cubic, regarded as a subinvariant, is decomposable, as we know is the case.*

So in general if we know that no perpetuant of the degree j is of lower weight than k, we may be assured that no invariant or semi-invariant to a quantic of the degree j can be absolutely indecomposable if the order of the quantic is less than $\frac{2k}{j}$.

Agreeing to call the weight of any subinvariant divided by its degree its relative weight, we may put this result into words, by saying no quantic can possess an absolutely indecomposable invariant or semi-invariant of a given degree unless its order is at least twice as great as the minimum relative weight of a perpetuant of that degree. We may see further that the quartic can have

* It may easily be collected from the course of the ensuing investigation that *every* binary discriminant is decomposable into subinvariants of lower degrees than its own.

no indecomposable invariant or semi-invariant of the degree 4, for its weight would be 8, but x^8 does not appear in the development of $\dfrac{x^7}{(1-x^2)(1-x^3)(1-x^4)}$.

Pass we on now to the case of the 5th degree.

The indefinite partitions of 5 (leaving 5 itself out of the number) are 4.1, 3.2, 3.1.1, 2.2.1 2.1^3 1^5 which obviously give rise to the subtrahends

$$\frac{x^7}{(1-x^2)(1-x^3)(1-x^4)},\ \frac{x^3}{(1-x^2)(1-x^3)}\cdot\frac{x^2}{1-x^2},\ \frac{x^3}{(1-x^2)(1-x^3)},\ \frac{x^4}{(1-x^2)(1-x^4)},\ \frac{x^2}{1-x^2},\ 1.$$

But from the mode in which the deduction has been carried on, it will be obvious on reflexion that the sum of all these except the second which corresponds to a partition not ending with a unit will be equal to the total generating function for the case of the degree 4. So that the total subtrahend is

$$\frac{1}{(1-x^2)(1-x^3)(1-x^4)}+\frac{x^5}{(1-x^2)(1-x^2)(1-x^3)}.$$

Hence the limited generating function for the degree 5 is

$$\frac{x^5}{(1-x^2)(1-x^3)(1-x^4)(1-x^5)}-\frac{x^5}{(1-x^2)(1-x^2)(1-x^3)},$$

i. e. is $$\frac{x^5(1-(1+x^2)(1-x^5))}{(1-x^2)(1-x^3)(1-x^4)(1-x^5)},\ \text{which is}\ \frac{-x^7+x^{10}+x^{12}}{(2)(3)(4)(5)},$$

where for brevity I use in general (q) to denote $1-x^q$.

Here, for the first time, a new feature presents itself, viz. the presence of a negative coefficient in the numerator, and consequently of a series of such in the development in an infinite series of the generating function.

Each negative term $-kx^t$ in the development will obviously indicate the existence of k general syzygies of the degree 5 and weight t, or as we might call them, *privative* groundforms. The number of such terms will be finite, and they will be most readily obtained by writing the $l.g.f.$ (limited generating function) under the form

$$\frac{-x^7(1-x^3)(1-x^5)+x^{15}}{(2)(3)(4)(5)}\ i.\ e.\ \frac{-x^7}{(2)(4)}+\frac{x^{15}}{(2)(3)(4)(5)};$$

To find them it will be observed that the number of ways of composing 0, 2, 4, 6, 8, 10, 12, 14, 16 with the elements 2 and 4 are respectively 1, 1, 2, 2, 3, 3, 4, 4, 5, and that 1, 1, 2, 3, 5 are the number of ways of composing 0, 2, 4, 6, 8, with the elements 2, 3, 4, 5. Hence there will exist the negative

terms $-x^7, -x^9, -2x^{11}, -2x^{13}, -2x^{15}, -2x^{17}, -2x^{19}, -x^{21}$,* the sum of which is

$$-\frac{x^7+x^{11}-x^{21}-x^{23}}{1-x^2}.$$

Adding this with its sign changed to $\frac{-x^7+x^{10}+x^{12}}{(2)(3)(4)(5)}$ there results

$$\frac{x^{18}+x^{20}-x^{21}-x^{23}+x^{24}+x^{25}+2x^{26}-x^{29}-2x^{30}-x^{31}-x^{32}+x^{33}+x^{35}}{(2)(3)(4)(5)},$$

which may be thrown under the form

$$x^{18}\left\{\frac{(3)+x^2(2)(3)+x^4+x^6(8)+x^8(3)(4)+x^8(4)(5)}{(2)(3)(4)(5)}\right\}.$$

It is therefore omni-positive in its development, which shows that no negative terms have been omitted, but that the 13 syzygies of odd weights ranging from 7 to 21 typically represented by $-\frac{x^7+x^{11}-x^{21}-x^{23}}{1-x^2}$ (say $-R_5$) constitute their entire aggregate. We see also that the minimum weight of a perpetuant of the 5th degree is 18, so that the double of the minimum relative weight is $\frac{36}{5}$, and accordingly there can exist no absolutely indecomposable binary subinvariants of the 5th degree, until we come to Quantics of the 8th order or upwards.

Proceeding to the degree 6, the total subtrahend from the *t.g.f.* (total generating function) for that degree would be *ut suprâ* the *t.g.f.* for the degree one below (here 5), less expressions depending on the partitions of 6 not concluding with a unit, were it not for the presence of the negative terms represented by $-R_5$; the quantity to be subtracted corresponding to the partition 5.1, being now not the *l.g.f.* for degree 5, $\frac{-x^7+x^{10}+{}^{12}}{(2)(3)(4)(5)}$, but this quantity rendered omni-positive in its development by the addition of R_5.

Hence the total subtrahend will be $\frac{1}{(2)(3)(4)(5)}+R_5+$ the quantities depending on the partitions 2.4 2.2.2 3.3.

To 2.4 will correspond the subtrahend $\frac{x^2}{(2)}\cdot\frac{x^7}{(2)(3)(4)}$.

To 3.3 will correspond $\frac{\varphi x^2+(\varphi x)^2}{2}$ where $\phi x=\frac{x^3}{(1-x^2)(1-x^3)}$, and to 2.2.2 by Crocchi's theorem,† will correspond the representative of the homogeneous

*The numbers 1 1 2 2 2 .2 1 are got by subtracting from the figures 1 1 2 2 3 3 4 4 5 the figures 1 1 2 3 5

† See for an instantaneous proof of this theorem, the Johns Hopkins University Circular for Nov. 1882.

products of the 3d order of the terms in $\psi x = \frac{x^2}{1-x^2}$, *i. e.* $\frac{(\psi x)^3 + 3\psi x \psi x^2 + 2\varphi x^3}{2.3}$.
There might for a moment be felt a hesitation in applying the formula for homogeneous products to ϕx, in consequence of the coefficients in its development being no longer exclusively unities; but the force of this objection vanishes as soon as it is borne in mind that we may replace any term kx^t in the development of ϕx by k, separate terms x^t, each of which corresponds to a distinct subinvariant.

Thus then to 3.3 will correspond the partial subtrahend

$$\frac{x^6}{2}\left\{\frac{1}{(1-x^2)^2(1-x^3)^2} + \frac{1}{(1-x^4)(1-x^6)}\right\} \text{ or } x^6 \frac{(1+x^3)(1+x^3)+(1-x^3)(1-x^8)}{2(2)(3)(4)(6)}$$

$$i.\ e.\ \frac{x^6+x^{11}}{(2)(3)(4)(6)},$$

and to 2.2.2 will correspond

$$x^6\frac{(1+x^2)(1+x^2+x^4)+3(1-x^6)+2(1-x^2)(1-x^4)}{6(2)(4)(6)}, \text{ or } \frac{x^6}{(2)(4)(6)}.$$

It may be remarked, in passing, that for any degree $2i$ the subtrahend corresponding to the partition consisting of i parts (each of the value 2), is $\frac{x^{2i}}{(2)(4)\ldots(2i)}$, as may be shown, *à priori*, thus: using y in place of x^2 we have to find the sum of all the quantities ky^t where k is the number of ways of generating y^t as a product of i of the powers 1, y, y^2, y^3 . . . , *i. e.* k is the number of ways of composing t with i or less than i of the indefinite series of natural numbers, which by Euler's theorem, already cited, is the same as that of compounding t out of any number of parts none exceeding i. Hence the denominator of the subtrahend required will be

$$\frac{1}{(1-y)(1-y^2)\ldots(1-y^i)}\ i.\ e.\ \frac{1}{(2)(4)\ldots(2i)}.$$

The numerator is obviously x^{2i}, and the complete value $\frac{x^{2i}}{(2)(4)\ldots(2i)}$ as was to be found.

I may add, that this theorem (which is one concerning homogeneous product-sums expressed as functions of power-sums of the same elements), by an easy deduction from Crocchi's theorem, serves to show if the i^{th} power-sum of a set of elements is $\frac{1}{1-c^i}$ (I substitute c for y) then the i^{th} elementary

symmetric function of the elements is

$$\frac{c^{\frac{i^2-i}{2}}}{(1-c)(1-c^2)\ldots(1-c^i)}$$

and reversing the terms of this proposition we may say, that if

$$z^q-\frac{1}{1-c}z^{q-1}+\frac{c}{(1-c)(1-c^2)}z^{q-2}\ldots\pm\frac{c^{\frac{n^2-n}{2}}}{(1-c)(1-c^2)\ldots(1-c^n)}z^{q-12}+\ldots=0,$$

then the sum of the i^{th} powers of z (q being not less than i) is $\frac{1}{1-c^i}$; to which may be added that the sum of the i^{th} homogeneous products of z is

$$\frac{1}{(1-c)(1-c^2)\ldots(1-c^i)},$$

as *ex gr.* if $i=2$ the first of these sums, viz.

$$\frac{1}{(1-c)^2}-2\frac{c}{(1-c)(1-c^2)}=\frac{1}{1-c^2}$$

and the other, viz.

$$\frac{1}{(1-c)^2}-\frac{c}{(1-c)(1-c^2)}=\frac{1}{(1-c)(1-c^2)}.$$

But this is a mere digression, a wild flower gathered on the wayside. Returning to the determination of the *l.g.f.** for the degree 6, we see that it will be

$$\frac{1}{(2)(3)(4)(5)(6)}-\frac{1}{(2)(3)(4)(5)}-\frac{x^9}{(2)(2)(3)(4)}-\frac{x^6+x^{11}}{(2)(3)(4)(6)}-\frac{x^6}{(2)(4)(6)}-R_5,$$

or $\frac{N}{(2)(3)(4)(5)(6)}-R_5$, where

$$\begin{aligned}N&=x^6-(1+x^2+x^4)(1-x^5)x^9-(x^6-x^{16})-x^6(1-x^3)(1-x^5)\\&=x^6+x^{14}+x^{16}+x^{18}+x^{16}+x^9+x^{11}\\&\quad-x^9-x^{11}-x^{13}-x^6-x^6-x^{14}\\&=-x^6-x^{13}+2x^{16}+x^{18}.\end{aligned}$$

Thus the *l.g.f.* for the degree 6 is

$$-R_5+\frac{-x^6-x^{13}+2x^{16}+x^{18}}{(2)(3)(4)(5)(6)}.$$

$-R_5$ represents the fourteen compound syzygants of the degree 6; the fraction to which $-R_5$ is annexed, when developed, will give rise to only a *limited* number

*I repeat that *t.g.f.* stands for total generating function, and *l.g.f.* for limited generating function.

of terms with negative coefficients corresponding to the ground-syzygies; the remainder of the terms, infinite in number, will represent the infinite succession of groundforms. It may be well here to notice, as a universal fact, that in the development of the fraction $\frac{R(x)}{(2)(3)\dots(n)}$ (where $R(x)$ is rational integral function of x), the number of negative terms or the number of positive terms will be finite according as $R(1)$ is positive or negative, and, as in the above fraction, $R(1)=1$, it follows that there are only a finite number of negative terms, and consequently only a limited number of ground-syzygies, an important conclusion which will easily be seen to apply not only to the use of the degree 5 (in which syzygies first make their appearance) and 6, as here shown, but for all higher degrees, it being a universal law that the irreducible syzygies for subinvariants of any given degree, and therefore of any degree not exceeding a given limit, are finite in number.

The law that the development of $\frac{R(x)}{(1-x^2)(1-x^3)\dots(1-x^n)}$, commencing from a certain point is omni-positive or omni-negative, according as $\phi 1$ is positive or negative when n exceeds 2, admits of easy proof. Of course the law could not be true when $n=2$, as *ex gr.* for $\frac{1-2x}{1-x^2}$ which remains *neutral*, *i. e.* neither omni-positive nor omni-negative (which latter, if the law did apply, it ought eventually to become) throughout its entire extent

Beginning with $\frac{Rx}{(1-x^2)(1-x^3)}$ the coefficient of x^i [where $i=6t+\tau(\tau<6)$] will be not less than t, and not greater than $t+1$ in the development of

$$\frac{1}{(1-x^2)(1-x^3)}.$$

Hence in the development $\frac{-K+(K+\varepsilon)x^\delta}{(1-x^2)(1-x^3)}$ the coefficient of x^i will be not less than $-K(t+1)+(K+\varepsilon)\left(t-\frac{\delta}{6}-1\right)$, and consequently for a sufficiently large value of i must be positive. *A fortiori* the same will be true for $\frac{R(x)}{(1-x^2)(1-x^3)}$ when $K+\varepsilon$ is the sum of the positive coefficients in Rx of powers of x none of whose indices are higher than δ, and K the sum of the negative coefficients of any powers of x; this proves the law for $\frac{R(x)}{(1-x^2)(1-x^3)}$ when $R(1)$

is supposed to be positive, and moreover the series will be omni-positive after a certain point in the strict sense of the following coefficients being neither negative nor zero.

Hence the law will be true for $\frac{Rx}{(1-x^2)(1-x^3)(1-x^4)}$, for we may divide $\frac{Rx}{(1-x^2)(1-x^3)}$, when expanded, into four series, whose indices $\equiv 0, 1, 2, 3$ respectively to modulus 4, and the negative terms in each of these being finite in number, it is clear that the effect of dividing any one of them by $1-x^4$ will be to give rise to a series omni-positive after a certain point, because each coefficient in the quotient of any one of the series divided by $\frac{1}{1-x^4}$ will at worst contain only the sum of a finite number of given negative coefficients, and a number of terms all greater than zero, whose sum, when that number is taken great enough, must exceed the arithmetical value of the former sum. Hence $\frac{R(x)}{(1-x^2)(1-x^3)(1-x^4)}$ will be the sum of four series, each omni-positive from a certain point, and will therefore be omni-positive from the most advanced of those points. In like manner $\frac{Rx}{(1-x^2)(1-x^3)(1-x^4)(1-x^5)}$ may be shown to be the sum of five series, each with an infinite omni-positive branch, and consequently will be itself of the same character, and so in general. Of course the same reasoning would show the truth of the law when $R(x^i)$ is negative, and that it may be extended to any denominator of the form $(1-x^i)(1-x^j)(1-x^k)\ldots$ provided any two of the indices $i, j, k \ldots$ are prime to one another. And of course a similar conclusion obtains (*mutatis mutandis*) when $R(x)$ is negative. The law might be proved more scientifically and more briefly as a consequence of the general algebraical representation of the denumerant of any equation in integers $[l_1x_1 + l_2x_2 + \ldots + l_ix_i = n]$ as a sum of a non-periodical and of periodical parts, whereof the former is always of a higher dimension in n than any of the latter, except when all the l quantities have a common factor. See the annexed Excursus.

I now proceed to find the lowest power of x in the fraction $\frac{-x^6-x^{13}+2x^{16}+x^{18}}{(2)(3)(4)(5)(6)}$, say F, in which the coefficient is positive, in order to ascertain the minimum weight of an absolutely irreducible subinvariant of the 6th degree.

I think the easiest practical mode of proceeding to effect this is to use the tables in my possession (having been previously calculated for me by Mr. Franklin for another purpose) which gives the coefficients of the powers of x in

$\frac{1}{(2)(3)(4)(5)(6)}$; those coefficients used as they stand, then advanced seven steps, then five steps further, then taken back two steps, and at the same time doubled, will give four series of numbers, the sum of the 1st and 2d of which subtracted from the sum of the 3d and 4th will give the successive coefficients of from x^6 upwards in the development of F.

The four series are as underwritten:

(0)	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)	(15)	(16)	(17)	(18)	(19)	(20)	(21)	(22)	(23)
1,	0,	1,	1,	2,	2,	4,	3,	6,	6;	9,	9,	14,	13,	19,	20,	26,	27,	36,	36;	47,	49,	60,	63
							1,	0,	1,	1,	2,	2,	4,	3,	6,	6;	9,	9,	14,	13,	19,	20,	26
												1,	0,	1,	1,	2,	2,	4,	3,	6,	6;	9,	9
										2,	0,	2,	2,	4,	4,	8,	6,	12,	12,	18,	18,	28,	26

(24)	(25)	(26)	(27)	(28)	(29)	(30)	(31)	(32)	(33)	(34)	(35)	(36)	(37)	(38)	(39)
78,	80,	97,	102,	120,	126;	149,	154,	180,	189,	216,	227,	260,	270,	307,	322;
27,	36,	36;	47,	49,	60,	63,	78,	80,	97,	102,	120,	126;	149,	154,	180
14,	13,	19,	20,	26,	27,	36,	36;	47,	49,	60,	63,	78,	80,	97,	102
38,	40,	52,	54,	72,	72;	94,	98,	120,	126,	156,	160,	194,	204,	240,	252;

(40)	(41)	(42)	(43)	(44)	(45)	(46)	(47)	(48)	(49)	(50)	(51)	(52)	(53)	(54)
361,	378,	424,	441,	492,	515,	568,	594,	656,	682;	750,	783,	854,	891,	972,
189,	216,	227,	260,	270,	307,	322;	361,	378,	424,	441,	492,	515,	568,	594
120,	126;	149,	154,	180,	189,	216,	227,	260,	270,	307,	322;	361,	378,	424
298,	308,	360,	378,	432,	454,	520,	540,	614,	644;	722,	756,	848,	882,	984

(55)	(56)	(57)	(58)	(59)	(60)	(61)	(62)	(63)	(64)	(65)	(66)	(67)
1010,	1098,	1144,	1236,	1287;	1391,	1443,	1555,	1617,	1734,	1802,	1932,	2002,
656,	682;	750,	783,	854,	891,	972,	1010,	1098,	1144,	1236,	1287;	1391,
441,	492,	515,	568,	594,	656,	682;	750,	783,	854,	891,	972,	1010,
1030,	1136,	1188,	1312,	1364;	1500,	1566,	1708,	1782,	1944,	2020,	2196,	2288,

(68)	(69)	(70)	(71)	(72)	(73)	(74)	(75)	(76)	(77)	(78)	(79)
2142,	2223;	2369,	2457,	2618,	2709,	2881,	2985,	3164,	3276,	3472,	3588;
1443,	1555,	1617,	1734;	1802,	1932,	2002,	2142,	2223;	2369,	2457,	2618
1098,	1144,	1236,	1287;	1391,	1443,	1555,	1617,	1734,	1802,	1932,	2002
2472,	2574;	2782,	2886,	3100,	3234,	3468,	3604,	3864,	4004,	4284,	4446;

The first group of four numbers in which the 3d and 4th terms combined exceed the 1st and 2d will easily be seen to be, $\left.\begin{array}{r}2369\\1617\\1236\\2782\end{array}\right\}$ which is 70 places from the first term, and for which the difference is 4018 less 3986 or 32. Starting from this point the series for F will be seen to be $32x^{76} - 18x^{77} + 81x^{78} + 36x^{79} + 138x^{80} + 94x^{81} + 211x^{82} + 161x^{83} + 287x^{84} + 242x^{85} + \ldots$; so that there can be no practical doubt of the series being omni-positive from and after the 78th power of x.*

The relative weight of any one of the irreducible subinvariants corresponding to $32x^{76}$ is $\frac{76}{6}$, the double of which is $25\frac{1}{3}$. Hence there can be no irreducible semi-invariant of the 6th degree to a quantic below the 26th order, and, on account of the coefficient of x^{77} being negative, we see that a quantic of the 26th order can have no groundforms of the 6th degree in the coefficients except such as are invariants or quart-invariants.

As regards the syzygies irrespective of the compound ones represented by $-R_5$, we see that there will be primitive ones of all weights from 6 to 77 inclusive, with the exception of the weights 7 and 76, but that there will be no syzygies, whether reducible or irreducible, of the same weights as the irreducible subinvariants. Let us now pass on to the case of the 7th degree.

The partitions of seven itself and those ending in unity excluded are 5.2 4.3 2.2.3.

Hence calling R_6 the sum of the negative terms in $\frac{-x^6 - x^{13} + 2x^{16} + x^{18}}{(2)(3)(4)(5)(6)}$, the $l.g.f.$ for 7 will be

$$\frac{x^7}{(2)(3)(4)(5)(6)(7)} - \frac{-x^7 + x^{10} + x^{12}}{(2)(3)(4)(5)}\frac{x^2}{(2)} - \frac{x^7}{(2)(3)(4)}\frac{x^3}{(2)(3)} - \frac{x^4}{(2)(4)}\frac{x^3}{(2)(3)} - R_5\frac{x^2}{1-x^2} - R^6.$$

If we call this $$\frac{x^7 + N}{(2)(3)(4)(5)(6)(7)} - R_5\frac{x^2}{1-x^2} - R_6,$$

$N = x^7(1-x^7)\{(1+x^2+x^4)(-x^9+x^{12}+x^{14}) + x^{10}(1+x+x^2+x^3+x^4)(1-x+x^2) + x^7(1-x^5)(1+x^2+x^4)\} = -(1-x^7)P$, where $P = \Sigma x^t - \Sigma x^\tau$, t having the values 12 14 16, 14 16 18; 10 11 12 13 14, 12 13 14 15 16; 7 9 11, and τ having the values 9 11 13; 11 12 13 14 15; 12 14 16.

* This conclusion will be strictly proved in the sequel with the aid of my general partition formulæ, in Section V.

Hence $$P = x^7 + x^{10} + x^{12} + 2x^{14} + 2x^{16} + x^{18},$$
and $$x^7 + N = -x^{10} - x^{12} - x^{14} - 2x^{16} - x^{18} + x^{17} + x^{19} + 2x^{21} + 2x^{23} + x^{25}.$$

The first term in the development of $\frac{x^7 + N}{(2)(3)\ldots(7)}$ is $-x^{12}$, indicating that the first irreducible syzygy is of the weight 12; it is not until a very high power of x is reached that a positive coefficient corresponding to a perpetuant makes its appearance.

The tables set out in a subsequent section exhibit *inter alia* the coefficients in the developments of $\frac{1}{(2)(3)\ldots(7)}$ and $\frac{1}{(2)(3)\ldots(6)}$, say F_7 and F_6 as far as the 174th power of x. Using instead of $\frac{x^7 + N}{(2)\ldots(7)}$ the equivalent value $x^7F_7 - PF_6$, if the coefficient of x^{q+7} in this is positive, the coefficient of x^q in F_7 must be greater than that of x^q in $(1 + x^3 + x^5 + 2x^7 + 2x^9 + x^{11})F_6$, and *à fortiori* greater than that of x^q in $8x^{11}F_6$, *i. e.* greater than 8 times that of x^{q-11} in F_6. But a glance at the tables* for the developments of F_7, F_6 will show that this is never the case within the limits of q, furnished by the tables, *i. e.* for any value of q not exceeding 174. It is certain, therefore, that the value of the lowest index of x^r, for which in $\frac{N}{(2)\ldots(7)}$ the coefficient is positive, must considerably exceed 181, as indeed one might have anticipated from the series of similar exponents 2, 3, 7, 18, 76 corresponding to the cases previously considered, the ratio of increase in these numbers going on continually increasing.† To ascertain the value of the exponent in question there is left no resource but to endeavor to elicit it (as I shall presently proceed to do) from the general algebraical value of the coefficient. But before doing so it will be well to notice a very important inference that may be drawn from the form of the generating function, viz.

$$\frac{N}{(2)(3)(4)(5)(6)(7)} - \frac{R_5}{(2)} - R_6$$

$\frac{R_5}{(2)}$ or $(1 + x^2 + x^4 + \ldots)(x^7 + x^9 + 2x^{11} + 2x^{13} + 2x^{15} + 2x^{17} + 2x^{19} + x^{21})$ will represent the deg-weights of the compound syzygies corresponding to the multiplication of the syzygies of the deg-weights 5.7 5.9 5.11 5.13 5.15 5.17 5.19 5.21 5.23 by the groundforms of every even weight.

---

* *Vide* the numerical tables at end of Section V of this Memoir.

† Subsequent calculations, however, have revealed to me that this ratio does not go on continually increasing.

There will thus be seen to exist compound syzygies of every odd weight (no less than 13 in fact of weight 21 or any higher odd number). If then ω' be the lowest power of x in $\frac{N}{(2)(3)(4)(5)(6)(7)}$ with a positive coefficient and with an odd exponent, there will coexist groundforms and syzygies of the same degree and weight appertaining to the quantic of an infinite order for every weight denoted by an odd number not less than ω'. From this it is easy to infer that there must exist syzygies and groundforms of the same deg-weight (and therefore of the same deg-order) for one or more quantics of an order not exceeding ω'; [and it may be added that ω' being a high number (not a number less than 23), there will be 13 syzygies of every odd weight equal to or greater than ω'.]

For suppose that Q is a quantic of order i. In determining its ground-semi-invariants of the successive degrees the same process may be applied as in calculating the perpetuants, *i. e.* the ground semi-invariants to a quantic of an unlimited order, except that in lieu of the complete development of the generating function $\frac{1}{(1-x^2)(1-x^3)\ldots(1-x^j)}$ only such powers of x must be retained as are not higher than x^i. For the number of linearly independent subinvariants of the weight w and degree j will now be the difference between the number of ways of making up w with j parts none greater than i, less the number of ways of so making up $(w-1)$ which will be the difference between the number of ways of making up w and of making up $(w-1)$ with i parts none greater than j, which, if w does not exceed i, will be the same as if i were infinite. So far then as weights not superior in value to i are concerned, the total generating function for a quantic of the order i will be the same as for a quantic of an unlimited order, and consequently up to the weight i (inclusive) the generating functions for the ground subinvariants (to be obtained, be it remembered, by combining the total generating functions in the same manner, whatever the value of i may be) will be the same for a quantic of the i^{th} as for the quantic of an unlimited order. Hence there must of necessity appertain irreducible covariants and compound syzygants of the same degree and order (viz. of the deg-order $7.5\omega'$) to a quantic of the order ω', and of course there is nothing to prevent such coexistence holding good for a quantic of an order very much lower than ω', the least value of which number say i, as far as I am able at present to see, can only be determined by putting each quantic of an order inferior to i successively upon its trial, a work of exceedingly great labor to undertake.

I use ω' to signify the lowest *odd* power of x in the development of the g. f. to perpetuants of the 7th degree affected with a positive coefficient, reserving ω to signify the lowest power (whether odd or even) so affected. Until further investigation we cannot say whether ω is equal to or less than ω', but we know that no absolutely irreducible subinvariant of the 7th degree can appertain to a quantic of an order lower than $\frac{2\omega}{7}$, a number whose exact value we shall eventually succeed in ascertaining with the aid of a partition formula obtained by the method which will form the subject of the annexed "excursus."

Inasmuch as the theory is precisely the same for fractions in general as for those which correspond to denumerants (the name I give to the number of solutions in integers of one or more linear equations), I shall show how to find the general term in the development of any rational fraction, limiting myself however, for the present, to the theory of rational functions of a single variable, which covers the case with which alone we are here concerned, of denumerants of a single linear equation, or which is the same thing, the problem of exhibiting the number of modes of composing a general number n with given smaller numbers as an algebraico-exponential function of n.

When analysis is sufficiently advanced to admit of a perfectly methodical distribution of its subject-matter, the theorem for the expansion of rational functions, about to be given, will, it seems to me, take its place immediately after Newton's binomial theorem, as the second leading theorem of Algebra; my method of partitions (as stated and applied in Tortolini's Ann. Vol. 8, 1856, and in the Quarterly Mathematical Journal, 1855, Vol. 1, p. 141, to neither of which I have at present means of access, but the latter of which is referred to by Prof. Cayley in the Ph. Tr. for 1880, footnote p. 47) virtually amounted to an enunciation of the theorem for the case of the reciprocal of a rational integral function all of whose roots are roots of unity, under such a form as almost of necessity to lead to the supposition of its remaining true (*mut. mut.*) in the general case; the actual averment of the generalization was, I believe, first made by Prof. Cayley.*

* On second thoughts, and after more deliberate reflection, it occurs to me that I may have overstated in the text above the importance of the general theorem viewed as a theorem *an sich;* and that it is only from its special application to rational fractions whose infinity-roots are all of them roots of unity, that it derives its claim to be regarded as a cardinal theorem in Algebra.

Excursus.

On Rational Fractions and Partitions.

The method of finding the general term in the development of a rational fraction of a single variable in a series of ascending powers of the same may be regarded as a corollary to the following lemma, the proof of which is an instantaneous consequence of the fact that the coefficient of $\frac{1}{x}$, or to use Cauchy's word, the residue of $\frac{1}{(1-e^x)^i}$ developed in ascending powers of x when i is any positive integer is always -1: that this is so will be seen at once from the fact that the effect of changing i into $i+1$ in the above fraction is to increase it by $\frac{e^x}{(1-e^x)^{i+1}}$, *i. e.* by the differential derivative of $\frac{1}{i(1-e^x)^i}$ whose residue is obviously zero, so that the residue of $\frac{1}{(1-e^x)^i}$ will be unaffected by continually decreasing i by a unit until it becomes unity; and obviously therefore the residue in question is always -1.

The lemma may be stated as follows:

The constant term in any proper algebraical fraction developed in ascending powers of its variable is the same as the residue with its sign changed of the sum of the fractions obtained by substituting in the given fraction in lieu of the variable its exponential multiplied in succession by each of its values (zero excepted, if there be such) which makes the given fractions infinite.

Any value of a variable which makes a function infinite may conveniently be called an infinity root, and if it is not zero, a finite-infinity root. So too, a factor whose vanishing makes a function vanish may be termed an infinity factor.

Suppose Fx is a *proper* Algebraical fraction, then we may write

$$Fx = \Sigma\Sigma \frac{c_{\lambda,\mu}}{(a_\mu - x)^\lambda} + \Sigma \frac{\gamma_\lambda}{x^\lambda},$$

where $\lambda = 1, 2, \ldots$; $\mu = 1, 2, \ldots j$ and of course any of the coefficients in either sum may be made zero, and then (using in general here and hereafter co_n to signify the coefficient of x^n in an ascending expansion of the function with which it is in regimen) we have

$$\mathrm{co}_{-1}\Sigma F(a_\nu e^x) \ [\text{where } \nu = 1, 2, \ldots . j]$$

$$= \mathrm{co}_{-1}\Sigma\Sigma\Sigma \frac{c_{\lambda,\mu}}{(a_\mu - a\,x)^\lambda} + \mathrm{co}_{-1}\Sigma\Sigma \frac{\gamma_\lambda}{a_\nu^\lambda e^{\lambda x}}$$

$$= \mathrm{co}_{-1}\Sigma\Sigma \frac{c_{\lambda,\mu}}{(a_\mu - a_\mu x)^\lambda} = -\frac{c_{\lambda,\mu}}{a_\mu^\lambda} = -\mathrm{co}_0 Fx$$

which proves the lemma.

Hence the coefficient of x^n in a rational function fx, which is the same as $\mathrm{co}_0 \frac{fx}{x^n}$ will be $-\mathrm{co}_{-1}\Sigma(r^{-n}e^{-nx}fre^x)$ or $\mathrm{co}_{-1}\Sigma(r^{-n}e^{nx}f(re^{-x}))$, [$r$ meaning each finite-infinity root of fx taken in turn], provided only that $\frac{fx}{x^n}$ is a proper algebraical function, *i. e.* provided that n is greater than the degree of $f(x)$.

As for instance, if the degree of the fraction is zero, the theorem will not give the constant, but will give every coefficient of positive powers in the ascending expansion of fx, and if it is negative, the theorem will give all but the coefficients of negative powers.

This theorem, as observed by Prof. Cayley, *Phil. Trans.* 1856, p. 139, may be obtained "from the known theorem," that if fx be resolved into simple partial fractions, the sum of those which have any power of $a - x$ in their denominator will be the residue of

$$\frac{f(a+\zeta)}{x - a - \zeta}.\ *$$

Prof. Cayley quotes as "a theorem of Cauchy's and Jacobi's, that the coefficient of $\frac{1}{z}$ in $Fz =$ coefficient of $\frac{1}{t}$ in $\psi' t F\psi t$."

This is obviously not true in general, for we might take $Fz = \frac{1}{z}$ and $\phi t = a + t$ or t^2 and the alleged equality would not exist. It is, however, true whenever ψt is of the form $at + bt^2 +$ etc., as may be proved instantaneously by supposing Fz resolved into partial fractions, and making $z = \psi t$, so that $\int dz Fz = \int dt \psi' t F\psi t$, and observing that if the expansion of $\psi' t F \psi t$ contains $\frac{k}{t}$, that of $\int dz Fz$ must contain $\frac{k}{z}$, since otherwise when this integral is expressed as a function of t, it would not contain (as it is bound to do) the term $k \log t$. The theorem so limited is sufficient for the purpose in view, since on writing in

* In his *Cours d'Algèbre*, Edition 1877, Vol. 1, pp. 497–499, M. Serret obtains the same result under the form of the value (for $\zeta =$ zero) of $\frac{1}{\pi(m-1)}\left[\left(\frac{d}{d\zeta}\right)^{m-1}\frac{\zeta^m f(a+\zeta)}{x-a-\zeta}\right]$, where m is the degree to which $(x-a)$ rises in the denominator of fx.

place of ζ, $-a(1-e^{-t})$ we see that the residue of $\frac{f(a+\zeta)}{x-a-\zeta}$ is the same as the residue of $\frac{f(ae^{-t})}{(1-ae^{-t}x)}$, and consequently the coefficient of x^n in so far as it depends on the infinity root a, will be the residue of $(a^{-n}e^{nt})f(ae^{-t})$ as has been shown above to be the case. It may, possibly, be thought somewhat surprising that those familiar with the known theorem referred to and the general principle of transformation of residues should not have recognized, previous to the divulgation of my theorem, that the two things put together were competent to give a complete solution of the much ventilated problem of simple denumeration. But, perhaps, even supposing the mental conjunction of the two facts to have taken place, there would still have been needed an act of imagination (such as Kant justly remarks is at the bottom of every advance in geometry, where in reality the proof lies in the construction*) to have led to the choice of the particular transformation employed in this case, and to have entailed the consequences that are implied in it.†

In applying this theorem to finding the value of the denumerant to the equation $ax+by+\ldots+lt=n$, which I denote by $\frac{n}{a, b, \ldots l,}$ and is the same thing as the coefficient of x^n in the expansion of the rational fraction

$$\frac{1}{(1-x^a)(1-x^b)\ldots(1-x^l)}$$

or more generally to finding the value of the denumerant

$$\frac{n}{a_1, a_2, \ldots a_\alpha, b_1, b_2, \ldots b_\beta, \ldots l_1, l_2, \ldots l_\lambda,}$$

(where each letter has a fixed value independent of its subindex), *i. e.* the coefficient of x^n in the development of $\frac{1}{(1-x^a)^\alpha(1-x^b)^\beta\ldots(1-x^l)^\lambda}$, say Fx, the first thing to be done is to determine and arrange in convenient groups the infinity roots of these functions. To effect this we have only to write down all the divisors of the set of numbers $a, b, \ldots. l$, *i. e.* all the integers which divide one or more of those numbers, say $\delta_1, \delta_2, \ldots \delta_\mu$. These divisors necessarily

*Take as an example the theorem that the sum of the three angles of a triangle is equal to right-angles: as soon as by a stroke of the imagination a line is conceived as drawn from one angle parallel to the opposite side, the truth of the proposition becomes virtually self-evident.

†Thus *ex gr.* the supposed investigator might have chosen to write sin t or log $(1+t)$ in lieu of $1-e^{-t}$ and the theorem thereby obtained would have been perfectly valid, but of little if any use, and the great bulk of transformations would certainly be of no use whatever; indeed, it is safe to say that the substitution practised, viz. that of $1-e^{\lambda t}$ [λ being taken at will] is the only one that would lead to a practical solution of the question.

include the indices $a, b, \ldots l$ and *unity*, which latter we may suppose to be δ_1.

Giving then i every value from 1 to μ, the primitive δ_i^{th} roots of unity will obviously be the infinity roots required, and we may separate the required function of n into μ distinct portions or waves, as I term them, where supposing $\nu_1, \nu_2, \ldots \nu^{\phi(\delta_i)}$ [$\phi(\delta_i)$ being the *totient* of δ_i, *i. e.* the number of integers less than δ_i and prime to it] to be the primitive δ_i^{th} roots of unity, the i^{th} period or wave, say W_i will be equal to the residue of $\Sigma r_q^{-n} e^{nt} F(r_q e^{-t})$ $[q = 1, 2, \ldots, (\phi\delta_i)]$.

Since every primitive root r_q is either equal to or is mated with its reciprocal, the above expression may be replaced by the somewhat more convenient one $\Sigma(r_q^n e^{nt}) F(r_q e^t)^{-1}$.

This again admits of a very important transformation, viz. we may write $\nu = n + \frac{1}{2}(\alpha a + \beta b + \ldots + \lambda l)$ and then

$$W_i = \text{co}_{-1}\Sigma \frac{r_q^\nu e^{\nu t}}{P(r_q^{\frac{a}{2}} e^{\frac{at}{2}} - r_q^{-\frac{a}{2}} e^{-\frac{at}{2}})}$$

(where P is used to signify that the product is to be taken of terms of like form to the one which is in regimen with it).

From this it follows that every wave W_i expressed as a function of ν, when ν is changed into $-\nu$, becomes $(-)^{\alpha+\beta+\cdots+\lambda-1} W_i$, *i. e.* retains its value absolutely or else merely changes its algebraic sign. To prove this it may be observed that whatever the index of the wave the above sum may be replaced by

$$\frac{1}{2}\,\text{co}_{-1}\Sigma \left\{ \frac{r_q^\nu e^{\nu t}}{P(r_q^{\frac{a}{2}} e^{\frac{at}{2}} - r_q^{-\frac{a}{2}} e^{-\frac{at}{2}})^\alpha} + \frac{r_q^{-\nu} e^{\nu t}}{P(r_q^{-\frac{a}{2}} e^{\frac{at}{2}} - r_q^{\frac{a}{2}} e^{-\frac{at}{2}})^\alpha} \right\}$$

This is a consequence of r being either identical with $\frac{1}{r}$ as is the case for W_1 and W_2, or else being mated with it as belonging to the same group of primitive roots of unity.

Hence r_q may be changed into r_q^{-1}, and the expression to be residuated will undergo no change.

Again if t is changed into $-t$, the residue changes its sign, and finally if r_q, t, and ν are simultaneously changed into r_q^{-1}, t^{-1}, $-\nu$ the expression to be residuated remains unaltered, except that it takes up a factor $(-)^{\Sigma\alpha}$. Consequently the effect of changing ν into $-\nu$, leaving everything else unaltered, will be to introduce the factor $(-)^{\Sigma\alpha-1}$; and this being true of every portion of

the value of $\dfrac{n,}{a\ldots, b\ldots, l\ldots,}$ it follows that when that denumerant is expressed under the form $F\nu$, where $\nu = n + \frac{1}{2}\Sigma \alpha a$, $F(-\nu) = (-)^{-1+\Sigma\alpha} F(\nu)$.

There is consequently an enormous advantage gained, as well as in the abbreviation of the calculations as in the conciseness of the result, by putting such a denumerant under the form of a function of the *augmented* argument ν instead of the original argument n; when so expressed I speak of the denumerant being in its canonical form.

In future, for greater simplicity, I shall disuse the indices $\alpha, \beta \ldots$ it being understood (unless the contrary is stated) that any of the indices $a, b, c \ldots$ in the denominator of the denumerant $\dfrac{n}{a, b, c, \ldots l,}$ or in its generating function $\dfrac{1}{(1-x^a)(1-x^b)\ldots(1-x^l)}$ may be made equal to one another.

It is perhaps not unworthy of notice that the denumerant $\dfrac{n}{a, b, \ldots, l,}$ may be expressed as the residue of a double sum without knowing the divisors of the indices. For it is obvious that we may express it as the sum of an infinite number of waves whose indices take in all values from unity up to infinity (since all those whose indices are non-divisors will be equal to zero),* and consequently as the residue of a sum of quantities obtained by substituting for r in the expression

$$\frac{r^\nu e^{\nu x}}{P\left(r^{\frac{a}{2}} e^{\frac{x}{2}} - r^{-\frac{a}{2}} e^{-\frac{x}{2}}\right)},$$

every primitive root of unity of every order up to the ω^{th} inclusive, where ω is any number not less than the greatest of the quantities a, and therefore, if we please, equal to Σa, which saves the necessity of distinguishing the relative magnitudes of the several quantities a (ω it should be noticed must not be taken infinity, because that would render the sum to be residuated infinite). Thus then we see that the denumerant $\dfrac{n}{a, b, \ldots, l,}$ is the residue of

$$\Sigma \frac{e^{(t+2\pi i k)\nu}}{P\left\{e^{a\left(\frac{t}{2}+\pi i k\right)} - e^{-a\left(\frac{t}{2}+\pi i k\right)}\right\}},$$

where k represents every distinct quantity expressible by a proper fraction whose denominator is equal to or less than Σa.†

* By a process, so to say, of *natural selection.*

† The number of terms in this sum will be the sum of the totients of all the numbers up to the limit, an empirical expression for which (if my memory is not in fault) has been recently investigated by Mr. Merrifield.

The result previously found concerning the relation of $F\nu$ to $F-\nu$ in accordance with the observation due, I believe, to Jacobi, that if ϕn, ψn be the coefficients of x^n [n positive or negative] in the ascending and descending expansions of a proper rational fraction, then $\psi n = -\phi n$. For, in the particular fraction we are considering, it is obvious that calling the number of the factors (our former $\alpha+\beta+\ldots+\lambda$) i and $a+b+\ldots+l=s$, we shall have

$$\psi(-n-s)=(-)^i\phi n.$$

Therefore $\phi n = (-)^{i-1}\phi(-n-s)$ by Jacobi's observation.

If then $\nu = n+\frac{s}{2}$ and $\phi n = F\nu$ so that $\phi(-n-s)=F\left(-n-\frac{s}{2}\right)=F(-\nu)$ we shall have $F\nu=(-)^{i-1}F(-\nu)$, as already shown.

It is also a part of the same observation and shown in the same way that ϕn, used in the same sense as above, is zero for all values of negative n between *zero* and the *degree* of the fraction (*exclusive*); hence $F(\pm\nu)$ is zero for all values of ν from 0 to $\frac{s}{2}-1$ inclusive if s be even, and from $\frac{1}{2}$ to $\frac{s}{2}-1$ inclusive if s be odd.*

This fact alone is sufficient to give exactly the number of homogeneous equations required to determine (to a numerical factor près) the algebraico-exponential form $F(\nu)$, *i. e.* the effective† *trivial* zero values of $F(\nu)$ are exactly equal in number to the number of terms which that form contains, as I will proceed to show.

The number of the indices $a, b, c, \ldots$ in which any divisor is contained may be termed its frequency in respect to those numbers, and it is a very simple arithmetical fact that if the totient of every divisor of a set of given numbers be multiplied by its frequency in respect to the set, the sum of the products so obtained will be equal to the sum of the given numbers. When the set reduces to a single term this theorem becomes the familiar one, that any number is equal to the sum of the totients of all its several divisors, and from this to the general case there is but a step, for we may suppose the set of numbers written out in a line, and under every one of them which contains a divisor j the totient of j to be written, and every value from 1 upwards as far as the highest number of the

* In order not to break up the text, the footnote (which ought to come here) regarding the two statements above, as to the coefficient-functions of any proper fraction, is transferred to the last page of this Excursus.

† I say *effective* because it will presently be seen that in a certain case one of the trivial zero values will be ineffective, *i. e.* will only lead to an identity and not to an equation between the coefficients in question.

set to be given to j. The rectangle (partly filled with totients and partly vacant) so formed, read off in columns, will, by the preceding case, give the sum of the set of numbers, and read off in lines, the sum of the products of each divisor by its frequency.

Let us now inquire into the number of the terms contained in the several waves. W_1, which always exists, will be the coefficient of $\frac{1}{t}$ in $\frac{e^{\nu t}}{P(e^{\frac{at}{2}}-e^{-\frac{at}{2}})}$, and therefore (always supposing the number of indices a to be i) will be the coefficient of t^{i-1} in the product of $\left(1+\nu t+\nu^2\frac{t^2}{1.2}+\ldots\right)$ into the ascending development of $\frac{1}{P\left(\frac{e^{\frac{at}{2}}-e^{-\frac{at}{2}}}{t}\right)}$, and will therefore be a function of ν consisting of multiples of ν^{i-1}, ν^{i-3}, ... until a multiple of ν or a constant is reached, and therefore containing $E\frac{i+1}{2}$ terms, the first of which it may be well to notice (using $a_1, a_2 \ldots a_i$ in lieu of $a, b, \ldots l$ as the indices) will obviously always be $\frac{1}{\Pi(i-1)a_1.a_2.\ldots a_i}$ *

In like manner it will be obvious that for W_2 the degree of ν will be the frequency of 2 diminished by a unit, and the form of W_2 will be $(-)^n$ into a polynomial function of ν of that degree.

Again, any other wave W_i of frequency f_i will consist of a set of products of polynomial functions of ν of the degree f_i-1 each multiplied by a sum of exponential quantities consisting of pairs of the form $c\Sigma(\rho^{\nu+\delta}+\rho^{\nu-\delta})$ or $c\Sigma(\rho^{\nu+\delta}-\rho^{\nu-\delta})$ according as $i-f_i$ is even or odd, where δ will be half the number of primitive i^{th} roots of unity, say $\frac{\tau(i)}{2}$, where the numerator is the totient of i.

* The highest power of ν in any other wave (which is its frequency diminished by unity) will *in general be* less than $i-1$, and consequently the sign of the terms in the development of any rational fraction beyond a certain point must be unvarying, and the development from that point omni-positive or omni-negative, according as the numerator, on substituting units for the variable, is positive or negative. The case of exception is when all the indices have a common numerant, say δ, for then the frequency of δ will be the same as of unity, and W_δ be of the same degree as W_1 in ν, so that the reason for uniformity of sign (at a sufficient distance from the origin) no longer subsists. This is the proof referred to at p. 113, in what precedes.

It is worth while imprinting on the memory the rule that the asymptotic value of

$$\frac{n}{a_1, a_2, \ldots a_i} \div n^{i-1} \text{ is } \frac{1}{(1.2.3\ldots(i-1))a_1.a_2\ldots a_i},$$

which ought, I imagine, to be susceptible of some simple proof or illustration by the method of nodes or cross-gratings, such as employed by Eisenstein to prove the law of reciprocity for quadratic residues, and by myself (Johns Hopkins Circulars, Nos. 13 and 14, pp. 179, 180, 209) to demonstrate the impossibility of the existence of trebly periodic functions.

Hence the total number of constants to be determined in the algebraico-exponential function representing $\frac{n}{a_1, a_2, \ldots a_i}$ will be $E\frac{f_1+1}{2}+E\frac{f_2+1}{2}+\Sigma\frac{\varphi\lambda f_\lambda}{2}$ $[\lambda=3, 4, \ldots\infty]$.

1°. Suppose that i and f_2 are not both even.

Then remembering that $\frac{f_1}{2}+\frac{f_2}{2}+\frac{f_3 . \tau_3}{2}+\frac{f_4 . \tau_4}{2}+\ldots=\frac{s}{2}$, the antecedent expression $=E\left(\frac{s}{2}+1\right)$, for when f_1, f_2 are both odd, the two first terms on the left-hand side of this equation exceed the corresponding ones in the equation above it by $\frac{1}{2}, \frac{1}{2}$ respectively, and $E\left(\frac{s}{2}+1\right)$ will exceed $\frac{s}{2}$ by unity (because f_1-f_2 the number of the odd elements in the sum of all of them being even, s is even). And if f_1, f_2 are one odd and the other even, the right as well as the left-hand side of each equation will be increased $\frac{1}{2}$, for s will be now odd.

2°. Suppose that f_1, f_2 are both even, then

$$E\frac{f_1+1}{2}+E\frac{f_2+1}{2}+\frac{f_3\tau(3)}{2}+\ldots=\frac{f_1}{2}+\frac{f}{2}+\frac{f_3\tau(3)}{2}+\ldots=\frac{s}{2}.$$

Hence the number of constants to be determined is $1+E\frac{s}{2}$; except when f_1, f_2 are both even, in which case it is $\frac{s}{2}$.

On the first supposition the trivial values of ν which make $F(\nu)$ zero are $0, 1, 2, \ldots \frac{s}{2}-1$ when s is even, and $\frac{1}{2}, \frac{3}{2}, \ldots\left(\frac{s}{2}-1\right)$ when s is odd, the number of such being $E\left(\frac{s}{2}\right)$ in either case, and there will be $E\left(\frac{s}{2}\right)$ homogeneous equations for finding the ratios of $E\left(\frac{s}{2}\right)+1$ coefficients, which is exactly the right number.

On the second supposition, *i. e.* when f_1, f_2 are both even, the number of the trivial values in question will be $\frac{s}{2}$, the same as the number of the coefficients, so that at first sight there would appear to be one superfluous equation —such, however, is not really the case—because the value 0 attributed to ν will lead not to a homogeneous equation between the coefficients but to the identity $0=0$. For evidently W_1, W_2 becoming odd functions of ν, will vanish when $\nu=0$, and every other wave will also vanish; for when $\nu=0$ it will consist exclusively of pairs of terms of the form $c(\rho^\delta-\rho^{-\delta})$ (because by hypothesis f_1 the number of the elements is even), and since ρ and $\frac{1}{\rho}$ may be interchanged,

it follows that the sum of such pairs must be zero. Hence whatever the relation of the number of odd and the number of even elements to the modulus 2, there will be just as many homogeneous equations as are required for determining the ratios of the coefficients in the form which expresses the denumerant. The absolute values of the coefficients may be found by writing $F\left(\frac{s}{2}\right) =$ coefficient of x^0 in the generating function $= 1$, or by virtue of the observation made above, that the leading coefficient in W_1 for the elements $a_1, a_2, \ldots a_i$ is $\frac{1}{\pi(i-1)a_1, a_2, \ldots a_i}$.

When the denumerant is regarded as a function of n and not of ν, it is obvious *à priori* that being a particular integral of an equation in finite differences of the order s, its coefficients must be determinable in relative magnitude by the knowledge of $(s-1)$ values of the variable for which it vanishes, and this is almost but not quite sufficient in itself to establish the preceding result regarding the canonical form.

I will illustrate this method presently by one or two easy examples, but previously it will, I think, be desirable to give greater precision and uniformity to the nomenclature of simple denumerants.

If any such be denoted by $\frac{n}{a, b, \ldots l}$, (I have sometimes here or elsewhere referred to n as the numerator or denumerator or partible number, and to $a, b, \ldots l$, variously as the denominators or as the indices or as the elements of the denumerant), in future I shall call n the componend, and $a, b, \ldots l$ the components of the denumerant.

A denumerant with a single component as $\frac{n}{a,}$ which I call an elementary denumerant, deserves special attention, for it will presently be seen that every given simple denumerant is expressible as a sum of powers of its componend multiplied respectively by linear functions of elementary denumerants whose several components are the divisors of the components of the given one.

The elementary denumerant $\frac{n}{a,}$ being the number of solutions in positive integers of the equation $ax = n$, is obviously 1 or 0 according as n does or does not contain. But we may also regard $\frac{n}{a,}$ as an analytical function and define it as the mean of the a values of ρ^n where ρ is any root of the equation $\rho^a - 1 = 0$, and so construed it will preserve a meaning even when n is taken a negative integer, and will mean 1 or 0, provided that n be an integer of either kind,

according as it does or does not contain a without a remainder. It is in this extended sense that $\frac{n}{a,}$ or $\frac{\nu}{a,}$ will be employed in what follows.

Supposing r to be a primitive i^{th} root of unity, W_i will consist of a sum of powers of ν each multiplied by the sum of quantities of the form $cr^{n+\delta}$ (where for the moment for greater clearness of elucidation I purposely retain n instead of using its augmentative ν). On giving n all values from $-\delta$ to $-\delta+i-1$ inclusive, this sum will take i successive values to be determined from the equation containing the primitive roots, say $\varepsilon_0, \varepsilon_1, \ldots \varepsilon_{i-1}$, so that its general value will be expressible under the form

$$\varepsilon_0 \frac{n+\delta}{i,} + \varepsilon_1 \frac{n+\delta-i}{i,} + \ldots + \varepsilon_{i-1} \frac{n+\delta-i+1}{i,}.$$

We may then replace n by $\nu - \frac{s}{2}$, and on so doing and further replacing (where requisite) any numerator by its residue in respect to i, shall obtain a sum of the form

$$\eta_0 \frac{\nu}{i,} + \eta_1 \frac{\nu-1}{i,} + \ldots + \eta_{i-1} \frac{\nu-i+1}{i,} \text{ when } s \text{ is even,}$$

and of the form

$$\eta_0 \frac{\nu-\frac{1}{2}}{i,} + \eta_1 \frac{\nu-\frac{3}{2}}{i,} + \ldots + \eta_{i-1} \frac{\nu-i+\frac{1}{2}}{i,} \text{ if } s \text{ is odd.}$$

On this being done, remembering the extension given to the sense of an elementary denumerant and the theorem that the analytical value $F\nu$ of a denumerant is equal to $\pm F(-\nu)$, we see that in either case the above sums will be reducible to a sum of pairs of terms of the form $\eta\left(\frac{\nu+k}{i,} \pm \frac{\nu-k}{i}\right)$ [the same $+$ or $-$ sign subsisting throughout the whole series for any specified power of ν] but subject to the exception that when i is even, two of the pairs will be replaced by single terms, multiples of $\frac{\nu \pm \frac{i}{2}}{i,}$ and of $\frac{\nu}{i}$ respectively, which become zero when the negative sign is the one to be employed.*

Thus taking $i=2$, W_2 takes the form $(-)^n R\nu$, *i. e.* $\frac{n}{2,} - \frac{n-1}{2}$. W_1 it is scarcely necessary to repeat will contain no elementary denumerants, being purely an algebraical function of the resolvent. W_2 is such a function multiplied by $(-1)^n$. This multiplier is expressible under the form $\left(\frac{n}{2,} - \frac{n \pm 1}{2,}\right)$ which is always a function of n that remains unchanged when n is changed into $-n$. But

*The sign is positive or negative according as the number of the components less the power of ν in question is odd or even, and it is easy also to see that the sum of all the coefficients of the elementary denumerants in the multiplier of each power of ν will be always zero.

when the two denumerants are expressed as functions of ν the case is different; if s (the sum of the components) is an even number, the above pair of terms becomes $(-)^{\frac{s}{2}}\left(\frac{\nu}{2}-\frac{\nu\pm 1}{2}\right)$ which is unaltered by the change of ν into $-\nu$, but when s is odd it becomes $(-)^{\frac{s-1}{2}}\left(\frac{\nu-\frac{1}{2}}{2}-\frac{\nu+\frac{1}{2}}{2}\right)$ which changes its sign when ν is changed into $-\nu$.

Before quitting the subject of nomenclature I may just observe that it will be convenient to call denumerants, when their resolvents are the natural numbers commencing with unity, *natural denumerants*, and when the natural numbers commencing with 2, curtate natural, or for greater brevity simply *curtate* denumerants, the highest number reached in either case being termed the order; D_i and Δ_i may then be used to denote natural and curtate denumerants of the order i.*

I now return to the application of the method of indeterminate coefficients to finding the value of denumerants whose components are given. This method is not practically applicable when the sum of the components is considerable, because that sum measures the number of linear equations to be solved. In the following section I shall work out in full, by the regular process, the case where the components are 2, 3, 4, 5, 6, 7, of which the result is more especially required for the purposes of the preceding section, and which has not previously been calculated. The other algebraical formulae for denumerants in their canonical form I shall give without exhibiting the work; the accuracy of most of them can be ascertained by comparison with Prof. Cayley's values of the same, exhibited as functions of the unaugmented componend in the *Phil. Trans.* for 1856 and 1858.

Let us suppose 1, 2, 3 to be the components,

we may write $\frac{n}{1,\,2,\,3,} = A\nu^2 + B + (-)^\nu C + \Sigma(\rho^{\nu+1}+\rho^{\nu-1})D,$

where $\rho^2+\rho+1=0$, or more simply, $A\nu^2+B+(-)^\nu C - D\Sigma\rho^\nu = 0$.

---

* It is curtate denumerants which are almost exclusively required in the applications to the theory of invariants. If necessary to bring into evidence the componend, we may use the more explicit notation $\overset{n}{\underset{i}{D}}$, $\overset{n}{\underset{i}{\Delta}}$ to signify natural and curtate denumerants of the order i with the componend n. Thus we may write $\overset{n}{D}_i - \overset{n-1}{D}_i = \overset{n}{\Delta}_i$ and $\overset{n}{D}_i - \overset{n}{D}_{i-1} = \overset{n-1}{\Delta}_i$.

It may be as well to notice that for curtate, as well as for natural denumerants, the divisors of the components are the natural numbers from unity to the order of the denumerant inclusive, so that the number of the waves for either of these sort of denumerants is equal to the order.

Hence making $\nu = 0, 1, 2$ we have
$$B + C - 2D = 0$$
$$A + B - C + D = 0$$
$$4A + B + C + D = 0,$$
so that $\quad 2C + 3A = 0 \quad 3D + 4A = 0 \quad B + \left(\frac{8}{3} - \frac{3}{2}\right) A = 0,$

or $\quad A = 6\rho \quad B = -7\rho \quad C = -9\rho \quad D = -8\rho;$

and to find ρ, making $\nu = 3$, we obtain

$$(54 - 7 + 9 + 16)\rho = 1 \text{ or } \rho = \frac{1}{72}.$$

Hence $\displaystyle \frac{n}{1, 2, 3,} = \frac{\nu^2}{12} - \frac{7}{72} - \frac{1}{8}\left(\frac{\nu}{2,} - \frac{\nu - 1}{2,}\right) + \frac{1}{9}\left(2\frac{\nu}{3,} - \frac{\nu + 1}{3} - \frac{\nu - 1}{3}\right)$ monomial denumerants being used to replace the exponential quantities $(-1)^\nu$; $\Sigma\rho^\nu$.

The leading coefficient $\frac{1}{12}$ it will be observed $= \frac{1}{(1.2)(1.2.3)}$, as it ought to be by the general rule.

The maximum negative value of $\frac{n}{1, 2, 3,} - \frac{\nu^2}{12}$ is $\frac{7}{72} + \frac{1}{8} - \frac{1}{9}$ or $\frac{1}{9}$, and its maximum positive value $\frac{2}{9} + \frac{1}{8} - \frac{7}{72}$ or $\frac{1}{4}$. Hence the value of $\frac{n}{1, 2, 3,}$ is always the nearest integer to $\frac{(n+3)^2}{12}$.

But by Euler's theorem of reciprocity $\frac{n}{1, 2, 3}$ is the number of ways of resolving n into three or less than three parts, and consequently $\frac{n-3}{1, 2, 3,}$ is the number of ways of resolving n into exactly three parts, this therefore is always the nearest integer to $\frac{n^2}{12}$, as first observed I believe by the late lamented Prof. DeMorgan.

Take as another case the components 1, 2, 3, 4 which gives $\nu = n + 5$. We may write

$$\frac{n}{1, 2, 3, 4,} = A\nu^3 + B\nu + (-)^\nu C\nu + D\Sigma(\rho^{\nu+1} - \rho^{\nu-1}) + E\Sigma(i^{\nu+1} - i^{\nu-1})$$

where $\rho^2 + \rho + 1 = 0$, $i^2 + 1 = 0$. Hence giving ν the successive values 1, 2, 3, 4, (omitting $\nu = 0$, which would lead to $0 = 0$) we obtain

$$A + B - C - 3D - 4E = 0$$
$$8A + 2B + 2C + 3D = 0$$
$$27A + 3B - 3C + 4E = 0$$
$$64A + 4B + 4C - 3D = 0$$

Hence $\quad 72A + 6B + 6C = 0$, and $36A + 6B - 2C = 0$,

consequently $\quad 2C + 9A = 0 \quad 2B + 15A = 0 \quad -3D + 16A = 0$

or $\quad A = 6\rho \quad B = -45\rho \quad C = -27\rho \quad D = 32\rho \quad E = -27\rho$.

Finally making $\nu = 5 \quad \rho(750 - 225 + 135 + 96 + 108) = 1$, or $\rho = \frac{1}{864}$,

and
$$\frac{n}{1,2,3,4,} = \frac{1}{144}\nu^3 - \frac{5}{96}\nu - \frac{1}{32}\left(\frac{\nu}{2,} - \frac{\nu-1}{2,}\right) + \frac{1}{9}\left(\frac{\nu-1}{3} - \frac{\nu+1}{3}\right) + \frac{1}{8}\left(\frac{\nu-1}{4} - \frac{\nu-3}{4}\right).$$

The principal coefficient is $\frac{1}{144}$ or $\frac{1}{\Pi 3.1.2.3.4}$, as it ought to be, according to the general rule, and this serves as a verification of the correctness of the whole work.

It will be found convenient to append here, instead of reserving for the following section, the analytical expression for the first wave of a general denumerant, which stands out markedly from the rest, inasmuch as it can be expressed once for all as an algebraical function of the componend and components without any regard being had to the arithmetical form of the latter.

Let $C(\tau_1\tau_2 \ldots \tau_j)$, $H(\tau_1\tau_2 \ldots \tau_j)$ or more briefly $C_j\tau$ $H_j\tau$ be understood to denote the perfectly well-known functions of $\tau_1, \tau_2, \ldots \tau_j$ which represent the elementary symmetric function and the sum of the homogeneous products of the j^{th} order of those quantities of which τ_q represents the sum of the q^{th} powers, so that *ex gr.* $C_2\tau$, $H_2\tau$ will serve to denote $\frac{\tau_1^2 - \tau_2}{2}$, $\frac{\tau_1^2 + \tau_2}{2}$ respectively, upon which supposition we may write

$$e^{\tau_1 t + \tau_2\frac{t^2}{2} + \tau_3\frac{t^3}{3} + \cdots} = 1 + \tau_1 t + \frac{\tau_1^2 + \tau_2}{2}t^2 + \ldots + H_q\tau t^q + \ldots$$

Also let it be observed preliminarily that as a direct inference from Macclaurin's theorem, if ϕ represent any function of x but does not contain ν,

$$\text{co}_j e^{\nu x + \phi} = \text{co}_j e^{\phi} + \text{co}_{j-1} e^{\phi}\nu + \text{co}_{j-2} e^{\phi}\frac{\nu^2}{1.2} + \ldots$$

Furthermore for greater brevity let us agree to express the W_1 for j components $a_1\, a_2 \ldots a_j$ under the form $W_{1,j}$, and write it equal to $\frac{V_j}{\pi_j}$ where π_j indicates the product of the j components.

We may then write
$$V_j = \pi_j\, \text{co}_{-1} \frac{e^{\nu x}}{P\left(e^{a\frac{x}{2}} - e^{-a\frac{x}{2}}\right)}.$$

Now from the known expression for log sin θ, we may write

$$\log\left(e^{\frac{\theta}{2}} - e^{-\frac{\theta}{2}}\right) = \log\theta + \beta_1\theta^2 - \beta_2\theta^4 + \ldots \pm \beta_q\theta^{2q} + \ldots$$

where $$\beta_q = \frac{1}{\Pi 2q} \cdot \frac{B_{2q-1}}{2q}.$$

Hence $$V_j = \mathrm{co}_{j-1} e^{\nu x - 2\tau_1 \frac{x^2}{2} + 2\tau_2 \frac{x^4}{4} - 2\tau_3 \frac{x^6}{6} \cdots}$$

where $2\tau_q = \frac{B_{2q-1}}{\Pi 2q}\sigma_{2q}$ and the latter factor indicates the sum of the $2q^{\text{th}}$ powers of the components.

Hence writing $x^2 = t$ we have $V_j = \mathrm{co}_{j-1} e^{\nu x - \tau_1 t + \tau_2 \frac{t^2}{2} - \tau_3 \frac{t^3}{3} \cdots}$ and consequently making $T = -\tau_1 t + \tau_2 \frac{t^2}{2} - \tau_3 \frac{t^3}{3} \cdots$

$$V_j = \mathrm{co}_{j-1} T + \mathrm{co}_{j-2} T.\nu + \mathrm{co}_{j-3} T.\frac{\nu^2}{1.2} + \mathrm{co}_{j-4} T.\frac{\nu^3}{1.2.3} \cdots$$
$$= \frac{\nu^{j-1}}{\Pi(j-1)} - H_1 \tau \frac{\nu^{j-3}}{\Pi(j-3)} + H_2 \tau \frac{\nu^{j-3}}{\Pi(j-3)} \cdots$$

the series ending with ν or with a constant according as j is even or odd.

Thus
$$V_2 = \nu,$$
$$V_3 = \frac{\nu^2}{2} - H_1(\tau),$$
$$V_4 = \frac{\nu^3}{6} - H_1(\tau)\nu,$$
$$V_5 = \frac{\nu^4}{24} - H_1(\tau)\frac{\nu^2}{2} + H_2(\tau),$$
$$V_6 = \frac{\nu^5}{120} - H_1(\tau)\frac{\nu^3}{6} + H_2(\tau)\nu, \text{ and so on,}$$

each V being an integral with respect to ν of the one which precedes it.

Substituting for each τ its value in terms of the Bernouillian numbers B and the σ's, and giving the former their arithmetical values we shall obtain

$$V_2 = \nu,$$
$$V_3 = \frac{\nu^2}{2} - \frac{\sigma_2}{24},$$
$$V_4 = \frac{\nu^3}{6} - \frac{\sigma_2}{24}\nu,$$
$$V_5 = \frac{\nu^4}{24} - \frac{\sigma_2}{48}\nu^2 + \left(\frac{\sigma_2^2}{1152} + \frac{\sigma_4}{2880}\right),$$
$$V_6 = \frac{\nu^5}{120} - \frac{\sigma_2}{144}\nu^3 + \left(\frac{\sigma_2^2}{1152} + \frac{\sigma_4}{2880}\right)\nu,$$
$$V_7 = \frac{\nu^6}{720} - \frac{\sigma_2}{576}\nu^4 + \left(\frac{\sigma_2^2}{2304} + \frac{\sigma_4}{5160}\right)\nu^2 - \left(\frac{\sigma_2^3}{82944} + \frac{\sigma_2\sigma_4}{103680} + \frac{\sigma_6}{181440}\right),$$
$$V_8 = \int_\gamma^0 d\nu\, V, \text{ and so on.}$$

Such are the expressions for V best adapted for actual use, since it is desirable to express $W_{1,j}$, *i. e.* $\frac{V_j}{a_1 . a_2 . \ldots a_j}$ explicitly in terms of powers of ν; but there is another somewhat noteworthy form which can be given to the V with an even subindex as follows:

It is obvious that

$$V_{2k} = \mathrm{co}_{-1} \frac{\frac{1}{2}(e^{\nu x} - e^{-\nu x}) + \frac{1}{2}(e^{\nu x} + e^{-\nu x})}{P(e^{\frac{a}{2}x} - e^{-\frac{a}{2}x})} = \mathrm{co}_{-1} \frac{\frac{1}{2}(e^{\nu x} - e^{-\nu x})}{P(e^{\frac{a}{2}x} - e^{-\frac{a}{2}x})}$$

for the neglected part of the numerator will contribute nothing to the residue.*

We may now calculate the logarithm of the entire quantity to be residuated instead of merely the denominator, and take the residue of its exponential ν; on so doing it will be obvious on reflection that we shall obtain the product of ν into a quantity of the very same form as the constant term in V_{2k-1}, when instead of σ_{2q} in the value of τ_q we substitute $-(2n)^{2q} + \sigma_{2q}$. If then we write $2U_q = \frac{B_{2q-1}}{\Pi 2q}((2n)^{2q} - \sigma_{2q})$ it is easy to see that we shall have $V_{2k} = \nu C_{k-1}(U)$.

Thus *ex gr.* suppose $2k = 6$, we may write V_6 under the form

$$\nu \left\{ \frac{(4\nu^2 - s_2)^2}{1152} - \frac{(16\nu^4 - s_4)}{2880} \right\}$$

to verify which it will be observed that

$$\frac{16}{1152} - \frac{16}{2880} = \frac{1}{72} - \frac{1}{180} = \frac{1}{120} \text{ and } \frac{8}{1152} = \frac{1}{144},$$

so that

$$V_6 = \frac{\nu^5}{120} - \frac{s_2}{144}\nu^3 + \left(\frac{s_2^2}{1152} + \frac{s_4}{2880}\right)\nu, \text{ as previously found.}$$

Before having done with this outline it may be well to call attention to the circumstance that the distribution of the infinity-roots into groups determined by the divisors of the components is not in all cases the best mode of grouping to adopt.

Thus suppose that the components $(a_1, a_2, \ldots a_i)$ are all prime relatively to each other, it will in such case be most expeditious, after taking out the algebra-

---

*For V_{2k} the *effective* numerator of the residuand is a *sine* form, and may be subjected to the same treatment as its fellows in the denominator. The case is different with V_{2k-1}, for which the effective numerator of the residuand is a *cosine* form. But we may write $V_{2k-1} = \frac{d}{d\nu} V_{2k} = C_{k-1} U + \nu\delta C_{k-1} U$, and if we turn to account the fact that in $C_{k-1} U$ along with $(2\nu)^2$, $(2\nu)^4, \ldots (2\nu)^q \ldots$ are associated $-s_2, -s_4, \ldots -s_{2q} \ldots$ and choose to write $-\nu^{2q}\frac{d}{ds_{2q}} = \Delta^q$, it will be found that the above expression may be transformed so as to give the symbolical equation (more curious perhaps than useful) $V_{2k-1} = \left(\frac{1+\Delta}{1-\Delta}\right)^2 C_{k-1} U$, whereas as previously found $V_{2k} = \nu C_{k-1} U$.

ical part W_1, to separate what remains into i portions, referring respectively to *all* the non-unity a_1^{th}; a_2^{th}, ... a_i^{th} roots of unity.*

This view enables us to give a concise answer to a question of some interest, viz. as to what is the number of solutions of the inequality $a_1x_1 + a_2x_2 + \ldots + a_ix_i < \mu(a_1a_2 \ldots a_i)$, say $\mu\pi_i$, where μ is any positive integer and the coefficients are relatively prime each to each.

Certainly this number is no other than the denumerant $\dfrac{\mu\pi_i}{1, a_1, a_2, \ldots a_i}$ which might be calculated by the general formula, but would give a result neither concise nor elegant; we may on the other hand regard it as a sum of denumerants, say $\Sigma \dfrac{\mu\pi_i - \delta}{a_1, a_2, \ldots a_i}$, where δ takes all values from 0 to $\mu\pi_i - 1$. Now each such denumerant will consist of a purely algebraical and a purely periodical part, and it is very easy to see according to the view just indicated that the sum of all the latter will be zero. Hence the number required will be

$$\Sigma^0_{\mu\pi_i - 1} \frac{V_1}{\pi_i}.$$

I may illustrate this by the very simplest imaginable case, where there are but two components p, q and the number required is that of the solutions in integers of the inequality $px + qy < pq$ where p and q are relative-primes.

Calling $pq = n$, the rule laid down will give for the number sought

$$\Sigma_0^{n-1} \frac{\nu}{n} \quad i.\ e.\ \Sigma_0^{n-1} \frac{n + \frac{p+q}{2}}{n} = \frac{pq - p - q - 1}{2}.$$

This result admits of a somewhat *piquant* verification. The number of integers less than pq and containing neither p nor q is $(p-1)(q-1)$, and if every two of these which are supplementary to one another (I mean whose sum is pq) be made into a pair, it is an easily demonstrable, but by no means an unimportant fact, that one of the pair will be a compound and the other a non-compound of p and q. Hence the total number of non-compounds is $\dfrac{(p-1)(q-1)}{2}$, and therefore the total number of solutions of $px + qy < pq$ will be the remainder when the above is subtracted from pq, *i. e.* $\dfrac{pq + p + q - 1}{2}$ as previously determined.

I will embrace this opportunity of noticing a correction that should be made to the long footnote in Section 3 given in the preceding number of the Journal.

* This is tantamount to blending into one all the waves corresponding to the non-unity divisors of each component.

In lieu of the words in the last paragraph of that note following the word *products*, line 3 and preceding the word *set*, line 8, read as follows:

Of the form $b^x\, Q^y R^z S^t\, T^u$ *such that no one of them could be* (*a power of one or*) *a product of powers of any of the others. If then it could be shown that there exists in the succession a set of quintuplets* x, y, z, t, u, *such that the quotient system of every other quintuplet in the succession is intermediate to the quotient system of that*

It may also be as well to notice here that the method of expressing in terms of ordinary space the intermediateness of a quadruplet, a triplet or a couplet, to four, three or two other such respective multiplets, may be profitably simplified by the use of quadriplanar, trilinear and bi-punctual coördinates, in flat spaces of three, two and one dimension respectively; for we may then without having recourse to quotient-systems regard each element of the multiplet as a coördinate of its representative point, inasmuch as the affection concerned being one relative exclusively to the inwardness or outwardness of a point in regard to a closed environment, obviously remains unchanged by projection.

What follows is the footnote referred to at foot of page 124 *where it was meant to be inserted.*

Each of the two statements regarding the coefficient-functions becomes next to self-evident when the coefficient of x^n in the reciprocal of $(1-\alpha x)(1-\beta x)\,.\,.\,.\,(1-\lambda x)$ is put under the form of a sum of terms similar to $\alpha^n \div \left(1-\frac{\beta}{\alpha}\right)\left(1-\frac{\gamma}{\alpha}\right)\,.\,.\,.\,\left(1-\frac{\lambda}{\alpha}\right)$ interpreted (when necessary) as meaning the function of $(n\,;\alpha\,,\beta\,.\,.\,.\,\lambda)$ indefinitely near to the value of what such sum becomes when any equal elements are made to undergo arbitrary infinitesimal variations. Jacobi's proof of the theorem, I rather think, is got by proving it directly for each of the simple partial fractions into which any given proper fraction may be supposed to have been resolved.

A third method is to form the equation between $u_n, u_{n-1},\,.\,.\,.\,u_{n+j-1}$, and between v_{-n}, v_{-n-1}, $.\,.\,.\,v_{-n-j-1}$ [u_n being the general coefficient in the ascending and v_{-n} in the descending development of $1\div R(x)$]; the two equations become identical on changing u and n into v and $-n$, and $j-1$ homogeneous equations which help to determine the constants will be the same in both, viz. those got by making $n=-1\,,-2\,,\,.\,.\,.-(j-1)$, consequently the two particular integrals u_n, v_n can differ only by a factor independent of n; if we write then $u_n : v_n :: P : Q$ and call the first and last coefficient in the denominator A and L. and pay attention to the fact that u_n, v_n can only become infinite when A, L vanish, and also to the indifference of the relation of R regarded as a quantic in x and 1 to the two sorts of development, it is plain to see that $P : Q :: A^\mu : \pm L^\mu$, but the x-weight of u_n is n and of v_{-n} is $-n$; hence $\mu=0$ and $u_n : v_n$ is independent, not only of n but of the coefficients in R, and to determine its value we may make $R=x^{j-1}-x^j$, which gives at once $u_n=-v_n$. This being true for all values of n, it is obvious that the relation will continue to subsist, when instead of unity any polynomial function of x of lower degree than that of the denominator (see below) is taken for the numerator.

Moreover, if the degree of the numerator be $j-\delta$, u_q and v_q will be seen (from what goes before) to vanish for every value of q common to the series $-1\,,-2\,,\,.\,.\,.-(j-1) : 0\,,-1\,,\,.\,.\,.-(j-2)$: $.\,.\,. : (j-\delta-1)\,,\,(j-\delta-2)\,,\,.\,.\,.-(\delta-1)$, viz. for the values $-1\,,-2\,,\,.\,.\,.-(\delta-1)$ or in other words either coefficient-function of the index of any power of the variable which appears neither in the ascending nor the descending development of a rational fraction is equal to zero.

Unless the fraction is a proper one u_n and v_n (the coefficient-functions) will not be continuous functions of n throughout; hence arises the necessity of this limitation in dealing with the generalized equation $u_n = -v_n$. Thus *ex gr.* for the improper fraction $\frac{1+2x^2}{(1-x)^2}$ u_0 and v_0 are 1 and 2, but for any positive or negative value of n other than 0, u_n and v_n will be $3n-1$ and $-(3n-1)$ respectively. It may be added that the theorem will continue to subsist even for an improper fraction, provided that on freeing its numerator from a power of the variable, it becomes a proper one, for then the coefficient-functions remain continuous throughout.

This last proof, although more labored than the preceding ones, seems to me the best because it goes straight to the heart of the question and does not depend on any apparently accidental results of calculation, but (so to say) compares the two twin functions in their nascent state, in the very act of birth.

The relation of the two coefficient-functions to one another and to the two general terms in the actual expansions becomes more clear if we use ϕn, ψn to denote the two former, reserving u_n, v_n for the two latter. Then besides the equation $\phi n + \psi n = 0$ which is absolute, we have the equations $u_n = \phi n$, $v_n = \psi n$, limited as follows. Call Δ the deficiency of the numerator of the generating proper fraction, *i. e.* the number of units that it stops short of its maximum possible value: then the first of these two equations holds good for all values of n not less than $-\Delta$, the latter for all values of n not greater than -1; if Δ is not zero, *i. e.* if the degree of the numerator is not the integer next below that of the denominator, these two ranges will overlap for the values $-1, -2, \ldots, -\Delta$ of n, and for those values $\phi n = u_n = 0$, $\psi n = v_n = 0$. In the use made of these theorems in the text, the numerator is a mere constant, so that Δ has its maximum value, namely it is one unit less than the sum of the components (that sum being the degree of the generating function to a denumerant).

The general theorem may be brought into more distinct relief as follows: A finite fraction may be conceived as containing any number of powers of x positive or negative in numerator and denominator, and its two developments may be supposed to touch or be separate or to intersect one another. In the *last* case two coefficient-functions ϕn, $-\phi n$ exist applicable to all terms outside but inapplicable to any term inside the overlap. In the second case such functions exist which (besides being applicable, as in the case of contact, to all terms belonging to either of the two developments) vanish for all values of n in the chasm which separates them.

A Memoir on the Abelian and Theta Functions.

BY PROFESSOR CAYLEY, *Cambridge, England.*

The present memoir is based upon Clebsch and Gordan's "*Theorie der Abelschen Functionen*," Leipzig, 1866 (here cited as C. and G.); the employment of differential rather than of integral equations is a novelty; but the chief addition to the theory consists in the determination which I have made for the cubic curve, and also (but not as yet in a perfect form) for the quartic curve, of the differential expression $d\Pi_{\xi\eta}$ (or as I write it $d\Pi_{12}$) in the integral of the third kind $\int_\alpha^\beta d\Pi_{\xi\eta}$ in the final normal form (endliche Normalform) for which we have (p. 117) $\int_\xi^\eta d\Pi_{\alpha\beta} = \int_\alpha^\beta d\Pi_{\xi\eta}$, the limits and parametric points interchangeable. The want of this determination presented itself to me as a *lacuna* in the theory during the course of lectures on the subject which I had the pleasure of giving at the Johns Hopkins University, Baltimore, U. S. A., in the months January to June, 1882, and I succeeded in effecting it for the cubic curve, but it was not until shortly after my return to England that I was able partially to effect the like determination in the far more difficult case of the quartic curve. The memoir contains, with additional developments, a reproduction of the course of lectures just referred to. I have endeavored to simplify as much as possible the notations and demonstrations of Clebsch and Gordan's admirable treatise; to bring some of the geometrical results into greater prominence; and to illustrate the theory by examples in regard to the cubic, the nodal quartic, and the general quartic curves respectively. The present three chapters are: I, Abel's Theorem; II, Proof of Abel's Theorem; III, The Major Function. The paragraphs of the whole memoir will be numbered continuously.

CHAPTER I. ABEL'S THEOREM.

The Differential Pure and Affected Theorems. Art. Nos. 1 to 5.

1. We have a fixed curve and a variable curve, and the differential pure theorem consists in a set of linear relations between the displacements of the intersections of the two curves; in the affected theorem a linear function of the displacements is equated to another differential expression. I state the two theorems, giving afterwards the necessary explanations.

The pure theorem is

$$\Sigma\,(x,\,y,\,z)^{n-3}\,d\omega = 0\,.$$

The affected theorem is

$$\Sigma\,\frac{(x,\,y,\,z)_{12}^{n-2}\,d\omega}{012} = -\,\frac{\delta\varphi_1}{\varphi_1} + \frac{\delta\varphi_2}{\varphi_2} \qquad \text{(See footnote.*)}$$

2. We have a fixed curve $f=0$, or say the curve f, or simply the fixed curve, of the order n, with δ dps, and therefore of the deficiency $\frac{1}{2}(n-1)(n-2) - \delta$, $= p$. The expression the dps means always the δ dps of f.

And we have a variable curve $\phi = 0$, or say the curve ϕ, or simply the variable curve, of the order m, passing through the dps and besides meeting the fixed curve in $mn - 2\delta$ variable points.

Moreover, $d\omega$ is the displacement of the current point 0, coordinates (x, y, z), on the fixed curve, viz. the equation $f=0$ gives

$$\frac{df}{dx}\,dx + \frac{df}{dy}\,dy + \frac{df}{dz}\,dz = 0\,,$$

$$\frac{df}{dx}\,x + \frac{df}{dy}\,y + \frac{df}{dz}\,z = 0\,,$$

and we thence have

$$\frac{df}{dx} : \frac{df}{dy} : \frac{df}{dz} = ydz - zdy : zdx - xdz : xdy - ydx\,,$$

so that we have three equal values each of which is put $= d\omega$, viz. we write

$$\frac{ydz - zdy}{\dfrac{df}{dx}} = \frac{zdx - xdz}{\dfrac{df}{dy}} = \frac{xdy - ydx}{\dfrac{df}{dz}}\,, = d\omega\,,$$

and $d\omega$ as thus defined is the displacement.

* For comparison with C. and G. observe that in the equation, p. 47, $V = \log\frac{\psi(\eta)\,\phi(\xi)}{\phi(\eta)\,\psi(\xi)}$, $= \log\frac{\psi_2\phi_1}{\phi_2\psi_1}$ suppose, their ψ belongs to the upper limit and corresponds to my ϕ: the equation gives therefore $dV = -\,\delta\,\frac{\psi_1}{\psi_1} + \delta\,\frac{\psi_2}{\psi_2}$, agreeing with the formula in the text.

$(x, y, z)^{n-3} = 0$ is the minor curve, viz. the general curve of the order $n-3$, which passes through the dps;* and the function $(x, y, z)^{n-3}$ is the minor function.

1 and 2 are fixed points on f, called the parametric points, coordinates (x_1, y_1, z_1) and (x_2, y_2, z_2) respectively; and 012 denotes the determinant

$$\begin{vmatrix} x, & y, & z \\ x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \end{vmatrix},$$

so that $012 = 0$ is the equation of the line joining the points 1 and 2: this line meets the fixed curve in $n-2$ other points, called the residues of 1, 2.

$(x, y, z)_{12}^{n-2} = 0$ is the major curve *quoad* the points 1 and 2; viz. this is the general curve of the order $n-2$, passing through the dps and also through the residues of 1, 2.

But further, the function $(x, y, z)_{12}^{n-2}$ is the proper major function; viz. the implicit factor of the function is so determined that taking $0 = 1$, the current point at 1 (that is writing (x_1, y_1, z_1) for (x, y, z)) the function $(x, y, z)_{12}^{n-2}$ reduces itself to the polar function $\left(x_2 \frac{d}{dx_1} + y_2 \frac{d}{dy_1} + z_2 \frac{d}{dz_1}\right) f_1$, afterwards written $n.1^{n-1}2$, of f: this implies that taking $0 = 2$, the current point at 2, the function reduces itself to the polar function $n.12^{n-1}$.

ϕ_1 is what ϕ becomes on writing therein (x_1, y_1, z_1) for (x, y, z): and similarly ϕ_2 is what ϕ becomes on writing therein (x_2, y_2, z_2) for (x, y, z).

δ denotes differentiation in regard only to the coefficients of ϕ; viz. writing $\phi = (\mathrm{a}, \ldots \between x, y, z)^m$ we have $\delta\phi = (d\mathrm{a}, \ldots \between x, y, z)^m$, and similarly $\delta\phi_1$ and $\delta\phi_2 = (d\mathrm{a}, \ldots \between x_1, y_1, z_1)^m$ and $(d\mathrm{a}, \ldots \between x_2, y_2, z_2)^m$ respectively.

The sum Σ extends to all the variable intersections of the two curves.

3. As to the meaning of the theorems, consider first the pure theorem. The variable intersections are not all of them arbitrary points on the fixed curve: a certain number of them taken at pleasure on the fixed curve will determine the remaining variable intersections; and there are thus a certain number of integral relations between the coordinates of the variable intersections; to each such integral relation there corresponds a linear relation between the displacements $d\omega$ of these points, or say a displacement-relation. It is precisely these displacement-relations which are given by the theorem, viz. the equation

$$\Sigma (x, y, z)^{n-3}\, d\omega = 0$$

*This definition implies that the number of dps is at most $= \frac{1}{2}(n-1)(n-2) - 1$, that is that the fixed curve is not unicursal. But see *post* No. 21.

breaks up into as many linear relations as there are constants in the function $(x, y, z)^{n-3}$ which equated to zero gives a curve of the order $n-3$ passing through the dps; for instance $n=3$, $\delta=0$, the equation gives the single relation $\Sigma d\omega=0$; but $n=4$, $\delta=0$, the equation gives the three relations $\Sigma x d\omega=0$, $\Sigma y d\omega=0$, $\Sigma z d\omega=0$.

4. It is of course important to show, and it will be shown, that the number of independent displacement-relations given by the theorem is equal to the number of independent integral relations between the variable intersections.

5. Observe that the pure theorem gives *all* the displacement-relations between the variable intersections; we are hereby led to see the nature of the affected theorem. Taking at pleasure on the fixed curve the sufficient number of variable intersections, the coefficients of ϕ are thereby determined in terms of the coordinates of the assumed variable intersections,* and hence the value of $-\frac{\delta\varphi_1}{\varphi_1}+\frac{\delta\varphi_2}{\varphi_2}$ is given as a linear function of the corresponding displacements $d\omega$; and, substituting this value, the affected theorem gives a linear relation between the displacements $d\omega$ of the several variable intersections. But any such linear relation must clearly be a mere linear combination of the displacement-relations $\Sigma (x, y, z)^{n-3} d\omega=0$ given by the pure theorem.

Examples of the Pure Theorem—The Fixed Curve a Cubic. Art. Nos. 6 to 12.

6. The pure theorem is not applicable to the case $n=2$, the fixed curve a conic: it in fact gives no displacement-relation; and this is as it should be, for the variable intersections are all of them arbitrary.

The next case is $n=3$, $\delta=0$, the fixed curve a cubic. For greater simplicity the equation is taken in Cartesian coordinates. In general for such an equation, writing in the homogeneous formulæ $z=1$, we have

$$d\omega=\frac{dx}{\frac{df}{dy}}=-\frac{dy}{\frac{df}{dx}},$$

(the two values being of course equal in virtue of $\frac{df}{dx}dx+\frac{df}{dy}dy=0$); taking the former value and considering $\frac{df}{dy}$ as expressed in terms of x, let this be called

* The coefficients are determined, except it may be as to some constants which remain arbitrary, but which disappear from the difference $-\frac{\delta\phi_1}{\phi_1}+\frac{\delta\phi_2}{\phi_2}$; this will be explained further on in the text.

P (of course P is an irrational function of x): then we have $d\omega = \frac{dx}{P}$; and similarly $d\omega_1 = \frac{dx_1}{P_1}$, etc.

The fixed curve being then a cubic, let the variable curve be a line; this meets the cubic in three points, say $1, 2, 3$; and any two of these determine the line, and therefore the third point; there should therefore be one integral relation, and consequently one displacement-relation; and this is what is given by the theorem, viz. we have $\Sigma d\omega = 0$, that is $d\omega_1 + d\omega_2 + d\omega_3 = 0$, or what is the same thing

$$\frac{dx_1}{P_1} + \frac{dx_2}{P_2} + \frac{dx_3}{P_3} = 0.$$

The corresponding integral equation is the equation which expresses that the points $1, 2, 3$ are in a line, viz. considering y_1, y_2, y_3 as given functions of x_1, x_2, x_3 respectively, this is

$$\begin{vmatrix} x_1, & y_1, & 1 \\ x_2, & y_2, & 1 \\ x_3, & y_3, & 1 \end{vmatrix} = 0,$$

or, in the notation already made use of for such a determinant, $123 = 0$.

7. This equation $d\omega_1 + d\omega_2 + d\omega_3 = 0$, where $d\omega$ denotes $\frac{dx}{P}$, has a peculiar interpretation when we consider the coefficients of the cubic as arbitrary constants, and therefore the cubic as a curve depending upon nine arbitrary constants.* In taking 1 a point on the curve we in effect determine y_1 as a function of x_1 and the nine constants; and similarly in taking 2 a point on the curve we determine y_2 as a function of x_2 and the nine constants; the points 1 and 2 determine the third intersection 3, and we have thus x_3 determined as a function of x_1, x_2 and the nine constants.

Considering x_3 as thus expressed, we have $dx_3 = \frac{dx_3}{dx_1} dx_1 + \frac{dx_3}{dx_2} dx_2$, an equation which must agree with $d\omega_1 + d\omega_2 + d\omega_3 = 0$, that is with $dx_3 = -\frac{P_3}{P_1} dx_1 - \frac{P_3}{P_2} dx_2$. It follows that we have $\frac{dx_3}{dx_1} \div \frac{dx_3}{dx_2} = \frac{P_2}{P_1}$, and taking the logarithms and differentiating with $\frac{d}{dx_1} \cdot \frac{d}{dx_2}$ we find $\frac{d}{dx_1} \frac{d}{dx_2} \log\left(\frac{dx_3}{dx_1} \div \frac{dx_3}{dx_2}\right) = 0$, a partial differential equation of the third order, independent of any particular

* This theory was communicated by me to Section A of the British Association at the York meeting. See B. A. Report, 1881, pp. 534–535, "A Partial Differential Equation connected with the Simplest Case of Abel's Theorem."

cubic curve, and satisfied by x_3 considered as a function of x_1, x_2 and the nine constants. Observe that starting from the expression for x_3, and proceeding to the differential coefficients of the third order, we have ten equations from which the nine constants can be eliminated, that is we ought to have a partial differential equation of the third order: and conversely that the equation for x_3, as containing nine arbitrary constants, is a complete solution of the partial differential equation: the complete solution of the partial differential equation in question is thus the equation which expresses that 3 is the remaining intersection of the line through 1 and 2 with a cubic.

8. The partial differential equation has a geometrical interpretation, or is at least very closely connected with a geometrical property. Consider three consecutive positions of the line, meeting the cubic in the points 1, 2, 3; 1′, 2′, 3′ and 1″, 2″, 3″ respectively: the three lines constitute a cubic curve: the nine points are thus the intersections of two cubic curves, or say they are an "ennead" of points: and any eight of the points thus determine uniquely the ninth point.

9. As a particular example let the cubic be $x^3+y^3-1=0$; then $y=\sqrt[3]{1-x^3}$, and $d\omega=\dfrac{dx}{y^2}, =\dfrac{dx}{(1-x^3)^{\frac{2}{3}}}$*; and with these values we have as before the differential relation $d\omega_1+d\omega_2+d\omega_3=0$, and the integral relation $123=0$. I give a direct verification. To find x_3, y_3 the coordinates of the third intersection, we may in the equation of the cubic write x_3, y_3, $1=\lambda x_1+\mu x_2$, $\lambda y_1+\mu y_2$, $\lambda+\mu$ respectively, and then writing for shortness $1^22=x_1^2x_2+y_1^2y_2-1$, $12^2=x_1x_2^2+y_1y_2^2-1$, we obtain for the determination of λ, μ the equation $\lambda.1^22+\mu.12^2=0$.

This being so, from the equation $123=0$, we obtain by differentiation

$$\Sigma\{dx_1(y_2-y_3)-dy_1(x_2-x_3)\}=0,$$

the sum consisting of three terms, the second and third of them being obtained from the one written down by the cyclical interchange of the numbers 1, 2, 3. But we have $x_1^2dx_1+y_1^2dy_1=0$, and the equation thus is

$$\Sigma\frac{dx_1}{y_1^2}\{y_1^2(y_2-y_3)+x_1^2(x_2-x_3)\}=0:$$

* Writing $f=x^3+y^3-1$ we should have $\frac{df}{dy}=3y^2$, and therefore $d\omega=\frac{dx}{3y^2}$; but the $\frac{1}{3}$ enters as a common factor in all the $d\omega$'s, and it may clearly be disregarded: the value in the text, $d\omega=\frac{dx}{y^2}$ could of course be obtained by writing, as we may do, $f=\frac{1}{3}(x^3+y^3-1)$, and so in other cases.

this will reduce itself to $\Sigma\,\frac{dx_1}{y_1^2}=0$, if only the three coefficients in { } are equal, that is we ought to have

$$y_1^2(y_2-y_3)+x_1^2(x_2-x_3)=y_2^2(y_3-y_1)+x_2^2(x_3-x_1)=y_3^2(y_1-y_2)+x_3^2(x_1-x_2).$$

Comparing for instance the first and second terms, the equation is

$$-y_3(y_1^2+y_2^2)-x_3(x_1^2+x_2^2)+(x_1^2x_2+y_1^2y_2+x_1x_2^2+y_1y_2^2)=0,$$

or as this may be written

$$-(\lambda y_1+\mu y_2)(y_1^2+y_2^2)-(\lambda x_1+\mu x_2)(x_1^2+x_2^2)+(\lambda+\mu)(x_1^2x_2+y_1^2y_2+x_1x_2^2+y_1y_2^2)=0,$$

where the whole coefficient of λ is $-x_1^3-y_1^3+x_1^2x_2+y_1^2y_2$, which in fact is $x_1^2x_2+y_1^2y_2-1, =1^22$; and similarly the whole coefficient of μ is 12^2; the equation is thus $\lambda.1^22+\mu.12^2=0$, which is right. The first and second coefficients are thus equal, and in like manner the first and third coefficients are equal; we have thus the required result, $\frac{dx_1}{y_1^2}+\frac{dx_2}{y_2^2}+\frac{dx_3}{y_3^2}=0$.

10. In all that follows, the cubic might be any cubic whatever, but to fix the ideas I take a particular form.

Let the cubic be $y^2-X=0$, X a cubic function $(x,1)^3$, or say even $X=x.1-x.1-k^2x$, then $y=\sqrt{X}$, $d\omega=\frac{dx}{y}, =\frac{dx}{\sqrt{X}}$; and with these values we have the differential relation $d\omega_1+d\omega_2+d\omega_3=0$, and the integral relation $123=0$. This last equation is an integral of the differential equation $d\omega_1+d\omega_2+d\omega_3=0$; as not containing any arbitrary constant it is a particular integral.

But regard one of the three points, say 3, as a fixed point, that is let the line pass through the fixed point 3 of the cubic, and besides meet it in the points 1 and 2. We write $d\omega_3=0$, and the differential equation thus is $d\omega_1+d\omega_2=0$, while the integral equation is as before $123=0$; this equation, as containing one arbitrary constant, is the general integral of $d\omega_1+d\omega_2=0$.

Let the variable curve be a conic; say the intersections with the cubic are 1, 2, 3, 4, 5, 6. Any five of these points determine the conic, and therefore the sixth point; there is thus one integral relation, the equation $123456=0$, which expresses that the six points are in a conic, and there should therefore be one displacement-relation, viz. this is the equation $\Sigma d\omega=0$, that is $d\omega_1+d\omega_2+d\omega_3+d\omega_4+d\omega_5+d\omega_6=0$.

We have thus $123456=0$, as a particular integral of $d\omega_1+d\omega_2+d\omega_3+d\omega_4+d\omega_5+d\omega_6=0$. If, however, we take 6 a fixed point on the cubic, then we have the same equation as the general integral of $d\omega_1+d\omega_2+d\omega_3+d\omega_4+d\omega_5=0$.

But taking also 5 a fixed point of the cubic we have as an integral of $d\omega_1 + d\omega_2 + d\omega_3 + d\omega_4 = 0$, the foregoing equation $123456 = 0$, which contains apparently two arbitrary constants; and so if we also fix the point 4, or the points 4 and 3, we have for the differential equations $d\omega_1 + d\omega_2 + d\omega_3 = 0$, and $d\omega_1 + d\omega_2 = 0$, integrals with apparently three arbitrary constants and four arbitrary constants respectively.

11. The explanation is contained in the theory of *Residuation* on a cubic curve. Take the case $d\omega_1 + d\omega_2 + d\omega_3 = 0$, with the integral $123456 = 0$, containing apparently three arbitrary constants, viz. the relation between the variable points 1, 2, 3, is here given by a construction depending on the three fixed points 4, 5, 6 on the cubic; it is to be shown that two of these points can always be regarded as no-matter-what* points. To see that this is so, take on the cubic any two no-matter-what points 4′, 5′, then according to the theory referred to, we can find on the cubic a determinate point 6′ such that the points 4′, 5′ and 6′ establish between the variable points 1, 2, 3, the same relation which is established between them by means of the points 4, 5 and 6, viz. whether in order to determine the point 3 we draw a conic through 1, 2, 4, 5 and 6, or a conic through 1, 2, 4′, 5′ and 6′, we obtain as the remaining intersection of the conic with the cubic one and the same point 3. The construction of 6′ is, through 4, 5 and 6 draw a conic meeting the cubic in any three points 1, 2, 3; through these points and 4′, 5′ draw a conic, the remaining intersection of this with the cubic will be the required point 6′, and the point 6′ thus obtained will be a determinate point, independent of the particular conic through 4, 5 and 6 used for the construction. Thus 4 and 5 are replaceable by the no-matter-what points 4′ and 5′, or what is the same thing, two of the points 4, 5 and 6 may be regarded as no-matter-what points, and the number of arbitrary constants is thus reduced to one. And so in other cases, all but one of the fixed points may be regarded as no-matter-what points, and the integral as containing in each case only one arbitrary constant.

But conversely, it being known that the integral of the differential equation contains but one arbitrary constant, we can thence arrive at the theory of residuation.

*The epithet explains, I think, itself; the point may be any point at pleasure, but it is quite immaterial what point, and for this reason it is not counted as an arbitrary point. The most simple instance is that of two constants presenting themselves in a combination such as $c + c'$, either of them may be regarded as a no-matter-what constant.

12. We might go on to the case where the variable curve is a cubic; there are here nine intersections; any eight of these do *not* determine the variable cubic, but they *do* determine the ninth intersection; and there is between the nine intersections one integral relation, and corresponding to it one displacement-relation $\Sigma d\omega = 0$, that is $d\omega_1 + d\omega_2 + \ldots + d\omega_9 = 0$, given by the pure theorem. But as to this see further on, where it is shown in general that the number of independent integral relations is equal to the number of independent displacement-relations given by the theorem.

Example of the Affected Theorem—Fixed Curve a Circle. Art. Nos. 13 and 14.

13. The fixed curve is taken to be the circle $x^2 + y^2 - 1 = 0$, and the parametric points 1 and 2 to be the points $(1, 0)$ and $(0, 1)$ on this circle. The variable curve is taken to be a line, say the line $ax + by - 1 = 0$, meeting the circle in the points 3 and 4, coordinates (x_3, y_3) and (x_4, y_4) respectively.

Starting from the formula

$$\Sigma \frac{(x, y, 1)^0_{12} d\omega}{012} = -\frac{\delta\varphi_1}{\varphi_1} + \frac{\delta\varphi_2}{\varphi_2},$$

where the summation extends to the points 3 and 4, $(x, y, 1)^0_{12}$ is here a constant, $= 2.12$, that is $2(x_1x_2 + y_1y_2 - 1)$, which for the points 1, 2 in question is $= -2$. We have 012 denoting the determinant

$$\begin{vmatrix} x, & y, & 1 \\ 1, & 0, & 1 \\ 0, & 1, & 1 \end{vmatrix},$$

which is $= -x - y + 1$; and $d\omega = \dfrac{dx}{2y}$. Also $\dfrac{\delta\varphi_1}{\varphi_1}, = \dfrac{xda + ydb}{ax + by - 1}$, is $= \dfrac{da}{a-1}$, and similarly $\dfrac{\delta\varphi_2}{\varphi_2}$ is $= \dfrac{db}{b-1}$. The formula thus is

$$\Sigma \frac{dx}{y(x + y - 1)} = -\frac{da}{a-1} + \frac{db}{b-1}.$$

The coefficients a and b are determined by means of the points 3 and 4, that is they are functions of x_3, x_4; and considering them as thus expressed, then (inasmuch as there is no linear relation between the displacements $\dfrac{dx_3}{y_3}$ and $\dfrac{dx_4}{y_4}$ of the two arbitrary points 3 and 4 on the circle) the equation must become an identity in regard to the terms in dx_3 and dx_4 respectively. It only remains to verify that this is so.

14. Writing $P, Q, R = -y_3 + y_4,\ x_3 - x_4,\ x_3y_4 - x_4y_3$; also L_3 and $L_4 = x_3 + y_3 - 1$ and $x_4 + y_4 - 1$ respectively, we have $a = P \div R$, $b = Q \div R$, and the equation is found to be

$$\frac{dx_3}{y_3L_3} + \frac{dx_4}{y_4L_4} = \frac{1}{(Q-R)(R-P)}\{(Q-R)\,dP + (R-P)\,dQ + (P-Q)\,dR\},$$

where, substituting for dy_3, dy_4 their values in terms of dx_3, dy_3, we have

$$dP,\ dQ,\ dR = \frac{1}{y_3}x_3dx_3 - \frac{1}{y_4}x_4dx_4,\ \ \frac{1}{y_3}y_3dx_3 - \frac{1}{y}y_4dx_4,$$
$$\frac{1}{y_3}(x_3x_4 + y_3y_4)\,dx_3 - \frac{1}{y_4}(x_3x_4 + y_3y_4)\,dx_4,$$

and with these values, and by aid of the relations $Q - R,\ R - P,\ P - Q = x_4L_3 - x_3L_4,\ y_4L_3 - y_3L_4,\ -L_3 + L_4$, the equation is found to be

$$\frac{dx_3}{y_3L_3} + \frac{dx_4}{y_4L_4} = \frac{L_3L_4(x_3x_4 + y_3y_4 - 1)}{(x_4L_3 - x_3L_4)(y_4L_3 - y_3L_4)}\left(\frac{dx_3}{y_3L_3} + \frac{dx_4}{y_4L_4}\right);$$

viz. this will be true if only

$$L_3L_4(x_3x_4 + y_3y_4 - 1) - (x_4L_3 - x_3L_4)(y_4L_3 - y_3L_4) = 0,$$

that is

$$-x_4y_4L_3^2 - x_3y_3L_4^2 + L_3L_4(x_3x_4 + y_3y_4 + x_3y_4 + x_4y_3 - 1) = 0.$$

But from the values of L_3, L_4 we have $x_4y_4 = \frac{1}{2}L_4^2 + L_4$, $x_3y_3 = \frac{1}{2}L_3^2 + L_3$, and the coefficient of L_3L_4 is $= L_3L_4 + L_3 + L_4$; the equation is thus verified.

The example would perhaps have been more instructive if the points 1 and 2 had been left arbitrary points on the circle, but the working out would have been more difficult.

The Variable Intersections of the Two Curves—Number of Independent Integral Relations. Art. Nos. 15 to 19.

15. Suppose $n = 3$, $\delta = 0$ $(p = 1)$, the fixed curve a cubic; and suppose successively $m = 1, 2, 3, \ldots$ the variable curve a line, conic, cubic, etc.

If $m = 1$, then two points on the cubic determine the line, and consequently the remaining intersection with the cubic; hence there is one integral relation.

If $m = 2$, then five points on the cubic determine the conic, and consequently the remaining intersection with the cubic; hence there is one integral relation.

If $m = 3$, then eight points on the fixed cubic do not determine the variable cubic, but they do determine the ninth intersection. For draw through the eight points a no-matter-what cubic $\chi = 0$, the general cubic through the eight

points is $\chi + \alpha f = 0$, and this meets the fixed cubic in the points $\chi = 0$, $f = 0$, that is in the eight points and in one other point independent of the constant α and therefore completely determinate. Hence in this case also there is one integral relation.

So if $m = 4$, then eleven points on the cubic do not determine the quartic, but they do determine the remaining intersection. For draw through the eleven points a no-matter-what quartic $\chi = 0$, the general quartic through the eleven points is $\chi + (x, y, z)^1 f = 0$, and this meets the cubic in the points $\chi = 0$, $f = 0$, that is in the eleven points and in one other point independent of the constants of the linear function $(x, y, z)^1$, and therefore completely determinate. Hence there is one integral relation.

And so in general, the fixed curve being a cubic, then whatever be the order of the variable curve, there is always one integral relation.

16. Suppose next $n = 4$, $\delta = 0\ (p = 3)$, the fixed curve a general quartic; and as before suppose successively $m = 1, 2, 3, \ldots$ the variable curve a line, conic, cubic, etc.

If $m = 1$, then two points on the quartic determine the line, and therefore the remaining two intersections; the number of integral relations is $= 2$.

If $m = 2$, then five points on the quartic determine the conic, and therefore the remaining three intersections; the number of integral relations is $= 3$, and similarly if $m = 3$, the number of integral relations is $= 3$.

If $m = 4$, then thirteen points on the fixed quartic do not determine the variable quartic, but they do determine the remaining three intersections. For draw through the thirteen points a no-matter-what quartic $\chi = 0$; the general quartic through the thirteen points is $\chi + \alpha f = 0$, and this meets the fixed quartic in the points $\chi = 0$, $f = 0$, that is in the thirteen points and in three other points, independent of α and thus completely determinate, and the number of integral relations is $= 0$; and so in general for any higher value of m, the number is still $= 3$.

17. Suppose $m = 5$, $\delta = 0\ (p = 6)$, the fixed curve a general quintic, and as before $m = 1, 2, 3 \ldots$ successively.

If $m = 1$, then two points on the quintic determine the line, and therefore the remaining three intersections; the number of integral relations is $= 3$.

If $m = 2$, then five points on the quintic determine the conic, and therefore the remaining five intersections; the number of integral relations is $= 5$.

If $m=3$, then 9 points on the quintic determine the cubic, and therefore the remaining six intersections; the number of integral relations is $=6$, and so if $m=4$, or if $m=5$ or any greater number, in the first case directly, and in the other cases by consideration of the quintic $\chi+\alpha f=0$, etc., we find that the number of integral relations is always $=6$.

18. The reasoning is scarcely altered in the case where the fixed curve has dps; thus considering the general case of a fixed curve n, δ, p:

If $m=1$, then $2-\delta$ points on the fixed curve (this implies $\delta \not> 2$) determine the line, and therefore the remaining $n-2\delta-(2-\delta)$, $=n-2-\delta$ intersections; the number of integral relations is $=n-2-\delta$.

If $m=2$, then $5-\delta$ points on the fixed curve (this implies $\delta \not> 5$) determine the conic, and therefore the remaining $2n-2\delta-(5-\delta)$, $=2n-5-\delta$ intersections; the number of integral relations is $=2n-5-\delta$, and so for any value of $m \not> n-3$, and indeed for the values $n-2$ and $n-1$; here $\frac{1}{2}m(m+3)-\delta$ points on the fixed curve (this implies $\delta \not> \frac{1}{2}m(m+3)$) determine the variable curve, and therefore the remaining $mn-2\delta-\left(\frac{1}{2}m(m+3)-\delta\right)$, $=mn-\frac{1}{2}m(m+3)-\delta$ intersections. Hence the number of integral relations is $=mn-\frac{1}{2}m(m+3)-\delta$, that is, $=p-\frac{1}{2}(n-m-1)(n-m-2)$. And thus in the cases $m=n-2$ and $n-1$ the number is $=p$.

If $m=n$, then $\frac{1}{2}n(n+3)-1-\delta$ points on the fixed curve do not determine the variable curve, but they do determine the remaining $n^2-2\delta-\left(\frac{1}{2}n(n+3)-1-\delta\right)$, $=\frac{1}{2}(n-1)(n-2)-\delta$, that is p intersections, and the number of integral relations is thus $=p$; and so for any higher value of m the number is still $=p$.

19. The conclusion thus is

$m \not> n-3$ number of integral relations $=p-\frac{1}{2}(n-m-1)(n-m-2)$,
$m=$ or $>n-2$ " " $=p$.

The integral equations spoken of throughout are of course independent relations.

The Variable Intersections of the Two Curves. Number of Independent Displacement Relations given by the Pure Theorem. Art. No. 20.

20. The number of displacement-relations given by the pure theorem is $=$ number of constants in minor function $(x, y, z)^{n-3}$, which equated to zero represents a curve through the dps, viz. this is always

$$\tfrac{1}{2}(n-1)(n-2)-\delta, =p.$$

But for $m > n-2$, these relations are not independent. For instance, for $n=4$, $\delta=0$, $m=1$, the displacement-relations are

$$\Sigma(x, y, z)^1 d\omega = 0, \text{ that is } \Sigma x d\omega = 0,\ \Sigma y d\omega = 0,\ \Sigma z d\omega = 0,$$

and conversely from these last equations we have $\Sigma(x, y, z)^1 d\omega = 0$. But in this case the variable curve is a line $ax+by+cz=0$; hence writing $(x, y, z)^1 = ax+by+cz$, the equation $(x, y, z)^1 = 0$ is satisfied for each of the intersections of the line with the quartic, and the corresponding equation $\Sigma(x, y, z)^1 d\omega = 0$ is an identity. Hence the number of independent displacement-relations is $3-1, =2$.

So for $n=5$, $\delta=0$, $m=1$, the displacement-relations are

$$\Sigma(x, y, z)^2 d\omega = 0, \text{ that is } \Sigma(x^2, y^2, z^2, yz, zx, xy)\, d\omega = 0,$$

and these six equations give conversely $\Sigma(x, y, z)^2 d\omega = 0$, and in particular they give $\Sigma x(x, y, z)^1 d\omega = 0$, $\Sigma y(x, y, z)^1 d\omega = 0$, $\Sigma z(x, y, z)^1 d\omega = 0$. But if $(x, y, z)^1$ denote $ax+by+cz$, then as before we have $(x, y, z)^1 = 0$, for each of the intersections of the two curves, and the last mentioned three equations are identities. The number of independent displacement-relations is thus $6-3, =3$.

So for $n=5$, $\delta=0$, $m=2$. Here if the variable curve is $\phi = (a, \ldots \between x, y, z)^2 = 0$, then taking $(x, y, z)^2 = (a, \ldots \between x, y, z)^2$, the equation $(x, y, z)^2 = 0$ is satisfied for each of the intersections of the two curves, and the corresponding equation $\Sigma(x, y, z)^2 d\omega = 0$ is an identity; the number of independent displacement-relations is $6-1, =5$.

The reasoning is the same when δ is not $=0$, and we see generally that for $m < n-2$, or say

$$m \not> n-3, \text{ number of independent displacement-relations}$$
$$= p - \tfrac{1}{2}(n-m-1)(n-m-2);$$

while for $m = \text{or} > n-2$, number is $=p$;

since in this case the relations given by the theorem are all of them independent. It thus appears *à posteriori*, that in every case the number of independent displacement-relations given by the pure theorem is equal to the number of independent integral relations.

As to the dps of the Fixed Curve. Art. No. 21.

I conclude with a general remark applicable to the whole of the three chapters. There is no necessity to attend to all or indeed to any of the dps of the fixed curve. Suppose that the fixed curve has $\delta+\delta'$ dps, where δ, δ' may

be either of them or each $=0$, but attend only to the δ dps, the δ' dps being wholly disregarded, and accordingly let the expression the dps mean as before the δ dps of the fixed curve. No alteration at all is required, only if δ' be not $=0$, then $p=\frac{1}{2}(n-1)(n-2)-\delta$ will no longer be the deficiency. To obtain the best theorems we use all the $\delta+\delta'$ dps, but disregarding the δ' dps, we obtain theorems, as for a curve with δ dps, which are true, and may frequently be useful either in their original form or with simplifications introduced therein by afterwards taking account of the δ' dps.

CHAPTER II. PROOF OF ABEL'S THEOREM.

Preparation. Art. Nos. 22 and 23.

22. Starting from the equation $\phi=(a,\ldots\between x, y, z)^m=0$ of the variable curve, we have

$$\frac{d\varphi}{dx}dx+\frac{d\varphi}{dy}dy+\frac{d\varphi}{dz}dz+\delta\phi=0,$$

$$\frac{d\varphi}{dx}x+\frac{d\varphi}{dy}y+\frac{d\varphi}{dz}z=0,$$

where $\delta\phi=(da,\ldots\between x, y, z)^m$. Let τ denote an arbitrary linear function, $=ax+by+cz$, multiply the two equations by $ax+by+cz$, $=\tau$, and $-(adx+bdy+cdz)$, $=-d\tau$ respectively, and add, we obtain

$$(ydz-zdy)\left(b\frac{d\varphi}{dz}-c\frac{d\varphi}{dy}\right)+(zdx-xdz)\left(c\frac{d\varphi}{dx}-a\frac{d\varphi}{dz}\right)$$
$$+(xdy-ydx)\left(a\frac{d\varphi}{dy}-b\frac{d\varphi}{dx}\right)+\tau\delta\phi=0;$$

introducing $d\omega$, this becomes

$$d\omega\left[\frac{df}{dx}\left(b\frac{d\varphi}{dz}-c\frac{d\phi}{dy}\right)+\frac{df}{dy}\left(c\frac{d\varphi}{dx}-a\frac{d\varphi}{dz}\right)+\frac{df}{dz}\left(a\frac{d\varphi}{dy}-b\frac{d\varphi}{dx}\right)\right]+\tau\delta\phi=0,$$

or observing that a, b, c are the differential coefficients $\frac{d\tau}{dx}, \frac{d\tau}{dy}, \frac{d\tau}{dz}$, the term in [] is $J(f, \tau, \phi)$, or say $J(\phi, f, \tau)$, and the equation is

$$d\omega J(\phi, f, \tau)+\tau\delta\phi=0,$$

where $J(\phi, f, \tau)$ is the Jacobian, or functional determinant

$$\begin{vmatrix} \frac{d\varphi}{dx}, & \frac{d\varphi}{dy}, & \frac{d\varphi}{dz} \\ \frac{df}{dx}, & \frac{df}{dy}, & \frac{df}{dz} \\ \frac{d\tau}{dx}, & \frac{d\tau}{dy}, & \frac{d\tau}{dz} \end{vmatrix}, =\frac{d(\varphi, f, \tau)}{d(x, y, z)};$$

and we hence have

$$d\omega = \frac{-\tau\delta\varphi}{J(\varphi, f, \tau)}.$$

23. The two theorems thus become

$$\Sigma\,(x,\,y,\,z)^{n-3}\frac{\tau\delta\varphi}{J(\varphi,f,\tau)} = 0\,,$$

$$\Sigma\frac{(x,\,y,\,z)_{12}^{n-2}}{012}\cdot\frac{-\tau\delta\varphi}{J(\varphi,f,\tau)} = -\frac{\delta\varphi_1}{\varphi_1}+\frac{\delta\varphi_2}{\varphi_2}.$$

But further, if in the first equation we write $\tau = z$, and in the second equation we retain τ, using it to denote the linear function 012, the equations become

$$\Sigma\,(x,\,y,\,z)^{n-3}\frac{z\delta\varphi}{J(\varphi,f)} = 0\,;$$

$$\Sigma\,(x,\,y,\,z)_{12}^{n-2}\cdot\frac{-\delta\varphi}{J(\varphi,f,\tau)} = -\frac{\delta\varphi_1}{\varphi_1}+\frac{\delta\varphi_2}{\varphi_2}\,;$$

where in the first equation $J(\phi,\,f)$ denotes the Jacobian

$$\left|\begin{matrix}\dfrac{d\varphi}{dx}, & \dfrac{d\varphi}{dy} \\ \dfrac{df}{dx}, & \dfrac{df}{dy}\end{matrix}\right|, = \frac{d(\varphi,f)}{d(x,y)}.$$

In these equations the only differential symbol is the δ affecting the coefficients a, b, ... of ϕ, ϕ_1, ϕ_2; the equations are in respect to the coördinates $(x,\,y,\,z)$ of the several variable intersections of the two curves, purely algebraical equations, which are in fact given by Jacobi's Fraction-theorem about to be explained. But for the further reduction of the affected theorem I interpose the next article.

The Coordinates $(\rho,\,\sigma,\,\tau)$. Art. Nos. 24 to 26.

24. The letter τ has just been used to denote the determinant 012: there is often occasion to consider three points 1, 2, 3 coordinates (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) respectively; and then writing ρ, σ, τ to denote the determinants 023, 031, 012 respectively, and Δ the determinant 123, we have identically

$$\begin{aligned}\Delta x &= x_1\rho + x_2\sigma + x_3\tau,\\ \Delta y &= y_1\rho + y_2\sigma + y_3\tau,\\ \Delta z &= z_1\rho + z_2\sigma + z_3\tau,\end{aligned}$$

which equations, regarding therein the point 0, coordinates $(x,\,y,\,z)$, as a current point, are in fact equations for transformation from the coordinates $(x,\,y,\,z)$ to the coordinates ρ, σ, τ belonging to the triangle of reference 123. The points

1 and 2 have been already taken to be on the fixed curve, and it will be assumed that 3 is also a point on this curve.

25. If the function f, which equated to zero gives the equation of the fixed curve, be transformed to the new coordinates (ρ, σ, τ), the coefficients of the transformed function are polar functions, each divided by ∇^n, viz. the coefficient of ρ^n is $\frac{1}{\nabla^n} 1^n$, which by reason that 1 is a point on the curve is $= 0$ (and similarly the coefficients of σ^n and of τ^n are each $= 0$), the coefficient of $\rho^{n-1}\sigma$ is $= \frac{1}{\nabla^n} n.1^{n-1}2$; that of $\rho^{n-1}\tau$ is $= \frac{1}{\nabla^n} n.1^{n-1}3$; that of $\rho^{n-2}\sigma^2$ is $= \frac{1}{\nabla^n} \frac{1}{2} n(n-1) 1^{n-2}2^2$; and so in other cases. I write this in the form

$$f = \frac{1}{\nabla^n} (1^n = 0, \ldots \between \rho, \sigma, \tau)^n;$$

or we might also use the symbolic form

$$f = \frac{1}{\Delta^n} (\rho 1 + \sigma 2 + \tau 3)^n.$$

The terms independent of τ contain, it is clear, the factor $\rho\sigma$, and separating these terms from the others, the equation may be written

$$f = \frac{1}{\nabla^n} \rho\sigma (n.1^{n-1}2, \ldots \dagger \between \rho, \sigma)^{n-2} + \&c.\,\tau.$$

26. The equations $\tau = 0$, $(\ldots \dagger \between \rho, \sigma)^{n-2} = 0$, determine the residues of the points 1, 2, and hence the major function $(x, y, z)_{12}^{n-2}$, expressed in terms of ρ, σ, τ, and writing therein $\tau = 0$, must reduce itself to $(\ldots \dagger \between \rho, \sigma)^{n-2}$ into a constant factor which is at once found to be $= \frac{1}{\nabla^{n-2}}$; for taking the current point at 1 we have $(\rho, \sigma, \tau) = (\Delta, 0, 0)$, and the corresponding value of the major function is thus $\frac{1}{\nabla^{n-2}} n.1^{n-1}2.\Delta^{n-2}$, $= n.1^{n-2}2$, as it ought to be. We have thus

$$(x, y, z)_{12}^{n-2} = \frac{1}{\nabla^{n-2}} (n.1^{n-1}2, \ldots \dagger \between \rho, \sigma)^{n-2} + \&c.\,\tau;$$

and we hence see that for $\tau = 0$,

$$(x, y, z)_{12}^{n-2} = \frac{\Delta^2 f}{\rho\sigma},$$

an equation which will be useful.

The Preparation for the Affected Theorem Resumed. Art. No. 27.

27. In the affected theorem instead of (x, y, z) we introduce the new coordinates (ρ, σ, τ). We have

$$J(\phi, f, \tau) = \frac{d(\varphi, f, \tau)}{d(\rho, \sigma, \tau)} \frac{d(\rho, \sigma, \tau)}{d(x, y, z)},$$

where the first factor is $=\frac{d(\varphi, f)}{d(\rho, \sigma)}$, say that this is $J(\phi, f)$ the Jacobian in regard to ρ, σ, and the second factor is at once found to be $=\Delta^2$. We have consequently

$$\frac{1}{J(\varphi, f, \tau)}=\frac{1}{\Delta^2 J(\varphi, f)},$$

and the equation for the affected theorem becomes

$$\Sigma\,(x,\,y,\,z)_{12}^{n-2}\,\frac{\delta\varphi}{J(\varphi, f)}=-\Delta^2\Big(-\frac{\delta\varphi_1}{\varphi_1}+\frac{\delta\varphi_2}{\varphi_2}\Big),$$

where $(x, y, z)_{12}^{n-2}$ is to be regarded as standing for its value in terms of (ρ, σ, τ).

Jacobi's Fraction Theorem. Art. Nos. 28 to 31.

28. This is the extension of a well-known theorem, which, in a somewhat disguised form, may be thus written: viz. if U be any rational and integral function $(x, 1)^m$, then we have

$$\frac{1}{U}=\Sigma\,\frac{1}{x-x'.J(U')},$$

or introducing an arbitrary constant A by the equation $AU=X$, say this is

$$\frac{A}{X}\Big(=\frac{1}{U}\Big)=\Sigma\,\frac{1}{x-x'.J(U')},$$

where U' is the same function $(x', 1)^m$ of x' that U is of x: $J(U'),=\frac{dU'}{dx'}$ is the Jacobian of U', and the summation extends to all roots x' of the equation $U'=0$: obviously this is nothing else than the formula for the decomposition of $\frac{1}{U}$ into simple fractions.

29. Take now $U=(x, y, 1)^m$, $V=(x, y, 1)^n$, functions of x, y of the degrees m and n respectively, and assume

$$\begin{aligned} AU+BV&=X, \text{ a function } (x, 1)^{mn}, \\ CU+DV&=Y, \quad \text{"} \quad (y, 1)^{mn}, \end{aligned}$$

viz. let $X=0$ and $Y=0$ be the equations obtained by elimination from $U=0$ and $V=0$ of the y and the x respectively. The forms are

$$\begin{aligned} A&=(x, y^{n-1}, 1)^{mn-m}, \quad & B&=(x, y^{m-1}, 1)^{mn-n}, \\ C&=(x^{n-1}, y, 1)^{mn-m}, \quad & D&=(x^{m-1}, y, 1)^{mn-n}, \end{aligned}$$

where these equations denote the first of them that A is a rational and integral function of the degree $mn-m$ in x and y jointly, but only of the degree $n-1$ in y: and so for the other equations. It follows that

$$AD-BC=(x^{mn-1}, y^{mn-1}, 1)^{2mn-m-n}.$$

The theorem now is

$$\frac{AD-BC}{XY}=\Sigma\frac{1}{x-x'.y-y'.J(U',\,V')},$$

where U', V' are the same functions of $(x',\,y')$ that U, V are of $(x,\,y)$; $J(U',\,V')$ is the Jacobian $\dfrac{d(U',\,V')}{d(x',\,y')}$; and the summation extends to all the simultaneous roots x', y' of the equations $U=0$, $V=0$.

30. For the proof, observe that $AD-BC$ is a sum of terms of the form $x^\alpha y^\beta$ where α and β are each of them at most $=mn-1$; hence X being of the degree mn we have $\dfrac{x^\alpha}{X}=$ a sum of fractions $\dfrac{L}{x-x'}$, where x' is any root of $X=0$; and similarly $\dfrac{y^\beta}{Y}=$ a sum of fractions $\dfrac{M}{y-y'}$, where y' is any root of $Y=0$; multiplying the two expressions and taking the sum for the several terms $\lambda x^\alpha y^\beta$ of $AD-BC$ we obtain a formula

$$\frac{AD-BC}{XY}=\Sigma\frac{K}{x-x'.y-y'},$$

where the summation extends to all the combinations of the mn values of x' with the mn values of y. But such a formula existing, the coefficients K may be determined in the usual manner, viz. multiplying by XY and then writing $x=x'$, $y=y'$, there is on the right-hand only one term which does not vanish, and we find

$$(AD-BC)_{x'y'}=K\left(\frac{X}{x-x'}\right)_{x'}\left(\frac{Y}{y-y'}\right)_{y'},=K\left(\frac{dX}{dx}\,\frac{dY}{dy}\right)_{x'y'},$$

where the factor which multiplies K does not vanish.

We distinguish the cases where $(x',\,y')$ are corresponding or non-corresponding roots of $X=0$, $Y=0$; viz. corresponding roots are those for which $U=0$, $V=0$, but for non-corresponding roots these equations do not hold good; there are obviously mn pairs of corresponding roots.

In the latter case $(AD-BC)\,U=DX-BY$; $(AD-BC)\,V=-CX+AY$, and since for the values in question X, Y each vanish, but U, V do not each of them vanish, we must for these values have $AD-BC=0$, and the foregoing equation for K gives then $K=0$.

31. The formula thus is

$$\frac{AD-BC}{XY}=\Sigma\frac{K}{x-x'.y-y'},$$

where the summation now extends only to corresponding roots x', y', for which we have $U=0$, $V=0$. We have for K the foregoing expression, which, to complete the determination, we write under the form

$$AD-BC=KJ(X,\ Y)_{x'y'};$$

this is allowable, for $J(X,\ Y),\ =\dfrac{d(X,\ Y)}{d(x,\ y)}$, differs from $\dfrac{dX}{dx}\dfrac{dY}{dy}$ only by the zero term $-\dfrac{dX}{dy}\dfrac{dY}{dx}$. Moreover, differentiating the expressions for X, Y, and considering $(x,\ y)$ as therein standing for a pair of corresponding roots $(x',\ y')$; the terms containing U, V will all vanish; we thus in effect differentiate as if A, B, C, D were constants, and the result is $(AD-BC)\ J(U,\ V)$, or say this is $(AD-BC)_{x'y'}\,J(U',\ V')$: hence, in the equation for K, the factor $(AD-BC)_{x'y'}$ divides out, and we have $1=KJ(U',\ V')$; hence the required formula is

$$\frac{AD-BC}{XY}=\Sigma\,\frac{1}{x-x'.\,y-y'}\,.\,J(U',\ V')$$

the summation extending to all the simultaneous roots $(x',\ y')$ of $U=0$, $V=0$.

Homogeneous Form of the Fraction Theorem. Art. Nos. 32 and 33.

32. For x, y, x', y' we write $\dfrac{x}{z},\ \dfrac{y}{z},\ \dfrac{x'}{z'},\ \dfrac{y'}{z'}$; supposing that U, V now denote homogeneous functions $(x,\ y,\ z)^m$, $(x,\ y,\ z)^n$, and that we have

$$\begin{aligned} AU+BV&=X,=(x,\ z)^{mn},=\alpha x^{mn}+\ldots\\ CU+DV&=Y,=(y,\ z)^{mn},=\beta y^{mn}+\ldots \end{aligned}$$

where the forms are

$$\begin{gathered} A=(x,\ y^{n-1},\ z)^{mn-m},\quad B=(x,\ y^{m-1},\ z)^{mn-n},\\ C=(x^{n-1},\ y,\ z)^{mn-m},\quad D=(x^{m-1},\ y,\ z)^{mn-n},\\ AD-BC=(x^{mn-1},\ y^{mn-1},\ z)^{2mn-m-n}, \end{gathered}$$

(viz. the degree of A in $(x,\ y,\ z)$ is $=mn-m$, but y rises only to the degree $n-1$; and so in other cases); then the theorem becomes

$$\frac{z^{m+n-2}(AD-BC)}{XY}=\Sigma\,\frac{z'^{m+n}}{xz'-x'z.\,yz'-y'z}\,.\,J(U',\ V'),$$

where $J(U',\ V')$ denotes the Jacobian $\dfrac{d(U',\ V')}{d(x',\ y')}$, and the summation extends to the simultaneous roots $(x',\ y',\ z')$ of $U=0$, $V=0$.

32. It is proper to introduce into the formula τ', an arbitrary linear function $ax' + by' + cz'$ of (x', y', z'): observe that in the Jacobian, (x', y', z') have always values for which $U' = 0$, $V' = 0$: we have therefore

$$x'\frac{dU'}{dx'} + y'\frac{dU'}{dy'} + z'\frac{dU'}{dz'} = 0,$$
$$x'\frac{dV'}{dx'} + y'\frac{dV'}{dy'} + z'\frac{dV'}{dz'} = 0,$$

and thence

$$x' : y' : z' = \frac{d(U', V')}{d(y', z')} : \frac{d(U', V')}{d(z', x')} : \frac{d(U', V')}{d(x', y')},$$

and if the expressions on the right-hand are for a moment called A', B', C', then writing $\tau' = ax' + by' + cz'$, we have $J(U', V', \tau') = aA' + bB' + cC'$, $= \frac{\tau'}{z'}C'$, $= \frac{\tau'}{z'}J(U', V')$, that is $\frac{1}{J(U', V')} = \frac{\tau'}{z'J(U', V', \tau')}$, or the equation becomes

$$\frac{z^{m+n-2}(AD - BC)}{XY} = \Sigma \frac{z'^{m+n-1}\tau'}{xz' - x'z \,.\, yz' - y'z} \cdot J(U', V', \tau'),$$

the summation being as before.

Resulting Special Theorems. Art. Nos. 33–35

33. Reverting to the Cartesian form, we have

$$\frac{xy(AD - BC)}{XY} = \Sigma \frac{1}{J(U', V')}\Big(1 + \frac{x'}{x} + \ldots\Big)\Big(1 + \frac{y'}{y} + \ldots\Big),$$
$$= \Sigma \frac{1}{J(U', V')}\Big\{1 + H_1\Big(\frac{x'}{x}, \frac{y'}{y}\Big) + H_2\Big(\frac{x'}{x}, \frac{y'}{y}\Big) + \ldots\Big\}$$

where H_m is the homogeneous sum of the order m, $H_1(u, v) = u + v$, $H_2(u, v) = u^2 + uv + v^2$, &c.

The left-hand side is

$$(AD - BC)\Big(\frac{1}{\alpha x^{mn-1}} + \frac{\&c.}{x^{mn}} \ldots\Big)\Big(\frac{1}{\beta y^{mn-1}} + \frac{\&c.}{y^{mn}} + \ldots\Big)$$

and in $AD - BC$ the terms of highest order in (x, y), say $(AD - BC)_0$ are $(AD - BC)_0 = (xy)^{mn-m-n+1}(a, b \ldots k\,)\!(\,x, y)^{m+n-2}$.

There is thus on the left-hand no term which is in (x, y) of a higher degree than $-(m + n - 2)$; hence on the right-hand every term of a higher degree than this in (x, y) must vanish, viz. we must have

$$0 = \Sigma \frac{x'^{\alpha}y'^{\beta}}{J(U', V')} \text{ so long as } \alpha + \beta \ngtr m + n - 3,$$

or what is the same thing, we must have

$$0 = \Sigma \frac{(x', y', 1)^{m+n-3}}{J(U', V')} \qquad (m+n-3) \text{ theorem.}$$

where $(x', y', 1)^{m+n-3}$ is the arbitrary function of the degree $m+n-3$.

34. Passing to the next lower degree $-(m+n-2)$ we have

$$\frac{1}{\alpha\beta(xy)^{m+n-2}}(a, b, \ldots k \dagger \between x, y)^{m+n-2} = \Sigma \frac{1}{J(U', V')} H_{m+n-2}\left(\frac{x'}{x}, \frac{y'}{y}\right)$$

and if in $(a, b, \ldots k \dagger \between x, y)^{m+n-2}$ we consider any term $gx^{m+n-2-p}y^{m+n-2-q}$, where $p+q=m+n-2$, then we have on the left-hand the term $\frac{g}{\alpha\beta x^p y^q}$, and the corresponding term on the right-hand must be $\Sigma \frac{1}{J(U', V')} \frac{x'^p y'^q}{x^p y^q}$; that is we have

$$\frac{g}{\alpha\beta} = \Sigma \frac{x'^p y'^q}{J(U', V')}.$$

But from the foregoing expression for $(AD-BC)_0$ it appears that $(AD-BC)_0$ contains the term $gx^{mn-1-p}y^{mn-1-q}$, and it hence appears that g is the constant term of the quotient $(AD-BC)_0$ divided by $x^{mn-1-p}y^{mn-1-q}$, or as this may be written

$$g = \text{const. of } \frac{(AD-BC)_0 x^p y^q}{(xy)^{mn-1}}$$

and comparing the two values of g we obtain

$$\text{const. of } \frac{(AD-BC)_0 x^p y^q}{\alpha\beta(xy)^{mn-1}} = \Sigma \frac{x'^p y'^q}{J(U', V')}, \quad (p+q=m+n-2),$$

and we hence derive

$$\text{Const. of } \frac{(AD-BC)_0 (x, y)^{m+n-2}}{\alpha\beta(xy)^{mn-1}} = \Sigma \frac{(x', y')^{m+n-2}}{J(U', V')},$$

where $(x, y)^{m+n-2}$ is the general function of the degree $m+n-2$, and, of course, $(x', y')^{m+n-2}$ is the same function of x', y'. The two functions may be written $(x, y, 0)^{m+n-2}$ and $(x', y', 0)^{m+n-2}$, and this being so we may on the right-hand write instead $(x', y', 1)^{m+n-2}$, for, by so doing we introduce in the numerator of the fraction new terms of an order not exceeding $m+n-3$, and by the $(m+n-3)$ theorem already obtained the sum Σ of the quotient of such terms by $J(U', V')$ is $=0$. We thus have

$$\text{Const. of } \frac{(AD-BC)_0 (x, y, 0)^{m+n-2}}{\alpha\beta(xy)^{mn-1}} = \Sigma \frac{(x', y', 1)^{m+n-2}}{J(U', V')}, \quad (m+n-2) \text{ theorem.}$$

where $(x', y', 1)^{m+n-2}$ is the general non-homogeneous function of the degree $m+n-2$, and $(x, y, 0)^{m+n-2}$ is obtained from it by attending only to the terms of the highest degree $m+n-2$, and therein substituting x, y for x', y'.

35. We may, it is clear, in the equations for the $(m+n-3)$ and for the $(m+n-2)$ theorems respectively, omit the accents on the right-hand sides; doing this, and moreover in each equation transposing the two sides, the two special theorems are

$$\Sigma\frac{(x, y, 1)^{m+n-3}}{J(U, V)}=0, \qquad (m+n-3) \text{ theorem.}$$

$$\Sigma\frac{(x, y, 1)^{m+n-2}}{J(U, V)}=\text{const. of }\frac{(AD-BC)_0(x, y, 0)^{m+n-2}}{\alpha\beta(xy)^{mn-1}}. \qquad (m+n-2) \text{ theorem.}$$

Homogeneous Form of the Special Theorems. Art. No. 36.

36. Writing $\frac{x}{z}$, $\frac{y}{z}$ for x, y, and introducing as before the arbitrary linear function $\tau=ax+by+cz$, we at once obtain, U, V being now homogeneous functions $(x, y, z)^m$ and $(x, y, z)^n$ respectively, and the A, B, C, D being also homogeneous functions accordingly,

$$\Sigma\frac{z(x, y, z)^{m+n-3}}{J(U, V)}=0, \qquad (m+n-3) \text{ theorem.}$$

$$\Sigma\frac{\tau(x, y, z)^{m+n-2}}{zJ(U, V, \tau)}=\text{const. of }\frac{(AD-BC)_0(x, y, 0)^{m+n-2}}{\alpha\beta(xy)^{mn-1}}, \quad (m+n-2) \text{ theorem.}$$

where the suffix 0 denotes that we are in $AD-BC$ to write $z=0$.

If in the last formula we change throughout the letters x, y, z into ρ, σ, τ (that is, consider U, V as given functions of ρ, σ, τ), but retain τ as standing for the particular function $0\rho+0\sigma+1\tau$, then the formula becomes

$$\Sigma\frac{(\rho, \sigma, \tau)^{m+n-2}}{\bar{J}(U, V)}=\text{const. of }\frac{(AD-BC)_0(\rho, \sigma, \tau)^{m+n-2}}{\alpha\beta(\rho\sigma)^{mn-1}}, \quad (m+n-2) \text{ theorem.}$$

where $\bar{J}(U, V)$ denotes $\frac{d(U, V)}{d(\rho, \sigma)}$, the Jacobian in regard to ρ, σ.

The effect of dps of the Curves $U=0$, $V=0$. Art. Nos. 37 and 38.

37. We must, in regard to the foregoing special theorems, consider the effect of any dps of the curves $U=0$, $V=0$.

Suppose one of the curves, say V, has a dp, but that the other curve U does not pass through it; the dp is not an intersection of U, V, and the theorems are in nowise affected.

If U passes through the dp then the dp counts twice among the intersections of U, V; at the dp we have $J(U', V')=0$, and (to fix the ideas attending to the $(m+n-3)$ theorem) the sum $\Sigma\frac{(x, y, z)^{m+n-3}}{J(U, V)}$ will contain two

infinite terms; these may very well, and indeed (assuming that the theorem remains true) must have a finite sum, but except by the theorem itself, this finite sum is not calculable, and the theorem thus becomes nugatory.

If, however, the curve $(x, y, z)^{m+n-3} = 0$ be a curve passing through the dp, then considering, instead, the case where the last-mentioned curve and U each approach indefinitely near to the dp of V; there are two intersections of U, V indefinitely near to each other and to the dp; at either intersection, the numerator $(x', y', z')^{m+n-3}$ and the denominator $J(U, V)$ are infinitesimals of the same order, say the first, and the fraction has a finite value; the finite values for the two intersections have not in general a zero sum, and consequently in the limit it would not be allowable to disregard the intersections belonging to the dp.

38. But if the numerator curve $(x, y, z)^{m+n-3} = 0$ passes twice through the dp (that is, has there a dp), then reverting to the two consecutive intersections, at either of these the denominator $J(U, V)$ is as before an infinitesimal of the first order, but the numerator $(x, y, z)^{m+n-3}$ is an infinitesimal of the second order, and in the limit the value of the fraction is $= 0$; we may in this case disregard the intersections belonging to the dp; and so in general, the curve $(x, y, z)^{m+n-3} = 0$ passing twice through each dp of U which lies upon V, we have

$$\Sigma \frac{z(x, y, z)^{m+n-3}}{J(U, V)} = 0,$$

the summation now extending to all the intersections of U, V other than the dps in question, which are to be disregarded. And the like in regard to the other theorem

$$\Sigma \frac{(x, y, z)^{m+n-2}}{J(U, V)} = \text{const. of } \frac{(AD - BC)_0 (x, y, 0)^{m+n-2}}{\alpha\beta (xy)^{mn-1}}.$$

The Pure Theorem.—Completion of the Proof. Art. No. 39.

39. The theorem was reduced to

$$\Sigma \frac{z(x, y, z)^{n-3} \delta\varphi}{J(\varphi, f)} = 0,$$

which is therefore the equation to be proved.

The $(m + n - 3)$ theorem, writing therein ϕ, f in place of U, V respectively (the degrees being as before m and n), is

$$\Sigma \frac{z(x, y, z)^{m+n-3}}{J(U, V)} = 0.$$

$(x, y, z)^{m+n-3}$ is here an arbitrary function of the degree $m+n-3$, and this may therefore be put $=(x, y, z)^{n-3}\delta\phi$, where $\delta\phi=(da, \ldots \between x, y, z)^m$, is a function of the degree m; and since the curve $\phi=0$ passes always through the dps of f, and varies subject to this condition, the curve $\delta\phi=0$ will also pass through the dps; hence taking $(x, y, z)^{n-3}=0$ a curve through the dps, the curve $(x, y, z)^{n-3}\delta\phi=0$ will be a curve passing twice through each of dps, and the $(m+n-3)$ theorem thus gives the equation which was to be proved. This completes the proof of the pure theorem

$$\Sigma\,(x, y, z)^{n-3}d\omega=0.$$

The Affected Theorem.—Completion of the Proof. Art. Nos. 40 and 41.

40. The theorem was reduced to

$$\Sigma\frac{(x, y, z)_{12}^{n-2}\delta\varphi}{J(\varphi, f)}=-\Delta^2\left(-\frac{\delta\varphi_1}{\varphi_1}+\frac{\delta\varphi_2}{\varphi_2}\right),$$

which is therefore the equation to be proved.

The $(m+n-2)$ theorem, written with (ρ, σ, τ) in place of (x, y, z), and putting therein ϕ, f for U, V, is

$$\Sigma\frac{(\rho, \sigma, \tau)^{m+n-2}}{J(\varphi, f)}=\text{const. of}\ \frac{(AD-BC)_0(\rho, \sigma, 0)^{m+n-2}}{\alpha\beta(\rho\sigma)^{mn-1}},$$

where it will be recollected that the suffix (0) denotes that τ is to be put $=0$. $(\rho, \sigma, \tau)^{m+n-2}$ is here an arbitrary function of the degree $m+n-2$, and this may therefore be put $=(x, y, z)_{12}^{n-2}\delta\phi$, the two factors being each of them considered as expressed in terms of (ρ, σ, τ); and since each of the curves $(x, y, z)_{12}^{n-2}=0$ and $\delta\phi=0$ passes through the dps of f, the curve $(x, y, z)_{12}^{n-2}\delta\phi=0$, is a curve passing twice through each of the dps. We have therefore

$$\Sigma\frac{(x, y, z)_{12}^{n-2}\delta\varphi}{J(\varphi, f)}=\text{const. of}\ \frac{(AD-BC)_0(x, y, z)_{12}^{n-2}d\varphi_0}{\alpha\beta(\rho\sigma)^{mn-1}},$$

where on the right-hand side $(x, y, z)_{12}^{n-2}$ is considered as a function of ρ, σ, τ, and we are to put therein $\tau=0$; it has been seen (No. 26) that the value is $=\dfrac{f_0\Delta^2}{\rho\sigma}$, where f_0 is what f considered as a function of ρ, σ, τ becomes on writing therein $\tau=0$; the right-hand side thus becomes

$$=\text{const. of}\ \frac{(AD-BC)_0 f_0\Delta^2\delta\varphi_0}{\alpha\beta(\rho\sigma)^{mn}}$$

41. But for $\tau=0$ we have

$$A_0\phi_0+B_0f_0=\alpha\rho^{mn},$$
$$C_0\phi_0+D_0f_0=\beta\sigma^{mn},$$

and hence $\qquad (AD-BC)_0 f_0 = A_0\beta\sigma^{mn} - C_0\alpha\rho^{mn}$,
and the right-hand side thus becomes, say

$$-\Delta^2 \text{ const. of } \left(-\frac{A_0}{\alpha\rho^{mn}} + \frac{C_0}{\beta\sigma^{mn}}\right)\delta\phi_0.$$

But in calculating the constant of $\frac{A_0}{\alpha\rho^{mn}}\delta\phi_0$, we may suppose not only $\tau=0$, but also $\sigma=0$: we then have $\phi_0=(x,y,z)^m, =\left(\frac{\rho}{\Delta}\right)^m(x_1,y_1,z_1)^m, =\left(\frac{\rho}{\Delta}\right)^m\phi_1$, and hence also $\delta\phi_0=\left(\frac{\rho}{\Delta}\right)^m\phi_1$.

Similarly in calculating the constant of $\frac{B_0}{\beta\sigma^{mn}}\delta\phi_0$, we may suppose not only $\tau=0$, but also $\rho=0$, we then have $\phi_0=(x,y,z)^m, =\left(\frac{\sigma}{\Delta}\right)^m(x_2,y_2,z_2)^m, =\left(\frac{\sigma}{\Delta}\right)^m\phi_2$, and hence $\delta\phi_0=\left(\frac{\sigma}{\Delta}\right)^m\delta\phi_2$.

Moreover, in the equations

$$A_0\phi_0+B_0f_0=\alpha\rho^{mn},$$
$$C_0\phi_0+D_0f_0=\beta\sigma^{mn},$$

writing in the first equation $\sigma=0$, we find $A_0\left(\frac{\rho}{\Delta}\right)^m\phi_1=\alpha\rho^{mn}$, that is $\frac{A_0}{\alpha\rho^{mn}} = \left(\frac{\Delta}{\rho}\right)^m\frac{1}{\varphi_1}$; and similarly writing in the second equation $\rho=0$, we find $C_0\left(\frac{\sigma}{\Delta}\right)^m\phi_2=\beta\sigma^{mn}$, that is $\frac{C_0}{\beta\sigma^{mn}}=\left(\frac{\Delta}{\sigma}\right)^m\frac{1}{\varphi_2}$: and the expression thus becomes

$$=-\Delta^2\left(-\frac{\delta\varphi_1}{\varphi_1}+\frac{\delta\varphi_2}{\varphi_2}\right),$$

giving the equation which was to be proved. This completes the proof of the affected theorem

$$\Sigma\frac{(x,y,z)_{12}^{n-2}d\omega}{012} = -\frac{\delta\varphi_1}{\varphi_1}+\frac{\delta\varphi_2}{\varphi_2}.$$

CHAPTER III. THE MAJOR FUNCTION $(x, y, z)_{12}^{n-2}$.

Analytical Expression of the Function. Art. Nos. 42 to 49.

42. The function has been defined by the conditions that the curve $(x,y,z)_{12}^{n-2}=0$, shall pass through the dps, and also through the $n-2$ residues of the parametric points 1, 2: and moreover, that on writing therein (x_1, y_1, z_1) for (x,y,z), the function shall become $=n.1^{n-1}2$. Obviously the function is not completely determined: calling it Ω (or when required Ω_{12}), then if Ω' be any particular form of it, the general form is $\Omega=\Omega'+(x,y,z)^{n-3}.012$, where

$(x, y, z)^{n-3}$ is the general minor function (viz. $(x, y, z)^{n-3}=0$ is a curve passing through the dps): the major function thus contains $\frac{1}{2}(n-1)(n-2)-\delta$, $=p$, arbitrary constants.

Agreeing with the definition we have the before-mentioned equation

$$\Omega=\frac{1}{\Delta^{n-2}}(n.1^{n-1}2, \ldots \dagger \rangle \rho, \sigma)^{n-2}+\&c. \tau,$$

viz. from this expression for Ω it appears that the curve $\Omega=0$ meets the line through 1, 2 in the $n-2$ residues of these points, and moreover, for $(x, y, z) =(x_1, y_1, z_1)$ and therefore $(\rho, \sigma, \tau)=(\Delta, 0, 0)$, the value of Ω is $=n.1^{n-1}2$.

43. We can without difficulty write down an equation determining Ω' as a function $(x, y, z)^{n-2}$, which on putting therein $\tau=0$, becomes equal to the foregoing expression $\frac{1}{\Delta^{n-2}}(n.1^{n-1}2, \ldots \dagger \rangle \rho, \sigma)^{n-2}$, and which is moreover such that the curve $\Omega'=0$ passes through the dps; which being so, we have as before, $\Omega=\Omega'+(x, y, z)^{n-3}.012$, for the general value of Ω.

To fix the ideas, consider the particular case $n=4$, the fixed curve a quartic: Ω', on putting therein $\tau=0$, should become

$$=\frac{1}{\Delta^2}(4.1^32,\ 6.1^22^2,\ 4.12^3 \dagger \rangle \rho, \sigma)^2;$$

and it is to be shown that this will be the case if we determine Ω' a quadric function of (x, y, z) by the equation

$$\left|\begin{array}{ll} (x,\ y,\ z)^2 & ,\ \Omega' \\ 1\,(x_1, y_1, z_1)^2 & ,\ 4.1^32 \\ 2\,(x_1, y_1, z_1 \rangle x_2, y_2, z_2), & 6.1^22^2 \\ 1\,(x_2, y_2, z_2)^2 & ,\ 4.12^3 \\ a,\ b,\ c,\ f,\ g,\ h & ,\ 0 \\ \cdot & \\ \cdot & \end{array}\right| =0,$$

where the left-hand side is a determinant of seven lines and columns, the top line being x^2, y^2, z^2, $2yz$, $2zx$, $2xy$, Ω' and similarly for the second line; the third line is $2x_1x_2$, $2y_1y_2$, $2z_1z_2$, $2(y_1z_2+y_2z_2)$, $2(z_2x_1+z_1x_2)$, $2(x_1y_2+x_2y_1)$, 6.1^22^2, and in each of the last three lines we have six arbitrary constants followed by a 0. The equation is of the form $\square+M\Omega'=0$, where $\square$ is a quadric function $(x, y, z)^2$, and M is a constant factor.

44. If the quartic curve has a dp, suppose at the point α, coordinates $(x_\alpha, y_\alpha, z_\alpha)$, then in order that the curve $\Omega'=0$ may pass through the dp, we must for one of the last three lines substitute $(x_\alpha, y_\alpha, z_\alpha)^2$, 0; and so for any

other dp or dps of the quartic curve. And the conditions as to the dp or dps (if any) being satisfied in this manner, we may if we please, taking $(x_\beta, y_\beta, z_\beta)$ as the coordinates of an arbitrary point β (not of necessity on the fixed curve), write any line not already so expressed, of the last three lines, in the form $(x_\beta, y_\beta, z_\beta)^2, 0$; the effect being to make the curve $\Omega'=0$ pass through the arbitrary point β.

45. To show that the equation on putting therein $\tau=0$ does in fact give the required value, $\Omega'=\frac{1}{\Delta^2}(4.1^32,\ 6.1^22^2,\ 4.12^3\dagger\!\!\between\rho,\ \sigma)^2, =\Phi$ suppose, it is to be observed that effecting a linear substitution upon the first six columns, the equation may be written

$$\left|\begin{array}{ll} (\rho,\ \sigma,\ \tau)^2 & ,\ \Omega' \\ 1\,(\rho_1,\ \sigma_1,\ \tau_1)^2 & ,\ 4.1^32 \\ 2\,(\rho_1,\ \sigma_1,\ \tau_1\between\rho_2,\ \sigma_2,\ \tau_2), & 6.1^22^2 \\ 1\,(\rho_2,\ \sigma_2,\ \tau_2)^2 & ,\ 4.12^3 \\ a',\ b',\ c',\ f',\ g',\ h' & ,\ 0 \end{array}\right| = 0,$$

where $(\rho_1, \sigma_1, \tau_1)$, $(\rho_2, \sigma_2, \tau_2)$ are what (ρ, σ, τ) become on writing therein for (x, y, z) the values (x_1, y_1, z_1) and (x_2, y_2, z_2) respectively; viz. we have $(\rho_1, \sigma_1, \tau_1)=(\Delta, 0, 0)$; $(\rho_2, \sigma_2, \tau_2)=(0, \Delta, 0)$; the equation thus is

$$\left|\begin{array}{ccccccc} \rho^2, & \sigma^2, & \tau^2, & 2\sigma\tau, & 2\tau\rho, & 2\rho\sigma, & \Omega' \\ \Delta^2, & 0, & 0, & 0, & 0, & 0, & 4.1^32 \\ 0, & 0, & 0, & 0, & 0, & 2\Delta^2, & 6.1^22^2 \\ 0, & \Delta^2, & 0, & 0, & 0, & 0, & 4.12^3 \\ a', & b', & c', & f', & g', & h', & 0 \\ \cdot & & & & & & \\ \cdot & & & & & & \end{array}\right| = 0,$$

and then by another linear substitution upon the columns, the last column can be changed into $\Omega'-\Phi, 0, 0, 0, 0, 0, 0$; whence writing $\tau=0$, the equation becomes

$$\left|\begin{array}{ccccccc} \rho^2, & \sigma^2, & 0, & 0, & 0, & 2\rho\sigma, & \Omega'-\Phi \\ \Delta^2, & 0, & 0, & 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0, & 0, & 2\Delta^2, & 0 \\ 0, & \Delta^2, & 0, & 0, & 0, & 0, & 0 \\ a', & b', & c', & f', & g', & h', & 0 \\ \cdot & & & & & & \\ \cdot & & & & & & \end{array}\right| = 0,$$

or, omitting a constant factor, it is

$$\begin{vmatrix} \rho^2, & \sigma^2, & 2\rho\sigma, & \Omega'-\Phi \\ \Delta^2, & 0, & 0, & 0 \\ 0, & 0, & 2\Delta^2, & 0 \\ 0, & \Delta^2, & 0, & 0 \end{vmatrix} = 0,$$

that is $\Omega'-\Phi=0$, or $\Omega'=\Phi$, $=\frac{1}{\Delta^2}(4.1^3 2,\ 6.1^2 2^2,\ 4.12^3 \dagger \between \rho,\ \sigma)^2$, the required value.

46. Considering the equation for Ω' as expressed in the before-mentioned form $\square + M\Omega' = 0$, the value of the constant factor M is

$$M = \begin{vmatrix} (x_1,\ y_1,\ z_1)^2 \\ (x_1,\ y_1,\ z_1 \between x_2,\ y_2,\ z_2) \\ (x_2,\ y_2,\ z_2)^2 \\ a,\ b,\ c,\ f,\ g,\ h, \\ . \\ . \end{vmatrix};$$

or if instead of each line such as $a,\ b,\ c,\ f,\ g,\ h$, we have a line $(x_\alpha,\ y_\alpha,\ z_\alpha)^2$ then we have

$$M = \begin{vmatrix} (x_1,\ y_1,\ z_1)^2 \\ (x_1,\ y_1,\ z_1 \between x_2,\ y_2,\ z_2) \\ (x_2,\ y_2,\ z_2)^2 \\ (x_\alpha,\ y_\alpha,\ z_\alpha)^2 \\ (x_\beta,\ y_\beta,\ z_\beta)^2 \\ (x_\gamma,\ y_\gamma,\ z_\gamma)^2 \end{vmatrix},$$

a value which is

$$= \begin{vmatrix} x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \\ x_\alpha, & y_\alpha, & z_\alpha \end{vmatrix} \begin{vmatrix} x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \\ x_\beta, & y_\beta, & z_\beta \end{vmatrix} \begin{vmatrix} x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \\ x_\gamma, & y_\gamma, & z_\gamma \end{vmatrix} \begin{vmatrix} x_\alpha, & y_\alpha, & z_\alpha \\ x_\beta, & y_\beta, & z_\beta \\ x_\gamma, & y_\gamma, & z_\gamma \end{vmatrix},$$

or say this is $= 12\alpha . 12\beta . 12\gamma . \alpha\beta\gamma$.

47. It is obvious that the foregoing process is applicable to the general case of the fixed curve of the order n with δ dps, and gives always Ω', by an equation of the foregoing form $\square + M\Omega' = 0$, where $\square$ is a function $(x,\ y,\ z)^{n-2}$ of the coordinates, and M is a constant factor. Supposing that in the deter-

minant for Ω', each of the lower lines is written in the form $(x_\alpha, y_\alpha, z_\alpha)^{n-2}$, 0, the number of the points α is $=\frac{1}{2}(n-1)(n-2)$, viz. these are the δ dps, and $\frac{1}{2}(n-1)(n-2)-\delta$, $=p$, other points α. The general expression of M is $M=12\alpha.12\beta\ldots(\alpha^{n-3}\beta^{n-3}\ldots)$, viz. equating to zero a factor such as 12α, this expresses that the point α is on the line 12; but equating to zero the last factor $(\alpha^{n-3}\beta^{n-3}\ldots)$, this expresses that the several points α, viz. the dps and the p other points α, are on a curve of the order $n-3$.

48. Preceding the case $n=4$, above considered, we have, of course, the case $n=3$, $\delta=0$, the fixed curve a cubic; the equation for Ω' is here

$$\begin{vmatrix} x, & y, & z, & \Omega' \\ x_1, & y_1, & z_1, & 3.1^22 \\ x_2, & y_2, & z_2, & 3.12^2 \\ x_\alpha, & y_\alpha, & z_\alpha, & \end{vmatrix} = 0,$$

giving

$$\tfrac{1}{3}\Omega' = \frac{1^22.02\alpha + 12^2.0\alpha1}{12\alpha},$$

or if we write herein 3 for α, this is

$$\tfrac{1}{3}\Omega' = \frac{1^22.023 + 12^2.031}{123},$$

and we have hence the general form

$$\tfrac{1}{3}\Omega = \frac{1^22.023 + 12^2.031}{123} + K.012,$$

where K is an arbitrary constant.

49. There is, however, a more simple particular solution $\frac{1}{3}\Omega' =$ polar function 012 ($f=x^3+y^3+z^3$, then $012 = xx_1x_2 + yy_1y_2 + zz_1z_2$), which, to avoid a confusion of notation, we may write $=\widetilde{012}$. We at once verify this, for expressing the coordinates (x, y, z) in terms of (ρ, σ, τ) we have $\frac{1}{3}\Omega' = \widetilde{012}$; $=\frac{1}{\Delta}(1^22.\rho + 12^2.\sigma + \widetilde{123}.\tau)$, which, for $\tau=0$ becomes $=\frac{1}{\Delta}\{1^22.\rho + 12^2.\sigma\}$.

We must, of course, have an identity of the form

$$\widetilde{012} = \frac{1^22.023 + 12^2.031}{123} + K.012,$$

and to find K, writing here $0=3$, we have $K = \frac{\widetilde{123}}{123}$, or we have the identity

$$123\,\widetilde{012} - \widetilde{123}\,012 = 1^22.023 + 12^2.031.$$

Single Letter Notation for the Polar Functions of the Cubic.
Art. Nos. 50 nd 51.

50. The notation of single letters for the polar functions is not much required in the case of the cubic, but, in the next following case of the quartic it can hardly be dispensed with, and I therefore establish it in the case of the cubic: viz. I write

$$2^23,\ 3^21,\ 1^22 = f,\ g,\ h,\quad 23^2,\ 31^2,\ 12^2 = i,\ j,\ k;\ \widetilde{123} = l,$$

or what is the same thing, the expression for the cubic function f, in terms of $\rho,\ \sigma,\ \tau$ is

$$\Delta^3.f = 3h\rho^2\sigma + 3j\rho^2\tau + 3k\rho\sigma^2 + 6l\rho\sigma\tau + 3g\rho\tau^2 + 3f\sigma^2\tau + 3i\sigma\tau^2;$$

an equation, which writing 0^3 instead of f, may also be written

$$\Delta^3.0^3 = (3h,\ 3j,\ 3k,\ 6l,\ 3g,\ 3f,\ 3i)(\rho^2\sigma,\ \rho^2\tau,\ \rho\sigma^2,\ \rho\sigma\tau,\ \rho\tau^2,\ \sigma^2\tau,\ \sigma\tau^2),$$

and I join to it the series of equations

$$\begin{array}{ll}
\Delta^2.0^21 = (0,\ 2h,\ 2j,\ k,\ 2l,\ g)(\rho^2,\ \rho\sigma,\ \rho\tau,\ \sigma^2,\ \sigma\tau,\ \tau^2), \\
\text{''}\ \ 0^22 = (h,\ 2k,\ 2l,\ 0,\ 2f,\ i)(\quad\text{''}\quad), \\
\text{''}\ \ 0^23 = (j,\ 2l,\ 2g,\ f,\ 2i,\ 0)(\quad\text{''}\quad), \\
\Delta.01^2 = (0,\ h,\ j)(\rho,\ \sigma,\ \tau), \\
\Delta\,\widetilde{012} = (h,\ k,\ l)(\ \text{''}\ ), \\
\text{''}\ \ \widetilde{013} = (j,\ l,\ g)(\ \text{''}\ ), \\
\text{''}\ \ 02^2 = (k,\ 0,\ f)(\ \text{''}\ ), \\
\text{''}\ \ \widetilde{023} = (l,\ f,\ i)(\ \text{''}\ ), \\
\text{''}\ \ 03^2 = (g,\ i,\ 0)(\ \text{''}\ ).
\end{array}$$

51. In particular we have $\Delta.\widetilde{012} = h\rho + k\sigma + l\tau$, and the above-mentioned identity $123\ \widetilde{012} - \widetilde{123}\ 012 = 1^22.023 + 12^2.031$ is simply $h\rho + k\sigma + l\tau - l\tau = h\rho + k\sigma$.

Single Letter Notation for the Polar Functions of the Quartic. Art. No. 52.

52. I write here

$$2^33,\ 3^31,\ 1^32 = f,\ g,\ h;\quad 23^3,\ 31^3,\quad 12^3 = i,\ j,\ k;$$
$$1^223,\ 12^23,\ 123^2 = l,\ m,\ n;\quad 2^23^2,\ 3^21^2,\ 1^22^2 = p,\ q,\ r;$$

so that the expression for the quartic function f in terms of $\rho,\ \sigma,\ \tau$ is

$$\begin{aligned}\Delta^4.f = {} & 4h\rho^3\sigma + 4j\rho^3\tau + 6p\rho^2\sigma^2 + 12l\rho^2\sigma\tau + 6q\rho^2\tau^2 \\ & + 4k\rho\sigma^3 + 12m\rho\sigma^2\tau + 12n\rho\sigma\tau^2 + 4g\rho\tau^3 + 4f\sigma^3\tau + 6r\sigma^2\tau^2 + 4i\sigma\tau^3,\end{aligned}$$

which, putting 0^4 for f, may also be written

$$\begin{aligned}\Delta^4.0^4 = {} & (4h,\ 4j;\ 6p,\ 12l,\ 6q:\ 4k,\ 12m,\ 12n,\ 4g:\ 4f,\ 6r,\ 4i\dagger) \\ & (\rho^3\sigma,\ \rho^3\tau;\ \rho^2\sigma^2,\ \rho^2\sigma\tau,\ \rho^2\tau^2,\ \rho\sigma^3,\ \rho\sigma^2\tau,\ \rho\sigma\tau^2,\ \rho\tau^3,\ \sigma^3\tau,\ \sigma^2\tau^2;\ \sigma\tau^3),\end{aligned}$$

and I join to it the series of equations

$$
\begin{array}{ll}
\Delta^3.0^3 1 & = (0;\ 3h,\ 3j;\ 3r,\ 6l,\ 3q;\ k;\ 3m,\ 3n,\ g \between \rho^3,\ \rho^2\sigma,\ \rho^2\tau,\ \rho\sigma^2,\ \rho\sigma\tau,\ \rho\tau^2,\ \sigma^3,\ \sigma^2\tau,\ \sigma\tau^2,\ \tau^3), \\
\text{''}\ \ 0^3 2 & = (h;\ 3r,\ 3l;\ 3k,\ 6m,\ 3n;\ 0,\ 3f,\ 3p,\ i \between \quad \text{''} \quad), \\
\text{''}\ \ 0^3 3 & = (j;\ 3l,\ 3q;\ 3n,\ 6n,\ 3g;\ f,\ 3p,\ 3i,\ 0 \between \quad \text{''} \quad), \\
\Delta^2.0^2 1^2 & = (0;\ 2h,\ 2j;\ r,\ 2l,\ q \between \rho^2;\ \rho\sigma,\ \rho\tau;\ \sigma^2,\ \sigma\tau,\ \tau^2), \\
\text{''}\ \ 0^2 12 & = (h;\ 2r,\ 2l;\ k,\ 2m,\ n \between \quad \text{''} \quad), \\
\text{''}\ \ 0^2 13 & = (j;\ 2l,\ 2q;\ m,\ 2n,\ g \between \quad \text{''} \quad), \\
\text{''}\ \ 0^2 2^2 & = (r;\ 2k,\ 2m;\ 0,\ 2f,\ p \between \quad \text{''} \quad), \\
\text{''}\ \ 0^2 23 & = (l;\ 2m,\ 2n;\ f,\ 2p,\ i \between \quad \text{''} \quad), \\
\text{''}\ \ 0^2 3^2 & = (q;\ 2n,\ 2g;\ p,\ 2i,\ 0 \between \quad \text{''} \quad), \\
\Delta.01^3 & = (0,\ h,\ j \between \rho,\ \sigma,\ \tau), \\
\text{''}\ \ 01^2 2 & = (h,\ r,\ l \between \quad \text{''} \quad), \\
\text{''}\ \ 01^2 3 & = (j,\ l,\ q \between \quad \text{''} \quad), \\
\text{''}\ \ 012^2 & = (r,\ k,\ m, \between \quad \text{''} \quad), \\
\text{''}\ \ 0123 & = (l,\ m,\ n \between \quad \text{''} \quad), \\
\text{''}\ \ 013^2 & = (q,\ n,\ g \between \quad \text{''} \quad), \\
\text{''}\ \ 02^3 & = (k,\ 0,\ f \between \quad \text{''} \quad), \\
\text{''}\ \ 02^2 3 & = (m,\ f,\ p \between \quad \text{''} \quad), \\
\text{''}\ \ 023^2 & = (n,\ p,\ i \between \quad \text{''} \quad), \\
\text{''}\ \ 03^3 & = (g,\ i,\ 0 \between \quad \text{''} \quad),
\end{array}
$$

which will be convenient in the sequel.

Major Function—The Fixed Curve a Cubic. Art. No. 53.

53. It has been already seen that a simple particular form is $\frac{1}{3}\,\Omega' = \widetilde{012}$: and that the general form is $\Omega = \Omega' + K.012$.

Major Function—The Fixed Curve a Quartic. Art. No. 54.

54. It is to be shown that a particular form is

$$\tfrac{1}{2}\,\Omega' = \frac{-01^3.02^3 + 01^2 2.012^2 + 0^2 12.1^2 2^2}{1^2 2^2}.$$

In fact by the foregoing values of $\Delta.01^3$, etc., the numerator of this expression, multiplied by Δ^2 is $=$

$$
\begin{aligned}
& -(h\sigma + j\tau)(k\rho + f\tau) \\
& + (h\rho + r\sigma + l\tau)(r\rho + k\sigma + m\tau) \\
& + r(h\rho^2 + 2r\rho\sigma + 2l\rho\tau + k\sigma^2 + 2m\sigma\tau + n\tau^2)
\end{aligned}
$$

which is

$$= 2hr\rho^2 + 3r^2\rho\sigma + (hm - jk + 3lr)\rho\tau$$
$$+ 2kr\sigma^2 + (-fh + kl + 3mr)\sigma\tau + (-fj + lm + nr)\tau^2$$

and this for $\tau = 0$, becomes

$$= r(2h\rho^2 + 3r\rho\sigma + 2k\sigma^2).$$

Hence for $\tau = 0$, we have

$$\Omega' = \frac{1}{\varDelta^2}(4h\rho^2 + 6r\rho\sigma + 4k\sigma^2), \text{ that is}$$
$$= \frac{1}{\varDelta^2}\{4.1^3 2.\rho^2 + 6.1^2 2^2.\rho\sigma + 4.12^3.\sigma^2\}.$$

and Ω' is thus a form of the major function $(x, y, z)^2_{12}$. Of course the general form is $\Omega = \Omega' + (x, y, z)^1.012^1$.

Syzygy of the Major Function. Art. No. 55.

55. Writing now $(x, y, z)^{n-2}_{12} = \Omega_{12}$; and taking on the fixed curve a new point 3, consider the like functions Ω_{23} and Ω_{31}: it is to be shown that we have identically

$\Omega_{23}.031.012 + \Omega_{31}.012.023 + \Omega_{12}.023.031 - (123)^2 f = 023.031.012\,(x, y, z)^{n-3}$, where $(x, y, z)^{n-3}$ is a *properly determined* minor function. Or considering herein 0 as a point on the fixed curve and writing therefore $f = 0$, the equation is

$$\frac{\varOmega_{23}}{023} + \frac{\varOmega_{31}}{031} + \frac{\varOmega_{12}}{012} = (x, y, z)^{n-3}. \quad \text{(See footnote*.)}$$

56. Write for a moment $X = \Omega_{23}.031.012 + \Omega_{31}.012.023 + \Omega_{12}.023.031$, then k being an arbitrary coefficient, we have $X - kf = 0$, a curve of the order n, passing through the points 1, 2, 3, and also through the residues of 2, 3, the residues of 3, 1, and the residues of 1, 2; in fact at the point 1 we have $012 = 0$, $031 = 0$, and therefore $X = 0$; also $f = 0$; and therefore 1 is a point on the curve. Again at any residue of 2, 3 we have $\Omega_{23} = 0$, $023 = 0$, and therefore $X = 0$; also $f = 0$; and hence the residue of 2, 3 is a point on the curve.

It is next to be shown that k can be so determined that the curve $X - kf = 0$ shall have a dp at each of the points 1, 2, 3. Supposing this to be so, we have the line 23 meeting the curve $X - kf = 0$ in the points 2 and 3, each counting twice, and in the $n - 2$ residues of 2, 3, that is in $n + 2$ points; hence the curve $X - kf = 0$ must contain as part of itself the line 23, and

* This is the differential theorem corresponding to C. and G.'s integral theorem, p. 26, viz. this is $S\xi\eta + S\eta\zeta + S\zeta\xi = I$, a sum of three integrals of the third kind = an integral of the first kind.

similarly it must contain as part of itself each of the other lines 31 and 12, viz. we shall then have $X - kf = 023.031.012.(x, y, z)^{n-3}$; and from this equation observing that the curves $\Omega_{23}=0$, $\Omega_{31}=0$, $\Omega_{12}=0$ each pass through the dps, it follows that the curve $(x, y, z)^{n-3}=0$ also passes through the dps; hence, k being found to be $=(123)^2$, the theorem will be proved.

57. Taking an arbitrary point α coordinates $(x_\alpha, y_\alpha, z_\alpha)$, and writing $D = x_\alpha \frac{d}{dx} + y_\alpha \frac{d}{dy} + z_\alpha \frac{d}{dz}$ we have to find k, so that the curve $D(X - kf) = 0$ shall pass through the point 1. Observing that $D023 = \alpha 23$, etc., we have

$$\begin{aligned} D(X-kf) = & D\Omega_{23}.031.012 + \Omega_{23}(031.\alpha 12 + \alpha 31.012) \\ & + \alpha 23\,(\Omega_{31}.012 + \Omega_{12}.031) \\ & + 023\{\Omega_{31}.\alpha 12 + \Omega_{12}.\alpha 31 + D\Omega_{31}.012 + D\Omega_{12}.031\} \\ & - kDf, \end{aligned}$$

and, to make the curve pass through 1, writing herein $0=1$, we have

$$0 = 123\,(\Omega_{31}^1.\alpha 12 + \Omega_{12}^1.\alpha 31) - k(Df)^1,$$

where the superfix (1) denotes that we are in Ω_{31}, Ω_{12} and Df respectively to write $0=1$. We have $\Omega_{31}^1 = n.1^{n-1}3$, $\Omega_{12}^1 = n.1^{n-1}2$, $(Df)^1 = n.1^{n-1}\alpha$, and the equation thus is

$$n.123\,(1^{n-1}3.\alpha 12 + 1^{n-1}2.\alpha 31) - kn.1^{n-1}\alpha = 0.$$

But we have identically $1^{n-1}1.\alpha 23 + 1^{n-1}2.\alpha 31 + 1^{n-1}3.\alpha 12 = 1^{n-1}\alpha.123$, where $1^{n-1}1, = 1^n$ is in fact $=0$; the factor $1^{n-1}\alpha$ thus divides out, and the equation becomes $k=(123)^2$; viz. k having this value, the curve $X - kf = 0$ will have a dp at 1; and clearly by symmetry, it will also have a dp at 2, and at 3; the theorem is thus proved.

The Syzygy, Fixed Curve a Cubic. Art. No. 58.

58. The syzygy may be verified independently in the case where the fixed curve is a cubic. Observe that the syzygy, if satisfied for any particular form of Ω will be generally satisfied; we may therefore take $\frac{1}{3}\Omega_{12} = \widetilde{012}$.

Writing then $$\frac{\frac{1}{3}\Omega_{12}}{012} = \frac{\widetilde{012}}{012}, = \{012\} \text{ suppose,}$$

and taking 0 to be a point on the cubic curve, we ought to have $\{023\} + \{031\} + \{012\} =$ a constant; the value of this constant comes out to be $=\{123\}$, and the syzygy in its complete form thus is

$$\{023\} + \{031\} + \{012\} = \{123\}.$$

We have

$$\Delta\widetilde{023},\ \Delta\widetilde{031},\ \Delta\widetilde{012} = l\rho + f\sigma + i\tau,\ j\rho + l\sigma + g\tau,\ h\rho + k\sigma + l\tau,$$

and the equation thus is

$$\frac{l\rho + f\sigma + i\tau}{\rho} + \frac{j\rho + l\sigma + h\tau}{\sigma} + \frac{h\rho + k\sigma + l\tau}{\tau} - l = 0\,;$$

this, multiplied by $\rho\sigma\tau$ becomes

$$h\rho^2\sigma + j\rho^2\tau + k\rho\sigma^2 + 2l\rho\sigma\tau + g\rho\tau^2 + f\sigma^2\tau + i\sigma\tau^2 = 0,$$

which is in fact $\frac{1}{3}f = 0$, the equation of the cubic curve.

Observe that the new symbol $\{012\}$ is in virtue of its determinant denominator, an alternate function, $\{012\} = -\{102\}$, $\{012\} = \{120\} = \{201\}$. The syzygy is a relation between any four points $1, 2, 3, 0$ of the curve, and it may be also expressed in the form

$$\{123\} - \{230\} + \{301\} - \{012\} = 0.$$

The Syzygy, Fixed Curve a Quartic. Art. No. 59.

59. Taking Ω_{12} as before, we have

$$\frac{\frac{1}{2}\Omega_{12}}{012} = \frac{-01^3.02^3 + 01^2 2.012^2 + 0^2 12.1^2 2^2}{012.1^2 2^2} = \{0^2 12\} \text{ suppose}:$$

and then taking 0 to be a point on the quartic curve, we ought to have

$$\{0^2 23\} + \{0^2 31\} + \{0^2 12\} = (x, y, z)^1 \text{ a linear function of } (x, y, z),$$

or what is the same thing, considering the left-hand side as expressed in terms of ρ, σ, τ, the sum should be

$$= (\rho, \sigma, \tau)^1, \text{ a linear function of } (\rho, \sigma, \tau).$$

By a preceding formula we have

$$\{0^2 12\} = \frac{1}{\Delta^2 r\tau}\{2hr\rho^2 + 3r^2\rho\sigma + (hm - jk + 3lr)\rho\tau$$
$$+ 2kr\sigma^2 + (-fh + kl + 3mr)\sigma\tau + (-fj + lm + nr)\tau^2\},$$

which is

$$= \frac{1}{\Delta^2}\left\{\left(3l + \frac{hm - jk}{r}\right)\rho + \left(3m + \frac{-fh + kl}{r}\right)\sigma + \left(n + \frac{-fj + lm}{r}\right)\tau\right\}$$
$$+ \frac{1}{\Delta^2}\frac{2h\rho^2 + 3r\rho\sigma + 2k\sigma^2}{\tau}.$$

And hence forming the sum $\{0^2 23\} + \{0^2 31\} + \{0^2 12\}$, we have first a fractional part which is found to be integral, viz. this is

$$\frac{1}{\Delta^2}\left\{\frac{2f\sigma^2 + 3p\sigma\tau + 2i\tau^2}{\rho} + \frac{2g\tau^2 + 3q\tau\rho + 2j\rho^2}{\sigma} + \frac{2h\rho^2 + 3r\rho\sigma + 2k\sigma^2}{\tau}\right\},$$

$$=\frac{1}{\Delta^2\rho\sigma\tau}\{2h\rho^3\sigma+2j\rho^3\tau+3r\rho^2\sigma^2+3q\rho^2\tau^2+2k\rho\sigma^3+2g\rho\tau^3+2f\sigma^3\tau+3p\sigma^2\tau^2+2i\sigma\tau^3\},$$

$$=\frac{1}{\Delta^2\rho\sigma\tau}\{\tfrac{1}{2}\Delta^4 f-6l\rho^2\sigma\tau-6m\rho\sigma^2\tau-6n\rho\sigma\tau^2\},$$

or since $f=0$, this is $\quad =\frac{1}{\Delta^2}(-6l\rho-6m\sigma-6n\tau),$

and then integral terms which are at once deduced from the above integral terms of $0^2 12$; and collecting the several terms we find

$$\{0^2 23\}+\{0^2 31\}+\{0^2 12\}=$$

$$\frac{1}{\Delta^2}\left\{\rho\left(l+\frac{mn-gk}{p}+\frac{jn-gh}{q}+\frac{hm-jk}{r}\right)+\sigma\left(m+\frac{fn-ik}{p}+\frac{ln-hi}{q}+\frac{kl-fh}{r}\right)\right.$$
$$\left.+\tau\left(n+\frac{lm-fg}{p}+\frac{gl-ij}{q}+\frac{lm-fj}{r}\right)\right\}$$

which is the required result.

Preparation for the Conversion—The Symbol ∂. Art. Nos. 60 to 63.

60. I use ∂ as the symbol of a quasi-differentiation, viz., U being any function of (x, y, z), ∂U denotes $\frac{1}{d\omega}$ into the differential $\frac{dU}{dx}dx+\frac{dU}{dy}dy+\frac{dU}{dz}dz$; in such a differential the increments dx, dy, dz do not in general present themselves in the combinations $ydz-zdy$, $zdx-xdz$, $xdy-ydx$; but they will do so if U is a function of the degree zero in the coordinates x, y, z (that is, if U be the quotient of two homogeneous functions of the same degree); and this being so, we can by the equations

$$\frac{ydz-zdy}{\frac{df}{dx}}=\frac{zdx-xdz}{\frac{df}{dy}}=\frac{xdy-ydx}{\frac{df}{dz}},=d\omega$$

get rid of the increments, and ∂U will denote a function of (x, y, z) derived in a definite manner from the function U; the symbol ∂ will be used only in the case in question of a function of the degree zero. Of course ∂_1 will denote the like operation in regard to (x_1, y_1, z_1); and so ∂_2, etc.; and we may for greater clearness write ∂_0 in place of ∂.

61. Consider then $\partial\frac{P}{Q}$, where P, Q are functions $(x, y, z)^m$ of the same degree, we have

$$\partial\frac{P}{Q}=\frac{1}{Q^2 d\omega}(QdP-PdQ),$$

and then

$$dP=\frac{dP}{dx}dx+\frac{dP}{dy}dy+\frac{dP}{dz}dz,\quad \frac{1}{m}P=\frac{dP}{dx}x+\frac{dP}{dy}y+\frac{dP}{dz}z,$$

with the like formulæ for Q. Substituting, we find

$$\partial\frac{P}{Q}=\frac{1}{mQ^2d\omega}\left\{\frac{d(Q,P)}{d(y,z)}(ydz-zdy)+\frac{d(Q,P)}{d(z,x)}(zdx-xdz)+\frac{d(Q,P)}{d(x,y)}(xdy-ydx)\right\},$$

that is

$$\partial\frac{P}{Q}=\frac{1}{mQ^2}\left\{\frac{df}{dx}\,\frac{d(Q,P)}{d(y,z)}+\text{\&c.}\right\}=\frac{1}{mQ^2}\,\frac{d(f,Q,P)}{d(x,y,z)},\ =\frac{1}{mQ^2}J(f,\ Q,\ P),$$

or say

$$\partial\frac{P}{Q}=-\frac{1}{mQ^2}J(P,\ Q,f).$$

62. As an example consider

$$\partial\{\widetilde{012}\},\ =-\frac{1}{(012)^2}J(\widetilde{012},\ 012,f).$$

The determinant is

$$\left|\begin{matrix}\dfrac{d}{dx}\widetilde{012}, & y_1z_2-y_2z_1, & \dfrac{df}{dx}\\ \dfrac{d}{dy}\widetilde{012}, & z_1x_2-z_2x_1, & \dfrac{df}{dy}\\ \dfrac{d}{dz}\widetilde{012}, & x_1y_2-x_2y_1, & \dfrac{df}{dz}\end{matrix}\right|,$$

and the coefficient herein of $\dfrac{d}{dx}\widetilde{012}$ is $(z_1x_2-z_2x_1)\dfrac{df}{dz}-(x_1y_2-x_2y_1)\dfrac{df}{dy}$, which is

$$=x_2\left(x_1\frac{df}{dx}+y_1\frac{df}{dy}+z_1\frac{df}{dz}\right)-x_1\left(x_2\frac{df}{dx}+y_2\frac{df}{dy}+z_2\frac{df}{dz}\right),\ =3\,(0^21.x_2-0^22.x_1);$$

and so for the other terms.

The determinant is thus

$$=3\left[0^21\left(x_2\frac{d}{dx}+y_2\frac{d}{dy}+z_2\frac{d}{dz}\right)-0^22\left(x_1\frac{d}{dx}+y_1\frac{d}{dy}+z_1\frac{d}{dz}\right)\right]\widetilde{012}$$

say this is

$$=3\,[0^21.\mathbb{D}_{\binom{2}{0}}-0^22\,\mathbb{D}_{\binom{1}{0}}]\,\widetilde{012}.$$

But we have $\mathbb{D}_{\binom{2}{0}}\widetilde{012}=12^2$, $\mathbb{D}_{\binom{1}{0}}\widetilde{012}=1^22$, and the determinant is then $=3\,(0^21.12^2-0^22.1^22)$; whence finally writing ∂_0 instead of ∂

$$\partial_0\{012\}=-3.\frac{0^21.12^2-0^22.1^22}{(012)^2}.$$

63. By cyclical interchange of the 0, 1, 2, we have

$$\partial_1\{012\}=-3.\frac{1^22.0^22-01^2.02^2}{(012)^2},$$

$$\partial_2\{012\}=-3.\frac{02^2.01^2-12^2.0^21}{(012)^2};$$

and thence adding, we find

$$(\partial_0+\partial_1+\partial_2)\{012\}=0,$$

an important property, which joined to the equation before obtained,

$$\{023\}+\{031\}+\{012\}=\{123\},$$

completes the theory of the function $\{012\}$.

Conversion of the Major Function (Interchange of Limits and Parametric Points). Art. No. 64.

64. Write in general

$$\frac{(x,\,y,\,z)_{12}^{n-2}}{012}=Q_{0,12},$$

$Q_{0,12}$ is an alternate function in regard to the points $1, 2\,(Q_{0,12}=-Q_{0,21})$, and it is in regard to the coordinates of the points $0, 1, 2$, rational, but not integral, of the degrees $n-3,\ 0,\ 0$ respectively: it can therefore be operated upon with ∂_1 or ∂_2, but (except in the case $n=3$) not with ∂_0.

The conversion relates not to the general major function $(x, y, z)_{12}^{n-2}$, but to this function *with the arbitrary constants properly determined,* and consists in a relation between two functions $Q_{4,12}$ and $Q_{1,45}$ (each of them a function of three out of four arbitrary points $1, 2, 4, 5$ on the fixed curve), viz. the conversion is

$$\partial_1 Q_{4,12}=\partial_4 Q_{1,45},$$

an equation which may be written in four different forms, viz. we may in the form written down interchange $1, 2$ and also $4, 5$.*

The determination of the constants is a very peculiar one, inasmuch as it is not algebraical, viz. in the case of the cubic curve, about to be considered, it appears that $Q_{0,12}$ contains the term $\int_2^1 d\omega\,\partial_3\{036\}$, which is a transcendental function of the coordinates of the parametric points 1 and 2.

The Conversion, Fixed Curve a Cubic. Art. No. 65.

65. We may write $Q_{0,12}=\{012\}+K$, where K is a constant, that is, it is independent of the point 0, but depends on the parametric points 1 and 2. I assume K to be properly determined, and give an *à posteriori* verification of the

* The meaning of the property is better seen from the integral form: $Q_{0,12}$ is a function of the points $0, 1, 2$ and $Q_{0,45}$ the like function of the points $0, 4, 5$ such that $\int_5^4 d\omega\, Q_{0,12}=\int_2^1 d\omega\, Q_{0,45}$: which equation operated upon with $\partial_1\partial_4$ gives the formula of the text. And there is thus the meaning (alluded to in the heading) that there exists for the integral of the third kind a canonical form (C. and G.'s endliche Normalform), such that the integral is not altered by the interchange of the limits and the parametric points. The expression for $Q_{0,12}$ mentioned further on in the text for the case, fixed curve a cubic, shows that in this case the canonical form of the integral of the third kind is $\int^4 d\omega\Big[\{012\}+\Big(\int_2^1 d\omega\,\partial_3\{036\}-\{123\}\Big)\Big]$.

equation $\partial_1 Q_{4,12} = \partial_4 Q_{1,45}$. The value is $K = \int_2^1 d\omega\, \partial_3\{036\} - \{123\}$, where 3, 6 are arbitrary points on the cubic curve, and where in the definite integral, regarded as an integral $\int Udu$ with a current variable u, the meaning is that this variable has at the limits the values u_1, u_2 which belong to the points 1 and 2 respectively: a fuller explanation might be proper, but the investigation will presently be given in a form not depending on any integral at all.

Substituting for K its value we have

$$Q_{0,12} = \{012\} + \left[\int_2^1 d\omega \partial_3\{036\} - \{123\}\right],$$

or as this may also be written

$$= -\{023\} - \{031\} + \int_2^1 d\omega \partial_3\{036\}.$$

We have thence

$$\partial_1 Q_{0,12} = -\partial_1\{031\} + \partial_3\{136\},$$

and consequently

$$\partial_1 Q_{4,12} = -\partial_1\{431\} + \partial_3\{136\},$$
$$\partial_4 Q_{1,45} = -\partial_4\{134\} + \partial_3\{436\},$$

and hence observing that $\{431\} = -\{134\}$ &c., we have

$$\partial_1 Q_{4,12} - \partial_4 Q_{1,45} = (\partial_1 + \partial_4)\{134\} + \partial_3\{136\} - \partial_3\{436\},$$
$$= -\partial_3\{134\} + \partial_3\{136\} - \partial_3\{436\},$$

which observing that we have $\partial_3\{641\} = 0$ is

$$= \partial_3(\{136\} - \{364\} + \{641\} - \{413\}), = 0,$$

the required theorem.

To avoid, in the proof, the use of the integral sign, we have only to consider the required function $Q_{0,12}$ as given by the foregoing differential formula

$$\partial_1 Q_{0,12} = -\partial_1\{031\} + \partial_3\{136\},$$

for we have then the values of $\partial_1 Q_{4,12}$ and $\partial_4 Q_{1,45}$, and the rest of the proof the same as before.

The Conversion, Fixed Curve a Quartic. Art. Nos. 66 to 73.

66. We have

$$Q_{0,12} = \{0^2 12\} + (x, y, z)^1,$$

where $(x, y, z)^1$ is a linear function of (x, y, z), but depending also on the parametric points 1 and 2, which is to be determined so as to satisfy the conversion equation

$$\partial_1 Q_{4,12} = \partial_4 Q_{1,45}.$$

Observing that we have $\{0^2 23\}+\{0^2 31\}+\{0^2 12\}=$ a linear function of (x, y, z), the linear function $(x, y, z)^1$ of $Q_{0,12}$ may be taken to be $=\Theta_{0,12}-\{0^2 23\}-\{0^2 31\}-\{0^2 12\}$; that is, we may assume

$$\begin{aligned} Q_{0,12} &= \{0^2 12\}+\Theta_{0,12}-(\{0^2 23\}+\{0^2 31\}+\{0^2 12\}),\\ &= -\{0^2 23\}-\{0^2 31\}+\Theta_{0,12}, \end{aligned}$$

where $\Theta_{0,12}$ is a linear function of (x, y, z), but depending also on the points 1 and 2, which has to be determined. We have

$$\partial_1 Q_{0,12} = -\partial_1\{0^2 31\}+\partial_1\Theta_{0,12},$$

and thence

$$\begin{aligned} \partial_1 Q_{4,12} &= -\partial_1\{4^2 31\}+\partial_1\Theta_{4,12},\\ \partial_4 Q_{1,45} &= -\partial_4\{1^2 34\}+\partial_4\Theta_{1,45}, \end{aligned}$$

giving an equation for Θ,

$$\partial_1\Theta_{4,12}-\partial_4\Theta_{1,45}=\partial_1\{4^2 31\}-\partial_4\{1^2 34\};$$

4 is here an arbitrary point of the quartic, and we may instead of it write 0, the equation thus becomes

$$\partial_1\Theta_{0,12}-\partial_0\Theta_{1,05}=\partial_1\{0^2 31\}-\partial_0\{1^2 30\}.$$

67. Of the terms on the left-hand side, the first is a linear function of (x, y, z), or say it is an integral function 0^1, and the second is a linear function of (x_1, y_1, z_1), or say it is an integral function 1^1: the given function on the right-hand side must therefore admit of expression in the form $\phi(0^1, 1, 3)-\phi(1^1, 0, 3)$, where $\phi(0^1, 1, 3)$ is a known function, integral and linear as regards the coordinates (x, y, z) of the point 0, but depending also on the points 1, 3; and $\phi(1^1, 0, 3)$ is the like known function, integral and linear as regards the coordinates (x_1, y_1, z_1) of the point 1, but depending also on the points 0, 3. Moreover, since 2 and 5 are arbitrary points entering only on the left-hand side, it is clear that $\partial_1\Theta_{0,12}$ must be independent of 2, and $\partial_0\Theta_{1,05}$ independent of 5 [reverting to the cubic case observe that here $\Theta_{0,12}=\int_2^1 d\omega\partial_3\{036\}$, whence $\partial_1\Theta_{1,12}=\partial_3\{136\}$, and so $\partial_0\Theta_{1,05}=\partial_3\{036\}$, and that the corresponding equation thus is $\partial_3\{136\}-\partial_3\{036\}=\partial_1\{031\}-\partial_0\{130\}$, where the left-hand side is $=\partial_3\{013\}$, and the equation itself $(\partial_0+\partial_3+\partial_1)\{031\}=0$]. We then have

$$\partial_1\Theta_{0,12}-\phi(0^1, 1, 3)=\partial_0\Theta_{1,02}-\phi(1^1, 0, 3),$$

where the one side is derived from the other by the interchange of the 0, 1. The solution therefore is

$$\partial_1\Theta_{0,12}-\phi(0^1, 1, 3)=X(\overline{0, 1}, 3),$$

a function symmetrical in regard to the points 0 and 1, and which, inasmuch as the left-hand is an integral function 0^1, must itself be an integral function $(0^1, 1^1)$, that is, integral and linear as regards the coordinates (x, y, z) and (x_1, y_1, z_1) of the points 0 and 1 respectively. We thus have

$$\partial_1\Theta_{0,12} = \phi(0^1, 1, 3) + X(\overline{0, 1}, 3),$$

and thence

$$\partial_2\Theta_{0,12} = -\phi(0^1, 2, 3) - X(\overline{0, 2}, 3),$$

viz. the second of these expressions is with its sign reversed the same function of 2 that the first is of 1.

68. It follows that taking a new symbol 7 for the variable of the definite integral (in the cubic case $\Theta_{0,12}$ was independent of 0, and there was nothing to prevent the use of 0 for the current point of the definite integral), we may write $\Theta_{0,12} = \int_2^1 d\omega_7 P(7, 0, 3)$, where $\partial_1 P(1, 0, 3) = \phi(0^1, 1, 3) + X(\overline{0, 1}, 3)$, an equation which implies $\partial_2 P(0, 2, 3) = \phi(0^1, 2, 3) + X(\overline{0, 2}, 3)$. But the first of these equations in P is nothing else than the first of the equations in $\Theta_{0,12}$.

70. I have succeeded in finding $\phi(0^1, 1, 3)$, but the calculation is a very tedious one, and I give only the principal steps, omitting all details. We have to bring $\partial_1 0^2 13 - \partial_0 1^2 03$ into the form $\phi(0^1, 1, 3) - \phi(1^1, 0, 3)$. From the value of $\{0^2 13\}, = \dfrac{-01^3.03^3 + 01^2 3.013^2 - 0^2 13.1^2 3^2}{013.1^2 3^2}$, we find by a process such as that of No. 62,

$$\begin{aligned}\partial_1\{0^2 13\} = \frac{1}{\sigma^2}\Bigg\{ &-0^2 3^2.01^3 - 0^3 3.j \\ &+\frac{1}{q}\begin{pmatrix} 2.01^3[03^3.01^2 3 - (013^2)^2] \\ +[2.0^2 13.013^2 + 1.0^2 3^2.01^2 3 - 3.0^2 1^2.0^2 3^2]j\end{pmatrix} \\ &+\frac{1}{q^2}\begin{pmatrix} 2.01^3(-01^3.03^3 + 013^2.01^2 3)g \\ -2.013^2(-01^3.03^3 + 013^2.01^2 3)j\end{pmatrix}\Bigg\}.\end{aligned}$$

Substituting herein the values $01^3 = \frac{1}{\Delta}(h\sigma + j\tau)$ &c. we have $\frac{1}{\sigma^2}$ into a cubic function $(\rho, \sigma, \tau)^3$, and writing down first the integral terms, and then the others, we have

$$\begin{aligned}\partial_1\{0^2 13\} = \frac{1}{\Delta^3}\Big\{ &\rho\Big[(-6hn + 5jm) + \frac{1}{q}(4ghl - 3gjr - 2hij + j^2 p + 2jnl) \\ &\qquad + \frac{1}{q^2}(-2g^2 h^2 + 4ghjn - 2j^2 n^2)\Big] \\ &+\sigma\Big[fj - hp + \frac{1}{q}(2hil - 2hn^2 - 3ijr + jlp + 2jmn) \\ &\qquad + \frac{1}{q^2}(2ghln - 2gh^2 i + 2hijn - 2jln^2)\Big]\end{aligned}$$

$$+\tau\Big[3jp+\frac{1}{q}(-2ghn+2gjm-2ijl-2jn^2)$$
$$+\frac{1}{q^2}(2g^2hl-2ghij-2gjln+2ij^2n)\Big]$$

(say this linear function of ρ, σ, τ is $=\Box$).

$$+\frac{1}{\varDelta^3\sigma^2}\{\rho^3.2j^2+\rho^2\sigma(6jl-3hg)+\rho^2\tau.3jq+\rho\sigma\tau(-2gh+6jn)+\rho\tau^2.2gj+\sigma\tau^2.2ij\}.$$

71. The expression of $\partial_0\{1^203\}$ is deduced from this by the interchange of 0, 1: and I write

$$\partial_1\{0^213\}-\partial_0\{1^203\}=\Box-*$$
$$+\frac{1}{\sigma^2\varDelta^3\rho^3}[\rho^3\{\rho^3.2j^2+\rho^2\sigma(-3hq+6jl)+\rho^2\tau.3jq$$
$$+\rho\sigma\tau(-2gh+6jn)+\rho\tau^2.2gj+\sigma\tau^2.2ij\}$$
$$-\Delta^3\{\Delta^3.2(0^23^2)^2-\Delta^2\sigma(-3.0^32.0^23^2+6.0^33.0^223)-\Delta^2\tau.3.0^33.0^23^2$$
$$+\Delta\sigma\tau(-2.03^3.0^32+6.0^33.023^2)+\Delta\tau^2.2.03^3.0^33-\sigma\tau^2.2i.0^33\}],$$

where, and in what follows, the $*$ denotes the function immediately to the left of it, interchanging therein the 0, 1. It will be observed that the $\Box$, *quâ* linear function of (ρ, σ, τ), that is of (x, y, z), is a term of the required function $\phi(0^1, 1, 3)$: the remaining portion has to be reduced by means of the expressions for $\Delta^2(0^23^2)$ etc. in terms of ρ, σ, τ.

72. We obtain $\quad\partial_1\{0^213\}-\partial_0\{1^203\}=\Box-*$

$$+\frac{1}{\varDelta^3}\{\sigma(2fj-3hp-3kq+18lm-9nr)+\tau(-3gr-3jp+9mq)\}$$
$$+\frac{1}{\varDelta^3\rho}\{\sigma^2(12fl-12kn+18m^2-9pr)+\sigma\tau(6fg-6gk-6ir+18mn)+\tau^2.6gm\}$$
$$+\frac{1}{\varDelta^3\rho^2}\{\sigma^3(18fm-9kp)+\sigma^2\tau(12fn-8ik+9mp)+\sigma\tau^2(4fg+6im)\}$$
$$+\frac{1}{\varDelta^3\rho^3}\{\sigma^4.4f^2+\sigma^3\tau.6fp+\sigma^2\tau^2.4fi\}.$$

The terms of the second line may be transformed as follows:

$$\frac{\sigma}{\varDelta^3}(2fj-3hp-3kq+18lm-9nr)$$
$$=\tfrac{1}{2}\frac{\sigma}{\varDelta^3}(2fj-3hp-3kq+18lm-9nr)-*$$
$$+\frac{1}{\varDelta^3\rho}\{\sigma^2(-12fl+12kn-18m^2+9pr)+\sigma\tau(-\tfrac{3}{2}fq+3gk+\tfrac{9}{2}ir-\tfrac{9}{2}lp-9mn)\}$$
$$+\frac{1}{\varDelta^3\rho^2}\{\sigma^3(-30fm+15kp)+\sigma^2\tau(-12fn+12ik-18mp)+\sigma\tau^2.-9np\}$$
$$+\frac{1}{\varDelta^3\rho^3}\{\sigma^4.-10f^2+\sigma^3\tau.-15fp+\sigma^2\tau^2.-9p^2+\sigma\tau^3.-3ip\}.$$

and

$$\frac{\tau}{\varDelta^3}(-3gr-3jp+9mq)$$

$$= \tfrac{1}{2}\frac{\tau}{\Delta^3}(-3gr - 3jp + 9mq) - *$$

$$+ \frac{1}{\Delta^3\rho}\{\sigma\tau(-\tfrac{9}{2}fq + 3gk + \tfrac{3}{2}ir + \tfrac{9}{2}lp - 9mn) + \tau^2.-6gm\}$$

$$+ \frac{1}{\Delta^3\rho^2}\{\sigma^2\tau(-9fn + 3ik) + \sigma\tau^2(-6fg - 6im) + \tau^3.-3gp\}$$

$$+ \frac{1}{\Delta^3\rho^3}\{\sigma^3\tau.-3fp + \sigma^2\tau^2.-6fi + \sigma\tau^3.-3ip\}$$

and substituting these values, the whole third line is destroyed, and we find

$$\partial_1\{0^213\} - \partial_0\{1^203\} = \square - *$$

$$+ \tfrac{1}{2}.\frac{1}{\Delta^3}\{\sigma(2fj - 3hq - 3kj + 18lm - 9nr) + \tau(-3gr - 3jp + 9mq)\} - *$$

$$+ \frac{1}{\Delta^3\rho^2}\{\sigma^3(-12fm + 6kp) + \sigma^2\tau(-9fn + 7ik - 9mp) + \sigma\tau^2(-2fg - 9np) + \tau^3.-3gp\}$$

$$+ \frac{1}{\Delta^3\rho^3}\{\sigma^4.-6f^2 + \sigma^3\tau.-12fp + \sigma^2\tau^2(-2fi - 9p^2) + \sigma\tau^3.-6ip\}.$$

And ultimately the last two lines of this expression are found to be

$$= \frac{1}{\Delta^3}\{\rho(-2hn + 4jm + 2l^2 - 2qr) + \sigma(-2fj + 2hp + 2kq - 10lm + 5nr) \\ + \tau(2gr + hi - jp + 7ln - 4mq)\} - *.$$

so that the whole is now a sum of three linear function of $(\rho, \sigma, \tau). - *$.

73. Collecting the terms, we have

$$\partial_1\{0^213\} - \partial_0\{1^203\} =$$

$$\begin{aligned}\frac{1}{\Delta^3}\Big[\rho\Big\{&(-8hn + 9jm + 2l^2 - 2qr) + \frac{1}{q}(4ghl - 3gjr - 2hij + j^2p + 2jln) \\ &+ \frac{1}{q^2}(-2g^2h^2 + 4ghjn - 2j^2n^2)\Big\} \\ +\sigma\Big\{&\tfrac{1}{2}(-hp + kq - 2lm + nr) + \frac{1}{q}(2hil - 2hn^2 - 3ijr + jlp + 2jmn) \\ &+ \frac{1}{q^2}(-2gh^2i + 2ghln + 2hijn - 2jln^2)\Big\} \\ +\tau\Big\{&\tfrac{1}{2}(gr + 2hi + jp + mq + 14ln) + \frac{1}{q}(-2ghn + 2gjm - 2ijl - 2jn^2) \\ &+ \frac{1}{q^2}(2g^2hl - 2ghij - 2gjln + 2ij^2n)\Big\}\Big]\end{aligned}$$

$$- *.$$

The right-hand side depends on the points 0, 1, 3 and 2: viz. we have therein $\rho = 023$, $\Delta = 123$, etc., but the left-hand side depending on only the points 0, 1 and 3, the right-hand side cannot really contain 2, and it must thus remain unaltered, if for 2 we substitute any other point on the quartic, say 6: the right-hand side may therefore be understood as a function of 0, 1, 3 and 6,

viz. ρ, Δ, f, etc., will mean 063, 163, 6^33, etc.: we have thus $\phi(0^1, 1, 3) =$ the above linear function with 2 thus replaced by 6; say

$$\phi(0^1, 1, 3) = \frac{1}{\varDelta^8}[\rho(\,) + \sigma(\,) + \tau(\,)],$$

a given function of the points 0, 1, 3 and the arbitrary point 6, on the quartic curve; we therefore write it $\phi(0^1, 1, 3, 6)$. There is no obvious value for $X(\overline{0, 1}, 3)$ which will produce any simplification, I therefore take this function to be $= 0$; and the final result is

$$Q_{0,12} = \{0^212\} + \Theta_{0,12} - (\{0^223\} + \{0^231\} + \{0^212\}),$$

where $\Theta_{0,12}$ is a function integral and linear as regards the coördinates (x, y, z) of the point 0, but transcendental as regards the parametric points 1, 2; and containing besides the arbitrary points 3, 6, of the quartic curve, its value being determined by the differential formulæ

$$\partial_1\Theta_{0,12} = \phi(0^1, 1, 3, 6),\ \partial_2\Theta_{0,12} = -\phi(0^1, 2, 3, 6),$$

where $\phi(0^1, 1, 3, 6)$ is a given function as above. I do not see the meaning of the very complicated linear function of (ρ, σ, τ), nor how to reduce it to any form such as the simple one $\partial_3\{036\}$, which presents itself in the case of the cubic curve.

END OF CHAPTER III.

CAMBRIDGE, ENGLAND, *October* 5, 1882.

On the Non-Euclidean Geometry.

By William E. Story.

In volume IV of this Journal, pp. 332–335, I showed how the formulae of any Non-Euclidean plane trigonometry could be deduced from those of the Euclidean spherical trigonometry, namely by the replacement of each side by a certain constant multiple of that side and each angle by a certain constant multiple of that angle. In the present paper I propose to make the corresponding deduction for any Non-Euclidean spherical trigonometry, and incidentally to give a number of other formulæ relating to distances, angles, areas and volumes. Some of these formulæ exhibit an important principle which seems to me to be new, and which may be roughly expressed thus: *the distance* (*or angle*) *between any two geometrical elements* (points, planes or straight lines) *is, to a constant factor près, the same, in whatever way it is measured.* For example, the formulæ will show that the distance of a given point from the nearest point of a given plane is always proportional to the angle between the given plane and the nearest plane (*i. e.* that which makes the least angle with it) through the given point. Again, the least (or greatest) distance from a point of one of two given straight lines to a point of the other is proportional to the least (or greatest) angle which a plane through one of the straight lines makes with a plane through the other; and, if the lines intersect, this is again proportional to the angle between the lines. A similar theorem holds for the least (or greatest) distance between a point and a straight line and between a plane and a straight line, etc. Such greatest or least distances are of such frequent occurrence that it is convenient to speak of them without distinguishing between greatest or least; I therefore call such a distance a max.–min. distance, and later, designate a constant multiple of it, simply as the *distance* of two elements.

I give also expressions for the area of any spherical polygon, the circumference and area of any circle (the former was given by Gauss), the area of the surface and the volume of any sphere, and show that the double plane is identical with a sphere of quasi-infinite radius.

I assume the absolute given in the form

(1) $$\Omega \equiv x^2 + y^2 + z^2 - 1 = 0,$$

in point-coordinates, and

(2) $$\mho \equiv u^2 + v^2 + w^2 - 1 = 0$$

in tangential coordinates. I also write

$$\Omega_{ii} \equiv x_i^2 + y_i^2 + z_i^2 - 1,\ \Omega_{ij} \equiv x_i x_j + y_i y_j + z_i z_j - 1,$$
$$\mho_{ii} \equiv u_i^2 + v_i^2 + w_i^2 - 1,\ \mho_{ij} \equiv u_i u_j + v_i v_j + w_i w_j - 1.$$

As there are three flat elements, namely the point, the plane, and the straight line, there are six species of distance, each having its own unitary constant. Throughout this paper I shall use T, T' to denote points, M, M' planes, L, L' straight lines, and denote the non-Euclidean distance of two elements by the letters which represent them bound by a vinculum. It is rather convenient to designate angles as "distances." Then, for instance, the non-Euclidean distance (angle) between the plane M and the straight line L will be $\overline{ML}$.

In accordance with the principles laid down by Dr. Klein in his paper "Ueber die sogenannte Nicht-Euklidische Geometrie,"* I define the distance between two points to be k times the natural logarithm of the anharmonic ratio of the points with respect to the intersections of the absolute with their junction, and the distance between two planes as k' times their anharmonic ratio with respect to the tangent planes to the absolute through their intersection. According to these definitions, if

$$T = (x_1, y_1, z_1),\ T' = (x_2, y_2, z_2),\ M = (u_1, v_1, w_1),\ M' = (u_2, v_2, w_2),$$

(3) $$\overline{TT'} = k \ln\left(\frac{\Omega_{12} + \sqrt{\Omega_{12}^2 - \Omega_{11}\Omega_{22}}}{\Omega_{12} - \sqrt{\Omega_{12}^2 - \Omega_{11}\Omega_{22}}}\right) = 2ik \cos^{-1}\left(\frac{\Omega_{12}}{\sqrt{\Omega_{11}\Omega_{22}}}\right),$$

(4) $$\overline{MM'} = k' \ln\left(\frac{\mho_{12} + \sqrt{\mho_{12}^2 - \mho_{11}\mho_{22}}}{\mho_{12} - \sqrt{\mho_{12}^2 - \mho_{11}\mho_{22}}}\right) = 2ik' \cos^{-1}\left(\frac{\mho_{12}}{\sqrt{\mho_{11}\mho_{22}}}\right),$$

the expressions as circular functions being those which I shall generally use.

The max.-min. distance of the point $T\ (x_1, y_1, z_1)$ from a point of the plane $M\ (u_1, v_1, w_1)$ is the max.-min. value of

$$2ik \cos^{-1}\left(\frac{\Omega_{12}}{\sqrt{\Omega_{11}\Omega_{22}}}\right)$$

where (x_2, y_2, z_2) satisfy the equation $u_1 x_2 + v_1 y_2 + w_1 z_2 + 1 = 0$;
i. e. x_2, y_2, z_2 are determined by the conditions that

$$\Omega_{22}(x_1 dx_2 + y_1 dy_2 + z_1 dz_2) - \Omega_{12}(x_2 dx_2 + y_2 dy_2 + z_2 dz_2) = 0$$

* Math. Annalen, Vol. IV, pp. 573-625.

shall be satisfied by every set of values of dx_2, dy_2, dz_2 which satisfy $u_1 dx_2 + v_1 dy_2 + w_1 dz_2 = 0$,

hence
$$\Omega_{22} x_1 - \Omega_{12} x_2 = \lambda u_1,$$
$$\Omega_{22} y_1 - \Omega_{12} y_2 = \lambda v_1,$$
$$\Omega_{22} z_1 - \Omega_{12} z_2 = \lambda w_1,$$
and
$$0 = \lambda(u_1 x_2 + v_1 y_2 + w_1 z_2 + 1) = \Omega_{22} - \Omega_{12} + \lambda,$$
$$\lambda = \Omega_{12} - \Omega_{22},$$
$$\lambda(u_1 x_1 + v_1 y_1 + w_1 z_1 + 1) = \Omega_{11}\Omega_{22} - \Omega_{12}^2 + \Omega_{22} - \Omega_{12} + \lambda,$$
$$\lambda \mho_{11} = \Omega_{22}(u_1 x_1 + v_1 y_1 + w_1 z_1 + 1) - \Omega_{22} + \Omega_{12} - \lambda,$$
i. e. writing $M_1 \equiv u_1 x_1 + v_1 y_1 + w_1 z_1 + 1$, we have
$$(\Omega_{12} - \Omega_{22}) M_1 = \Omega_{11}\Omega_{22} - \Omega_{12}^2, \; (\Omega_{12} - \Omega_{22}) \mho_{11} = \Omega_{22} M_1,$$
whence
$$\frac{\Omega_{12}}{\Omega_{22}} = \frac{\Omega_{11} + M_1}{\Omega_{12} + M_1} = \frac{M_1 + \mho_{11}}{\mho_{11}}, \; \frac{\Omega_{12}}{\Omega_{11}} = \frac{\Omega_{11}\mho_{11} - M_1^2}{(\mho_{11} + M_1)\Omega_{11}},$$
and the max.–min. value required is
$$2ik \cos^{-1}\sqrt{\frac{\Omega_{11}\mho_{11} - M_1^2}{\Omega_{11}\mho_{11}}} = 2ik \sin^{-1}\left(\frac{\pm M_1}{\sqrt{\Omega_{11}\mho_{11}}}\right).$$

Similarly the max.–min. distance between the plane M and a plane through T is
$$2ik' \sin^{-1}\left(\frac{\pm M_1}{\sqrt{\Omega_{11}\mho_{11}}}\right),$$
as is also evident from the symmetry of this form. These distances being always proportional, I take

(5) $$\overline{MT} = 2ik'' \sin^{-1}\left(\frac{M_1}{\sqrt{\Omega_{11}\mho_{11}}}\right).$$

On a nearer consideration of formulæ (3) and (4), it appears that the distance between two points and the max.–min. distance between two planes one through one point and one through the other are proportional, as are also the distance between two planes and the max.–min. distance between a point of the one and a point of the other.

If a point, plane, or straight line move continuously from a certain initial position to a certain final position, according to a definite law, the aggregate of the successive positions may be called a "course," and the measure of the total motion or "length of the course" may be defined to be the sum of the infinitesimal distances between successive positions of the point, plane, or straight line.

To obtain the length of any course we must express the distance between the points x, y, z and $x+dx, y+dy, z+dz$, between the planes u, v, w and $u+du, v+dv, w+dw$, or between the infinitely near positions of the straight line, and integrate between the two extremities of the course, taking account of the equation of the course. A course may be called "flat" or "doubly flat" according as its elements satisfy one or two linear equations. Thus a flat course from one point to another is a plane course, a doubly flat course from one point to another is a straight line, a flat course from one plane to another is an aggregate of positions of a plane always passing through a fixed point, and a doubly flat course from one plane to another is an aggregate of positions of a plane rotating about a fixed straight line in it. Then, as I prove below, the length of a course between two given points or planes is a max.–min. when the course is doubly flat.

The max.–min. distance of a given point $T(x_1, y_1, z_1)$ from a point of a given straight line L whose equation is

$$\frac{x-x_2}{A'}=\frac{y-y_2}{B'}=\frac{z-z_2}{C'}$$

is the distance of x_1, y_1, z_1 from the farthest or nearest point, say ξ, η, ζ of the line. Let

$$\xi=x_2+\lambda A',\ \eta=y_2+\lambda B',\ \zeta=z_2+\lambda C',$$

then the condition for λ is

$$[\Omega_{22}+2\lambda(A'x_2+B'y_2+C'z_2)+\lambda^2(A'^2+B'^2+C'^2)](A'x_1+B'y_1+C'z_1)d\lambda$$
$$-[\Omega_{12}+\lambda(A'x_1+B'y_1+C'z_1)][(A'x_2+B'y_2+C'z_2)+\lambda(A'^2+B'^2+C'^2)]d\lambda=0$$

for all values of $d\lambda$, and the max.–min. distance is then

$$2ik\cos^{-1}\left(\frac{\Omega_{12}+\lambda(A'x_1+B'y_1+C'z_1)}{\sqrt{\Omega_{11}[\Omega_{22}+2\lambda(A'x_2+B'y_2+C'z_2)+\lambda^2(A'^2+B'^2+C'^2)]}}\right)$$

$$=2ik\cos^{-1}\sqrt{\frac{[\Omega_{12}+\lambda(A'x_1+B'y_1+C'z_1)](A'x_1+B'y_1+C'z_1)}{\Omega_{11}[(A'x_2+B'y_2+C'z_2)+\lambda(A'^2+B'^2+C'^2)]}}.$$

Writing

$$A'x_1+B'y_1+C'z_1=N_1',\ A'x_2+B'y_2+C'z_2=N_2',\ A'^2+B'^2+C'^2=R'^2,$$

the condition for λ is

$$\lambda(R'^2\Omega_{12}-N_1'N_2')=N_1'\Omega_{22}-N_2'\Omega_{12},$$

and the max.–min. distance is

$$2ik\cos^{-1}\sqrt{\frac{R'^2\Omega_{12}^2+N_1'^2\Omega_{22}-2N_1'N_2'\Omega_{12}}{\Omega_{11}(R'^2\Omega_{22}-N_2'^2)}}.$$

The max.–min. distance of T from the farthest or nearest plane through L is 0; but the max.–min. distance of L from a straight line through T, as I shall show further on, is proportional to the value just obtained for the max.–min. distance of T from a point of L. I take therefore

$$(6)\ \overline{LT} = 2ik''' \cos^{-1}\sqrt{\frac{R'^2\Omega_{12}^2 + N_1'^2\Omega_{22} - 2N_1'N_2'\Omega_{12}}{\Omega_{11}(R'^2\Omega_{22} - N_2'^2)}} = 2ik''' \sin^{-1}\sqrt{\frac{\begin{vmatrix} \Omega_{11} & \Omega_{12} & N_1' \\ \Omega_{21} & \Omega_{22} & N_2' \\ N_1' & N_2' & R'^2 \end{vmatrix}}{\Omega_{11}(R'^2\Omega_{22} - N_2'^2)}}$$

$$= 2ik''' \cos^{-1}\sqrt{\frac{\Omega_{12}^2\Omega_{33} + \Omega_{13}^2\Omega_{22} - 2\Omega_{12}\Omega_{13}\Omega_{23}}{\Omega_{11}(\Omega_{22}\Omega_{33} - \Omega_{23}^2)}} = 2ik''' \sin^{-1}\sqrt{\frac{\begin{vmatrix} \Omega_{11} & \Omega_{12} & \Omega_{13} \\ \Omega_{21} & \Omega_{22} & \Omega_{23} \\ \Omega_{31} & \Omega_{32} & \Omega_{33} \end{vmatrix}}{\Omega_{11}(\Omega_{22}\Omega_{33} - \Omega_{23}^2)}}$$

$$= 2ik''' \sin^{-1}\sqrt{\frac{N_T''^2\mho_{22} + R''^2T_2^2 - 2N_T''N_2''T_2}{\Omega_{11}(R''^2\mho_{22} - N_2''^2)}} = 2ik''' \cos^{-1}\sqrt{\frac{\begin{vmatrix} \Omega_{11} & T_2 & N_T'' \\ T_2 & \mho_{22} & N_2'' \\ N_T'' & N_2'' & R''^2 \end{vmatrix}}{\Omega_{11}(R''^2\mho_{22} - N_2''^2)}}$$

$$= 2ik''' \sin^{-1}\sqrt{\frac{T_2^2\mho_{33} + T_3^2\mho_{22} - 2T_2T_3\mho_{23}}{\Omega_{11}(\mho_{22}\mho_{33} - \mho_{23}^2)}} = 2ik''' \cos^{-1}\sqrt{\frac{\begin{vmatrix} \Omega_{11} & T_2 & T_3 \\ T_2 & \mho_{22} & \mho_{23} \\ T_3 & \mho_{23} & \mho_{33} \end{vmatrix}}{\Omega_{11}(\mho_{22}\mho_{33} - \mho_{23}^2)}}$$

In the third and fourth of these expressions L is supposed to be given as the junction of (x_2, y_2, z_2) and (x_3, y_3, z_3), in the fifth and sixth by equations of the form $\frac{u-u_2}{A''} = \frac{v-v_2}{B''} = \frac{w-w_2}{C''}$, and in the seventh and eighth as the intersection of (u_2, v_2, w_2) and (u_3, v_3, w_3); also
$N_T'' = A''x_1 + B''y_1 + C''z_1$, $T_2 = x_1u_2 + y_1v_2 + z_1w_2 + 1$, $R''^2 = A''^2 + B''^2 + C''^2$, $N_2'' = A''u_2 + B''v_2 + C''w_2$.

The third and fourth forms are obtained from the first and second by putting $A' = x_3 - x_2$, $B' = y_3 - y_2$, $C' = z_3 - z_2$; the fifth and sixth are obtained from the first and second on eliminating A', B', C', x_2, y_2, z_2 by means of the equations

$$A' : B' : C' = B''w_2 - C''v_2 : C''u_2 - A''w_2 : A''v_2 - B''u_2,$$
$$u_2x_2 + v_2y_2 + w_2z_2 + 1 = 0,\quad A''x_2 + B''y_2 + C''z_2 = 0;$$

and the seventh and eighth are obtained from the fifth and sixth by putting

$$A'' : B'' : C'' = u_3 - u_2 : v_3 - v_2 : w_3 - w_2.$$

If (u_2, v_2, w_2) is the plane of T and L, $T_2 = 0$, and

$$(7)\quad \overline{LT} = 2ik''' \sin^{-1}\left(\frac{\pm N_T''\sqrt{\mho_{22}}}{\sqrt{\Omega_{11}(R''^2\mho_{22} - N_2''^2)}}\right) = 2ik''' \sin^{-1}\left(\frac{\pm T_3\sqrt{\mho_{22}}}{\sqrt{\Omega_{11}(\mho_{22}\mho_{33} - \mho_{23}^2)}}\right).$$

The max.–min. distance between the plane $M(u_1, v_1, w_1)$ and a plane through the straight line L whose equation is

$$\frac{u-u_2}{A''}=\frac{v-v_2}{B''}=\frac{w-w_2}{C''}$$

is

$$2ik' \cos^{-1}\sqrt{\frac{R''^2\mho_{12}^2+N_1''^2\mho_{22}-2N_1''N_2''\mho_{12}}{\mho_{11}(R''^2\mho_{22}-N_2''^2)}},$$

where

$$N_1''=A''u_1+B''v_1+C''w_1,\ N_2''=A''u_2+B''v_2+C''w_2,\ R''^2=A''^2+B''^2+C''_2.$$

If the equation of L is given in the form

$$\frac{x-x_2}{A'}=\frac{y-y_2}{B'}=\frac{z-z_2}{C'},$$

then $\quad A'':B'':C''=B'z_2-C'y_2:C'x_2-A'z_2:A'y_2-B'x_2;$

and putting $\quad A'u_1+B'v_1+C'w_1=N'_M,\ u_1x_2+v_1y_2+w_1z_2+1=M_2,$

the max.–min. distance will be found to be

$$2ik' \sin^{-1}\sqrt{\frac{N_M'^2\Omega_{22}+R'^2M_2^2-2N'_MM_2N'_2}{\mho_{11}(R'^2\Omega_{22}-N_2'^2)}}.$$

If the straight line L were given as the intersection of the planes (u_2, v_2, w_2) and (u_3, v_3, w_3), the max.–min. distance between the plane M and a plane through L is the max.–min. value of

$$2ik' \cos^{-1}\left(\frac{\mho_{12}+\lambda\mho_{13}}{\sqrt{\mho_{11}(\mho_{22}+2\lambda\mho_{23}+\lambda^2\mho_{33})}}\right);$$

the condition for λ is

$$\lambda(\mho_{12}\mho_{33}-\mho_{13}\mho_{23})+(\mho_{12}\mho_{23}-\mho_{13}\mho_{22})=0,$$

and the max.–min. distance required is

$$2ik' \cos^{-1}\sqrt{\frac{\mho_{12}^2\mho_{33}+\mho_{13}^2\mho_{22}-2\mho_{12}\mho_{13}\mho_{23}}{\mho_{11}(\mho_{22}\mho_{33}-\mho_{23}^2)}}.$$

The max.–min. distance between M and a point of L is 0, as is also the max.–min. distance between L and a point of M. As I shall afterwards show the max.–min. distance between L and a line in M is proportional to the above given max.–min. distance between M and a plane through L. I take, therefore,

$$(8)\quad \overline{ML}=2ik^{\text{IV}} \cos^{-1}\sqrt{\frac{\mho_{12}^2\mho_{33}+\mho_{13}^2\mho_{22}-2\mho_{12}\mho_{13}\mho_{23}}{\mho_{11}(\mho_{22}\mho_{33}-\mho_{23}^2)}}=2ik^{\text{IV}} \sin^{-1}\sqrt{\frac{\begin{vmatrix}\mho_{11}&\mho_{12}&\mho_{13}\\ \mho_{21}&\mho_{22}&\mho_{23}\\ \mho_{31}&\mho_{32}&\mho_{33}\end{vmatrix}}{\mho_{11}(\mho_{22}\mho_{33}-\mho_{23}^2)}}$$

$$=2ik^{\text{IV}} \cos^{-1}\sqrt{\frac{R''^2\mho_{12}^2+N_1''^2\mho_{22}-2N_1''N_2''\mho_{12}}{\mho_{11}(R''^2\mho_{22}-N_2''^2)}}=2ik^{\text{IV}} \sin^{-1}\sqrt{\frac{\begin{vmatrix}\mho_{11}&\mho_{12}&N_1''\\ \mho_{21}&\mho_{22}&N_2''\\ N_1''&N_2''&R''^2\end{vmatrix}}{\mho_{11}(R''^2\mho_{22}-N_2''^2)}}$$

$$= 2ik^{\text{IV}} \sin^{-1}\sqrt{\frac{N_M'^2 \Omega_{22} + R'^2 M_2^2 - 2N_M' N_2' M_2}{\mho_{11}(R'^2 \Omega_{22} - N_2'^2)}} = 2ik^{\text{IV}} \cos^{-1} \sqrt{\frac{\begin{vmatrix} \mho_{11} & M_2 & N_M' \\ M_2 & \Omega_{22} & N_2' \\ N_M' & N_2' & R'^2 \end{vmatrix}}{\mho_{11}(R'^2 \Omega_{22} - N_2'^2)}}$$

$$= 2ik^{\text{IV}} \sin^{-1}\sqrt{\frac{M_2^2 \Omega_{33} - 2M_2 M_3 \Omega_{23} + M_3^2 \Omega_{22}}{\mho_{11}(\Omega_{22}\Omega_{33} - \Omega_{23}^2)}} = 2ik^{\text{IV}} \cos^{-1} \sqrt{\frac{\begin{vmatrix} \mho_{11} & M_2 & M_3 \\ M_2 & \Omega_{22} & \Omega_{23} \\ M_3 & \Omega_{32} & \Omega_{33} \end{vmatrix}}{\mho_{11}(\Omega_{22}\Omega_{33} - \Omega_{23}^2)}}.$$

In the fifth and sixth of these expressions L is supposed to be given in the form $\frac{x - x_2}{A'} = \frac{y - y_2}{B'} = \frac{z - z_2}{C'}$, and in the seventh and eighth expressions L is determined as the junction of (x_2, y_2, z_2) and (x_3, y_3, z_3).

The fifth and sixth are obtained from the first and second by eliminating $u_2, v_2, w_2, u_3, v_3, w_3$ by means of the equations

$$u_2 x_2 + v_2 y_2 + w_2 z_2 + 1 = 0, \qquad u_3 x_2 + v_3 y_2 + w_3 z_2 + 1 = 0,$$
$$A' u_2 + B' v_2 + C' w_2 = 0, \qquad A' u_3 + B' v_3 + C' w_3 = 0;$$

inasmuch as (u_2, v_2, w_2) and (u_3, v_3, w_3) are any planes through L, it is allowable to assume $v_2 = 0$, $u_3 = 0$, which simplifies the reduction. The seventh and eighth forms of (8) are obtained from the fifth and sixth by writing $A' = x_3 - x_2$, $B' = y_3 - y_2$, $C' = z_3 - z_2$.

If x_2, y_2, z_2 is the point in which the plane M is met by the line L, $M_2 = 0$, and

$$\overline{ML} = 2ik^{\text{IV}} \sin^{-1}\left(\frac{\pm M_3 \sqrt{\Omega_{22}}}{\sqrt{\mho_{11}(\Omega_{22}\Omega_{33} - \Omega_{23}^2)}}\right) \tag{9}$$
$$= 2ik^{\text{IV}} \sin^{-1}\left(\frac{\pm N_M' \sqrt{\Omega_{22}}}{\sqrt{\mho_{11}(R'^2 \Omega_{22} - N_2'^2)}}\right).$$

The max.–min. distance between the line L and a point of the line L', where L is the junction of (x_1, y_1, z_1) and (x_2, y_2, z_2), and L' the junction of (x_3, y_3, z_3) and (x_4, y_4, z_4) is the max.–min. distance between L and the point

$$\left(\frac{x_3 + \lambda x_4}{1 + \lambda}, \frac{y_3 + \lambda y_4}{1 + \lambda}, \frac{z_3 + \lambda z_4}{1 + \lambda}\right)$$

i. e. by (6) the max.–min. value of

$$2ik''' \cos^{-1}\sqrt{\frac{P + 2\lambda Q + \lambda^2 R}{A(\Omega_{33} + 2\lambda\Omega_{34} + \lambda^2 \Omega_{44})}},$$

where

$$P = \Omega_{13}^2 \Omega_{22} + \Omega_{23}^2 \Omega_{11} - 2\Omega_{13}\Omega_{23}\Omega_{12},$$
$$Q = \Omega_{13}\Omega_{14}\Omega_{22} + \Omega_{23}\Omega_{24}\Omega_{11} - \Omega_{13}\Omega_{24}\Omega_{12} - \Omega_{14}\Omega_{23}\Omega_{12},$$
$$R = \Omega_{14}^2 \Omega_{22} + \Omega_{24}^2 \Omega_{11} - 2\Omega_{14}\Omega_{24}\Omega_{12},$$
$$A = \Omega_{11}\Omega_{22} - \Omega_{12}^2,$$

I also put

$$\begin{aligned} B &= \Omega_{13}\Omega_{24} - \Omega_{14}\Omega_{23}, & C &= \Omega_{33}\Omega_{44} - \Omega_{34}^2, \\ \alpha &= \Omega_{11}\Omega_{23} - \Omega_{12}\Omega_{13}, & \beta &= \Omega_{11}\Omega_{24} - \Omega_{12}\Omega_{14}, \\ \gamma &= \Omega_{12}\Omega_{23} - \Omega_{13}\Omega_{22}, & \delta &= \Omega_{12}\Omega_{24} - \Omega_{14}\Omega_{22}, \\ \alpha' &= \Omega_{23}\Omega_{44} - \Omega_{24}\Omega_{34}, & \beta' &= \Omega_{24}\Omega_{33} - \Omega_{23}\Omega_{34}, \\ \gamma' &= \Omega_{14}\Omega_{34} - \Omega_{13}\Omega_{44}, & \delta' &= \Omega_{13}\Omega_{34} - \Omega_{14}\Omega_{33}; \end{aligned}$$

namely A, B, α, β, γ, δ are the minors of the second order formed from the first and second rows of the determinant

$$\begin{vmatrix} \Omega_{11} & \Omega_{12} & \Omega_{13} & \Omega_{14} \\ \Omega_{21} & \Omega_{22} & \Omega_{23} & \Omega_{24} \\ \Omega_{31} & \Omega_{32} & \Omega_{33} & \Omega_{34} \\ \Omega_{41} & \Omega_{42} & \Omega_{43} & \Omega_{44} \end{vmatrix},$$

and B, C, α', β', γ', δ' are the minors formed from the third and fourth rows, *i. e.* this determinant, expressed as a sum of products of minors formed from the first two and last two rows, is

$$AC + B^2 - \alpha\alpha' - \beta\beta' - \gamma\gamma' - \delta\delta';$$

but the determinant is evidently $-\Delta^2$, where

$$\Delta = \begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}$$

(*i. e.* $\Delta = 0$ is the condition that L and L' intersect, or lie in one plane), hence

(10) $$-\Delta^2 = AC + B^2 - \alpha\alpha' - \beta\beta' - \gamma\gamma' - \delta\delta'.$$

Now the condition for λ is

$$(\Omega_{33} + 2\lambda\Omega_{34} + \lambda^2\Omega_{44})(Q + \lambda R) - (P + 2\lambda Q + \lambda^2 R)(\Omega_{34} + \lambda\Omega_{44}) = 0,$$

and from this follows

$$\frac{P + 2\lambda Q + \lambda^2 R}{\Omega_{33} + 2\lambda\Omega_{34} + \lambda^2\Omega_{44}} = \frac{Q + \lambda R}{\Omega_{34} + \lambda\Omega_{44}},$$

and hence the max.–min. value required is

$$2ik''' \cos^{-1}\sqrt{\frac{Q + \lambda R}{A(\Omega_{34} + \lambda\Omega_{44})}}.$$

Put

$$t = \frac{Q + \lambda R}{\Omega_{34} + \lambda\Omega_{44}} = \frac{P + \lambda Q}{\Omega_{33} + \lambda\Omega_{34}},$$

by the previous equation; hence

$$\lambda = -\frac{t\Omega_{34} - Q}{t\Omega_{44} - R} = -\frac{t\Omega_{33} - P}{t\Omega_{34} - Q},$$

i. e. t is determined by the quadratic

$$Ct^2-(\Omega_{33}R+\Omega_{44}P-2\Omega_{34}Q)t+PR-Q^2=0;$$

but

$$\Omega_{33}R-\Omega_{34}Q=\beta\beta'+\delta\delta',$$
$$\Omega_{44}P-\Omega_{34}Q=\alpha\alpha'+\gamma\gamma',$$

hence

$$\Omega_{33}R+\Omega_{44}P-2\Omega_{34}Q=\alpha\alpha'+\beta\beta'+\gamma\gamma'+\delta\delta'=AC+B^2+\Delta^2,$$

by (10); also P, Q, R may be written

$$P=\Omega_{23}\alpha-\Omega_{13}\gamma,\ R=\Omega_{24}\beta-\Omega_{14}\delta,$$
$$Q=\Omega_{23}\beta-\Omega_{13}\delta=\Omega_{24}\alpha-\Omega_{14}\gamma,$$

hence

$$PR-Q^2=\begin{vmatrix}\Omega_{23}\alpha-\Omega_{13}\gamma, & \Omega_{24}\alpha-\Omega_{14}\gamma\\ \Omega_{23}\beta-\Omega_{13}\delta, & \Omega_{24}\beta-\Omega_{14}\delta\end{vmatrix}=B\begin{vmatrix}\alpha & \gamma\\ \beta & \delta\end{vmatrix}=AB^2;$$

the quadratic in t then becomes

$$Ct^2-(AC+B^2+\Delta^2)t+AB^2=0,$$

i. e. the max.–min. distance required is

$$2ik'''\cos^{-1}\sqrt{\frac{t}{A}}=2ik'''\cos^{-1}\sqrt{\frac{AC+B^2+\Delta^2\pm\sqrt{(AC+B^2+\Delta^2)^2-4AB^2C}}{2AC}}$$
$$=2ik'''\cos^{-1}\sqrt{\frac{AC+B^2+\Delta^2\pm\sqrt{(AC-B^2+\Delta^2)+4B^2\Delta^2}}{2AC}}$$
$$=2ik'''\sin^{-1}\sqrt{\frac{AC-B^2-\Delta^2\mp\sqrt{(AC-B^2-\Delta^2)^2+4AC\Delta^2}}{2AC}};$$

the symmetry of this expression shows that it is also the max.–min. distance between L' and a point of L.

It is very evident, from the investigation leading to (6), that the max.–min. distance between a point of L and a point of L' will be found from this max.–min. distance between L and a point of L' by changing k''' to k, *i. e.* this distance is

$$2ik\sin^{-1}\sqrt{\frac{AC-B^2-\Delta^2\mp\sqrt{(AC-B^2-\Delta^2)^2+4AC\Delta^2}}{2AC}}.$$

Similarly, if L is the intersection of the planes (u_1, v_1, w_1) and (u_2, v_2, w_2) and L' the intersection of (u_3, v_3, w_3) and (u_4, v_4, w_4), the max.–min. distance between L and a plane through L', or between L' and a plane through L is

$$2ik^{\mathrm{IV}}\sin^{-1}\sqrt{\frac{\forall\rotatebox[origin=c]{180}{C}-\rotatebox[origin=c]{180}{B}^2-\nabla^2\mp\sqrt{(\forall\rotatebox[origin=c]{180}{C}-\rotatebox[origin=c]{180}{B}^2-\nabla^2)^2+4\forall\rotatebox[origin=c]{180}{C}\nabla^2}}{2\forall\rotatebox[origin=c]{180}{C}}},$$

where

$$\forall = \mho_{11}\mho_{22} - \mho_{12}^2, \quad \rotatebox[origin=c]{180}{B} = \mho_{13}\mho_{24} - \mho_{14}\mho_{23}, \quad \rotatebox[origin=c]{180}{C} = \mho_{33}\mho_{44} - \mho_{34}^2,$$

$$\nabla = \begin{vmatrix} u_1 & v_1 & w_1 & -1 \\ u_2 & v_2 & w_2 & -1 \\ u_3 & v_3 & w_3 & -1 \\ u_4 & v_4 & w_4 & -1 \end{vmatrix}.$$

The max.–min. dista between a plane through L and a plane through L' is then

$$2ik' \sin^{-1}\sqrt{\frac{\forall\rotatebox[origin=c]{180}{C} - \rotatebox[origin=c]{180}{B}^2 - \nabla^2 \mp \sqrt{(\forall\rotatebox[origin=c]{180}{C} - \rotatebox[origin=c]{180}{B}^2 - \nabla^2)^2 + 4\forall\rotatebox[origin=c]{180}{C}\nabla}}{2\forall\rotatebox[origin=c]{180}{C}}}.$$

The max.–min. distance between a point on L_1, the junction of (x_1, y_1, z_1) and (x_2, y_2, z_2), and a plane through L_2, the intersection of (u_3, v_3, w_3) and (u_4, v_4, w_4), is the max.–min. value of

$$2ik'' \sin^{-1} \frac{M_1''' + \lambda M_2''' + \mu M_1^{\mathrm{IV}} + \lambda\mu M_2^{\mathrm{IV}}}{\sqrt{(\Omega_{11} + 2\lambda\Omega_{12} + \lambda^2\Omega_{22})(\mho_{33} + 2\mu\mho_{34} + \mu^2\mho_{44})}},$$

where $\quad M_i''' = u_3x_i + v_3y_i + w_3z_i + 1, \; M_i^{\mathrm{IV}} = u_4x_i + v_4y_i + w_4z_i + 1,$
but this expression is obtained from

$$2ik \cos^{-1} \frac{\Omega_{13} + \lambda\Omega_{23} + \mu\Omega_{14} + \lambda\mu\Omega_{24}}{\sqrt{(\Omega_{11} + 2\lambda\Omega_{12} + \lambda^2\Omega_{22})(\Omega_{33} + 2\mu\Omega_{34} + \mu^2\Omega_{44})}}$$

by replacing Ω_{13}, Ω_{23}, Ω_{14}, Ω_{24}, Ω_{33}, Ω_{34}, Ω_{44}, k, $\cos^{-1}$ by M_1''', M_2''', M_1^{IV}, M_2^{IV}, $\mho_{33}$, $\mho_{34}$, $\mho_{44}$, k'', $\sin^{-1}$; now the max.–min. value of the latter expression is the max.–min. distance between a point of the junction of (x_1, y_1, z_1) and (x_2, y_2, z_2) and a point of the junction of (x_3, y_3, z_3) and (x_4, y_4, z_4); but this distance has been found to be

$$2ik \sin^{-1}\sqrt{\frac{AC - B^2 - \Delta^2 \mp \sqrt{(AC - B^2 - \Delta^2)^2 + 4AC\Delta^2}}{2AC}}$$

where A, C, B, Δ are defined as above; hence the max.–min. distance between a point on L and a plane through L' is

$$2ik'' \cos^{-1}\sqrt{\frac{A\rotatebox[origin=c]{180}{C} - \mathfrak{B}^2 - \mathfrak{D}^2 \mp \sqrt{(A\rotatebox[origin=c]{180}{C} - \mathfrak{B}^2 - \mathfrak{D}^2)^2 + 4A\rotatebox[origin=c]{180}{C}\mathfrak{D}^2}}{2A\rotatebox[origin=c]{180}{C}}}$$

$$= 2ik'' \sin^{-1}\sqrt{\frac{A\rotatebox[origin=c]{180}{C} + \mathfrak{B}^2 + \mathfrak{D}^2 \pm \sqrt{(A\rotatebox[origin=c]{180}{C} + \mathfrak{B}^2 + \mathfrak{D}^2)^2 - 4A\rotatebox[origin=c]{180}{C}\mathfrak{B}^2}}{2A\rotatebox[origin=c]{180}{C}}},$$

where $A, \mathcal{C}$ are defined as above, and

$$\mathfrak{B} = M_1''' M_2^{\text{IV}} - M_1^{\text{IV}} M_2''',$$

$$-\mathfrak{D}^2 = \begin{vmatrix} \Omega_{11} & \Omega_{12} & M_1''' & M_1^{\text{IV}} \\ \Omega_{21} & \Omega_{22} & M_2''' & M_2^{\text{IV}} \\ M_1''' & M_2''' & \Omega_{33} & \Omega_{34} \\ M_1^{\text{IV}} & M_2^{\text{IV}} & \Omega_{43} & \Omega_{44} \end{vmatrix} = -\begin{vmatrix} x_1, & y_1, & z_1, & 1 \\ x_2, & y_2, & z_2, & 1 \\ u_3, & v_3, & w_3, & -1 \\ u_4, & v_4, & w_4, & -1 \end{vmatrix}^2,$$

i. e.

$$\mathfrak{D} = \begin{vmatrix} x_1, & y_1, & z_1, & 1 \\ x_2, & y_2, & z_2, & 1 \\ u_3, & v_3, & w_3, & -1 \\ u_4, & v_4, & w_4, & -1 \end{vmatrix},$$

and $\mathfrak{D} = 0$ is the condition that each of the lines L and L' should intersect the polar line of the other with respect to the absolute, *i. e.* the condition for what I shall afterwards call the perpendicularity of the lines. The proportionality of the last three distances can be best proven by expressing them in terms of the coordinates of the lines; $\lambda, \mu, \rho, \sigma$ are simply proportional factors, let the coordinates of L and L' be respectively $a_1, b_1, c_1, f_1, g_1, h_1$ and $a_2, b_2, c_2, f_2, g_2, h_2$; then

$$\lambda(y_1 z_2 - z_1 y_2) = \mu(u_1 - u_2) = a_1, \ \lambda(z_1 x_2 - x_1 z_2) = \mu(v_1 - v_2) = b_1,$$
$$\lambda(x_1 y_2 - y_1 x_2) = \mu(w_1 - w_2) = c_1, \ \lambda(x_1 - x_2) = \mu(v_1 w_2 - w_1 v_2) = f_1,$$
$$\lambda(y_1 - y_2) = \mu(w_1 u_2 - u_1 w_2) = g_1, \ \lambda(z_1 - z_2) = \mu(u_1 v_2 - v_1 u_2) = h_1,$$
$$\rho(y_3 z_4 - z_3 y_4) = \sigma(u_3 - u_4) = a_2, \ \rho(z_3 x_4 - x_3 z_4) = \sigma(v_3 - v_4) = b_2,$$
$$\rho(x_3 y_4 - y_3 x_4) = \sigma(w_3 - w_4) = c_2, \ \rho(x_3 - x_4) = \sigma(v_3 w_4 - w_3 v_4) = f_2,$$
$$\rho(y_3 - y_4) = \sigma(w_3 u_4 - u_3 w_4) = g_2, \ \rho(z_3 - z_4) = \sigma(u_3 v_4 - v_3 u_4) = h_2;$$

and

$$\lambda^2 A = a_1^2 + b_1^2 + c_1^2 - f_1^2 - g_1^2 - h_1^2, \ \lambda\rho B = a_1 a_2 + b_1 b_2 - c_1 c_2 - f_1 f_2 - g_1 g_2 - h_1 h_2,$$
$$\rho^2 C = a_2^2 + b_2^2 + c_2^2 - f_2^2 - g_2^2 - h_2^2, \ \lambda\rho\Delta = a_1 f_2 + b_1 g_2 + c_1 h_2 + f_1 a_2 + g_1 b_2 + h_1 c_2,$$
$$\mu^2 \mathcal{F} = f_1^2 + g_1^2 + h_1^2 - a_1^2 - b_1^2 - c_1^2, \ \mu\sigma\mathcal{G} = f_1 f_2 + g_1 g_2 + h_1 h_2 - a_1 a_2 - b_1 b_2 - c_1 c_2,$$
$$\sigma^2 \mathcal{C} = f_2^2 + g_2^2 + h_2^2 - a_2^2 - b_2^2 - c_2^2, \ \mu\sigma\nabla = -a_1 f_2 - b_1 g_2 - c_1 h_2 - f_1 a_2 - g_1 b_2 - h_1 c_2,$$
$$\lambda\sigma\mathfrak{B} = a_1 f_2 + b_1 g_2 + c_1 h_2 + f_1 a_2 + g_1 b_2 + h_1 c_2, \ \lambda\sigma\mathfrak{D} = f_1 f_2 + g_1 g_2 + h_1 h_2 - a_1 a_2 - b_1 b_2 - c_1 c_2,$$

from which follow

$$\mu^2 \mathcal{F} = -\lambda^2 A, \ \mu\sigma\mathcal{G} = \lambda\sigma\mathfrak{D} = -\lambda\rho B, \ \sigma^2\mathcal{C} = -\rho^2 C, \ \mu\sigma\nabla = -\lambda\sigma\mathfrak{B} = -\lambda\rho\Delta;$$

hence

$$\mu^2\sigma^2(\mathcal{F}\mathcal{C} - \mathcal{G}^2 - \nabla^2) = \lambda^2\rho^2(AC - B^2 - \Delta^2),$$
$$\mu^4\sigma^4 \mathcal{F}\mathcal{C}\nabla^2 = \lambda^4\rho^4 AC\Delta^2, \ \mu^2\sigma^2\mathcal{F}\mathcal{C} = \lambda^2\rho^2 AC,$$
$$\lambda^2\sigma^2(A\mathcal{C} + \mathfrak{B}^2 + \mathfrak{D}^2) = -\lambda^2\rho^2(AC - B^2 - \Delta^2),$$
$$\lambda^4\sigma^4 A\mathcal{C}\mathfrak{D}^2 = -\lambda^4\rho^4 ACB^2, \ \lambda^2\sigma^2 A\mathcal{C} = -\lambda^2\rho^2 AC,$$

and therefore

$$\frac{\rotatebox[origin=c]{180}{CA}-\rotatebox[origin=c]{180}{B}^2-\nabla^2\mp\sqrt{(\rotatebox[origin=c]{180}{CA}-\rotatebox[origin=c]{180}{B}^2-\nabla^2)^2+4\rotatebox[origin=c]{180}{CA}\nabla^2}}{2\rotatebox[origin=c]{180}{CA}}$$

$$=-\frac{A\rotatebox[origin=c]{180}{C}+\mathfrak{B}^2+\mathfrak{D}^2\pm\sqrt{(A\rotatebox[origin=c]{180}{C}+\mathfrak{B}^2+\mathfrak{D}^2)^2-4A\rotatebox[origin=c]{180}{C}\mathfrak{B}^2}}{2A\rotatebox[origin=c]{180}{C}}$$

$$=\frac{AC-B^2-\Delta^2\mp\sqrt{(AC-B^2-\Delta^2)^2+4AC\Delta^2}}{2AC},$$

and the proportionality of the three distances is established. I therefore take

$$(11)\qquad \overline{LL'}=2ik^{\mathrm{v}}\sin^{-1}\sqrt{\frac{AC-B^2-\Delta^2\mp\sqrt{(AC-B^2-\Delta^2)^2+4AC\Delta^2}}{2AC}},$$

in which the quantity under the outer radical sign may be replaced by either of the above given equal expressions. It is to be noticed that, on account of the double sign before the inner radical, there are two max.–min. distances between two given straight lines, corresponding, as I shall show below, to the geometrical fact that two straight lines can be drawn to intersect four given straight lines.

If the straight lines L and L' intersect, Δ, ∇ and $\mathfrak{B}$ vanish, one of the values of $\overline{LL'}$ is 0, and the other is

$$(12)\qquad \overline{LL'}=2ik^{\mathrm{v}}\sin^{-1}\sqrt{\frac{AC-B^2}{AC}}=2ik^{\mathrm{v}}\cos^{-1}\frac{B}{\sqrt{AC}}=2ik^{\mathrm{v}}\cos^{-1}\frac{-\rotatebox[origin=c]{180}{B}}{\sqrt{\rotatebox[origin=c]{180}{CA}}}$$

$$=2ik^{\mathrm{v}}\cos^{-1}\frac{-\mathfrak{D}}{\sqrt{-A\rotatebox[origin=c]{180}{C}}};$$

now

$$2i\cos^{-1}\frac{B}{\sqrt{AC}}=\ln\frac{B\pm\sqrt{B^2-AC}}{B\mp\sqrt{B^2-AC}}=\ln\alpha,\text{ say,}$$

where α is easily seen to be the anharmonic ratio of L and L' referred to the two tangents to the absolute from their intersection, which lie in their plane; namely, the above angle is evidently independent of the particular choice of the points T_1, T_2, T_3, T_4 in either line, hence the value of α will be unaffected by the assumption that T_2 and T_4 coincide with the intersection of the lines; now α is the ratio of the roots of the equation

$$A+2\lambda B+\lambda^2 C=0,$$

and if $T_4=T_2$ this is

$$(\Omega_{11}\Omega_{22}-\Omega_{12}^2)+2\lambda(\Omega_{13}\Omega_{22}-\Omega_{12}\Omega_{23})+\lambda^2(\Omega_{33}\Omega_{22}-\Omega_{23}^2)=0,$$

i. e.

$$\Omega_{22}(\Omega_{11}+2\lambda\Omega_{13}+\lambda^2\Omega_{33})-(\Omega_{12}+\lambda\Omega_{23})^2=0,$$

but this is the condition that the point $(x_1+\lambda x_3,\ y_1+\lambda y_3,\ z_1+\lambda z_3)$ shall lie on a tangent from T_2 to the absolute, *i. e.* α is the anharmonic ratio of T_1, T_3 with respect to the intersections of the line T_1T_3 with the tangent cone from T_2 to the

absolute, but this is the anharmonic ratio of the lines L and L', to which reference is made above.

We can now find the max.–min. distances, above mentioned, between a given straight line and the nearest or farthest line through a given point, and between a given straight line and the nearest or farthest line in a given plane. It suffices to determine the max.–min. value of (11) when L and (x_3, y_3, z_3) are constant and (x_4, y_4, z_4) variable, and when L and (u_3, v_3, w_3) are constant and (u_4, v_4, w_4) variable. In the former case, evidently

$$\frac{B^2+\Delta^2}{AC} \mp \sqrt{\left(1-\frac{B^2+\Delta^2}{AC}\right)^2+\frac{4\Delta^2}{AC}}$$

is to be a max.–min.; but there are two independent parameters involved in the determination of a straight line through a given point, and taking $\frac{B^2+\Delta^2}{AC}$ and $\frac{\Delta^2}{AC}$ to be the two parameters it appears that $\Delta = 0$ or $AC = 0$, but the latter condition makes the max.–min. distance indeterminate, and assuming $\Delta = 0$ there is still a parameter to be determined, which may be done thus: because the lines intersect we may put $x_4 : y_4 : z_4 : 1 = x_1 + \lambda x_2 : y_1 + \lambda y_2 : z_1 + \lambda z_2 : 1 + \lambda$, and the expression whose max.–min. value is to be determined is $2ik^{\mathrm{v}} \cos^{-1} \frac{B}{\sqrt{AC}}$, for which we easily find $\lambda = -\frac{\Omega_{13}}{\Omega_{23}}$, and the max.–min. value is

$$2ik^{\mathrm{v}} \cos \sqrt{\frac{\Omega_{11}\Omega_{23}^2 + \Omega_{22}\Omega_{13}^2 - 2\Omega_{12}\Omega_{13}\Omega_{23}}{\Omega_{33}(\Omega_{11}\Omega_{22} - \Omega_{12}^2)}},$$

which is proportional to the value which would be obtained from (6) for the distance between the line and the point, namely by an interchange of the suffices 1 and 3. In the same manner the proportionality of the max.–min. distance between a given line and a line in a given plane to the expression obtained from (8) for the distance between the line and the plane.

All the above given formulæ for distance may easily be applied to the determination of the distance of two infinitely near elements, but the most interesting case of this kind, from the point of view of the Euclidean Geometry, is that in which the elements are points, say $T(x, y, z)$ and $T'(x + dx, y + dy, z + dz)$. For convenience I write

$$\delta\Omega = \frac{1}{2}\left(\frac{\partial \Omega}{\partial x} dx + \frac{\partial \Omega}{\partial y} dy + \frac{\partial \Omega}{\partial z} dz\right),$$

$$\delta^2\Omega = \frac{1}{2}\left(\frac{\partial^2 \Omega}{\partial x^2} dx^2 + \frac{\partial^2 \Omega}{\partial y^2} dy^2 + \frac{\partial^2 \Omega}{\partial z^2} dz^2 + 2\frac{\partial^2 \Omega}{\partial y \partial z} dydz + 2\frac{\partial^2 \Omega}{\partial z \partial x} dzdx + 2\frac{\partial^2 \Omega}{\partial x \partial y} dxdy\right);$$

then the distance is

$$(13)\qquad d\overline{TT'} = 2ik\cos^{-1}\frac{\Omega+\delta\Omega}{\sqrt{\Omega(\Omega+2\delta\Omega+\delta^2\Omega)}}$$

$$= 2ik\sin^{-1}\sqrt{\frac{\Omega\delta^2\Omega-(\delta\Omega)^2}{\Omega(\Omega+2\delta\Omega+\delta^2\Omega)}}$$

$$= 2ik\frac{1}{\Omega}\sqrt{\Omega\delta^2\Omega-(\delta\Omega)^2},$$

neglecting higher powers of dx, dy, dz. This may be put into the short symbolic form

$$d\overline{TT'} = 2ik\sqrt{\delta\left(\frac{\delta\Omega}{\Omega}\right)} = 2ik\sqrt{\delta^2\ln\Omega}.$$

Similarly the distance between the planes $M(u, v, w)$ and $M'(u+du, v+dv, w+dw)$, where du, dv, dw are infinitesimal, is

$$(14)\qquad d\overline{MM'} = 2ik'\frac{1}{\varpi}\sqrt{\mho\delta^2\mho-(\delta\mho)^2} = 2ik'\sqrt{\delta^2\ln\mho},$$

where
$$\delta\mho = \frac{1}{2}\left(\frac{\partial\varpi}{\partial u}du+\frac{\partial\varpi}{\partial v}dv+\frac{\partial\varpi}{\partial w}dw\right)$$

$$\delta^2\mho = \frac{1}{2}\left(\frac{\partial^2\varpi}{\partial u^2}du^2+\frac{\partial^2\varpi}{\partial v^2}dv^2+\frac{\partial^2\varpi}{\partial w^2}dw^2+2\frac{\partial^2\varpi}{\partial v\partial w}dv\,dw\right.$$
$$\left.+2\frac{\partial^2\varpi}{\partial w\partial u}dw\,du+2\frac{\partial^2\varpi}{\partial u\partial v}du\,dv\right).$$

The distance between two infinitely near straight lines L and L', given by their coordinates (a, b, c, f, g, h) and $(a+da, b+db, c+dc, f+df, g+dg, h+dh)$ is obtained from (11) by substituting for A, B, C, Δ their values in terms of line coordinates. Then

$$B = A+\delta A,$$
$$C = A+2\delta A+\delta^2 A,$$
$$\Delta = 2\delta O,$$

where

$$O \equiv af+bg+ch = 0,$$
$$\delta \equiv \frac{1}{2}\left(\frac{\partial}{\partial a}da+\frac{\partial}{\partial b}db+\frac{\partial}{\partial c}dc+\frac{\partial}{\partial f}df+\frac{\partial}{\partial g}dg+\frac{\partial}{\partial h}dh\right),$$
$$\delta^2 \equiv \frac{1}{2}\left(\frac{\partial^2}{\partial a^2}da^2+\ldots+2\frac{\partial^2}{\partial a\partial f}da\,df+\ldots\right).$$

Now $O=0$ for any line, and $2\delta O = -\delta^2 O$ is a differential of the second order; and it will be found that

$$(15)\qquad d\overline{LL'} = ik^{\text{v}}\left(\frac{A\delta^2 A-(\delta A)^2 \pm \sqrt{[A\delta^2 A-(\delta A)^2]^2+4A^2(\delta^2 O)^2}}{A^2}\right),$$

when terms of a higher than the second order are neglected.

The distance between any two infinitely near positions in a "course" will be given by (13), (14) or (15), and the "*length of the course*" will be obtained by integrating this expression between the extreme positions. It will be found by the usual method of the calculus of variations that the shortest course between any two given points or planes is "*doubly flat*"; the integrals in these cases are (3) and (4) respectively.

The same thing may be proved thus: the condition that the sum of the distances of a variable point (x_3, y_3, z_3) from two fixed points (x_1, y_1, z_1) and (x_2, y_2, z_2) as determined by (3) shall be a maximum or minimum is given by three equations of the fourth degree in (x_3, y_3, z_3), but they may be written as linear homogeneous equations in

$$\left(\frac{\partial\Omega_{11}}{\partial x_1}, \frac{\partial\Omega_{22}}{\partial x_2}, \frac{\partial\Omega_{33}}{\partial x_3}\right), \left(\frac{\partial\Omega_{11}}{\partial y_1}, \frac{\partial\Omega_{22}}{\partial y_2}, \frac{\partial\Omega_{33}}{\partial y_3}\right) \text{ and } \left(\frac{\partial\Omega_{11}}{\partial z_1}, \frac{\partial\Omega_{22}}{\partial z_2}, \frac{\partial\Omega_{33}}{\partial z_3}\right)$$

respectively, whose coefficients are identical; it is therefore necessary for a maximum or minimum that

$$\begin{vmatrix} \frac{\partial\Omega_{11}}{\partial x_1}, & \frac{\partial\Omega_{22}}{\partial x_2}, & \frac{\partial\Omega_{33}}{\partial x_3} \\ \frac{\partial\Omega_{11}}{\partial y_1}, & \frac{\partial\Omega_{22}}{\partial y_2}, & \frac{\partial\Omega_{33}}{\partial y_3} \\ \frac{\partial\Omega_{11}}{\partial z_1}, & \frac{\partial\Omega_{22}}{\partial z_2}, & \frac{\partial\Omega_{33}}{\partial z_3} \end{vmatrix} = 0;$$

with the special form (1) of Ω the constituents of either row of this determinant are the same linear homogeneous functions of (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) respectively, hence the determinant is the product of a constant determinant and

$$\begin{vmatrix} x_1, & x_2, & x_3 \\ y_1, & y_2, & y_2 \\ z_1, & z_2, & z_3 \end{vmatrix},$$

hence we may write

$$x_3 = \lambda x_1 + \mu x_2, \quad y_3 = \lambda y_1 + \mu y_2, \quad z_3 = \lambda z_1 + \mu z_2,$$

and these substituted in the above-mentioned equations give

$$\lambda + \mu = 1,$$

i. e. the point (x_3, y_3, z_3) lies on the junction of (x_1, y_1, z_1) and (x_2, y_2, z_2). The same method gives the condition for a plane moving by the shortest course from one position to another. This method applied to the case of the variable

straight line $L_3(a_3, b_3, c_3, f_3, g_3, h_3)$ moving by the shortest course from $L_1(a_1, b_1, c_1, f_1, g_1, h_1)$ to $L_2(a_2, b_2, c_2, f_2, g_2, h_2)$ gives

$$\begin{vmatrix} a_1, & b_1, & c_1, & -f_1, & -g_1, & -h_1 \\ f_1, & g_1, & h_1, & a_1, & b_1, & c_1 \\ a_2, & b_2, & c_2, & -f_2, & -g_2, & -h_2 \\ f_2, & g_2, & h_2, & a_2, & b_2, & c_2 \\ a_3, & b_3, & c_3, & -f_3, & -g_3, & -h_3 \\ f_3, & g_3, & h_3, & a_3, & b_3, & c_3 \end{vmatrix} = 0,$$

where also $$a_3 f_3 + b_3 g_3 + c_3 h_3 = 0,$$
and we may write

$$\begin{aligned} a_1\alpha + b_1\beta + c_1\gamma - f_1\phi - g_1\chi - h_1\psi &= 0, \\ f_1\alpha + g_1\beta + h_1\gamma + a_1\phi + b_1\chi + c_1\psi &= 0, \\ a_2\alpha + b_2\beta + c_2\gamma - f_2\phi - g_2\chi - h_2\psi &= 0, \\ f_2\alpha + g_2\beta + h_2\gamma + a_2\phi + b_2\chi + c_2\psi &= 0, \\ a_3\alpha + b_3\beta + c_3\gamma - f_3\phi - g_3\chi - h_3\psi &= 0, \\ f_3\alpha + g_3\beta + h_3\gamma + a_3\phi + b_3\chi + c_3\psi &= 0; \end{aligned}$$

and it will be found that the conditions for maximum or minimum will be satisfied when $$\alpha\phi + \beta\chi + \gamma\psi = 0,$$
i. e. when $\alpha, \beta, \gamma, \phi, \chi, \psi$ are the coordinates of a straight line, which, as is evident from the equations just written, intersects L_1, and its polar with respect to the absolute, and L_2 and its polar, and L_3 and its polar. This line must then be one of the two straight lines which meets each of the lines L_1, L_2 and their polars, and the conditions show that it may be either. Hence, in moving by the shortest course from L_1 to L_2, the variable line L_3 and its polar continually intersect both the straight lines intersecting L_1, L_2 and both their polars. This may be stated, as will be shown below, in other words, thus: *in passing by the shortest course from L_1 to L_2 the variable straight line L_3 slides along the two common perpendiculars to L_1 and L_2, and is itself constantly perpendicular to these common perpendiculars.*

I now find the conditions for zero-distance or "*parallelism.*" By (3), two points T_1 and T_2 are parallel when

(16) $$\Omega_{11}\Omega_{22} - \Omega_{12}^2 = 0,$$
i. e. when their junction is tangent to the absolute.

By (4), two planes M_1 and M_2 are parallel when

(17) $$\mho_{11}\mho_{22} - \mho_{12}^2 = 0,$$
i. e. when their intersection is tangent to the absolute.

By (5), a point and a plane are parallel when the point lies in the plane, *i. e.* when

$$M_1 = 0. \tag{18}$$

By (6), a point T_1 and a straight line given as the junction of T_2 and T_3 are parallel when

$$\begin{vmatrix} \Omega_{11} & \Omega_{12} & \Omega_{13} \\ \Omega_{21} & \Omega_{22} & \Omega_{23} \\ \Omega_{31} & \Omega_{32} & \Omega_{33} \end{vmatrix} = 0, \tag{19}$$

i. e. when the plane of the point and line is tangent to the absolute. The same condition can be obtained more directly from (7), which gives as the condition that the point T_1 shall be parallel to the intersection L of the planes M_2 and M_3, where M_2 is the plane containing T_1 and L, and M_3 is any *other* plane through L_2,

$$T_3\sqrt{\mho_{22}} = 0, \tag{20}$$

where T_3 is the result of substituting the coordinates of M_3 in the (tangential) equation of T_1. Hence either T_1 lies on L, or their common plane is tangent to the absolute. The former is only a special case of the later when differently stated, thus: a point and line are parallel when it is possible to pass through them both a plane tangent to the absolute.

Similarly a plane M_1 and a line L given as the intersection of two planes M_2, M_3 are parallel, by (8), when

$$\begin{vmatrix} \mho_{11} & \mho_{12} & \mho_{13} \\ \mho_{21} & \mho_{22} & \mho_{23} \\ \mho_{31} & \mho_{32} & \mho_{33} \end{vmatrix} = 0, \tag{21}$$

or, if the line L is given as the junction of T_2 and T_3, where T_2 is the intersection of M_1 and L,

$$M_3\sqrt{\Omega_{22}} = 0, \tag{22}$$

where M_3 is the result of substituting the coordinates of T_3 in the equation of M_1; *i. e.* the line and plane will be parallel when they intersect on the absolute, or when the line lies in the plane, in which latter case they have two common points on the absolute. It may be remarked that the condition $M_3 = 0$ is equivalent to the condition

$$AA' + BB' + CC' = 0$$

that the line

$$\frac{x - x'}{A'} = \frac{y - y'}{B'} = \frac{z - z'}{C'}$$

shall be parallel *in the Euclidean sense* to the plane

$$Ax + By + Cz + D = 0,$$

but in the non-Euclidean sense it is the condition that the line shall lie wholly in the plane.

By (11) the two straight lines L and L' are parallel only when $A=0$, or $C=0$, or $\Delta=0$; if $A=0$, or $C=0$, the distance $\overline{LL'}$ is indeterminate, and the lines cannot strictly be called parallel, indeed either of these conditions affects only one of the lines, it is the condition that the line in question shall be tangent to the absolute; the condition $\Delta=0$, *i. e.* that the lines intersect, makes one of the distances $\overline{LL'}=0$, the other value of $\overline{LL'}$ is then $2ik\cos^{-1}\frac{B}{\sqrt{AC}}$, and this also becomes 0 when $AC-B^2=0$, which is therefore the condition that L and L' shall be parallel in the strictest sense; and I shall call them parallel only when the two conditions are satisfied:

(23) $$\Delta=0,\ AC-B^2=0,$$

i. e. when the lines meet on the absolute.

I pass now to the conditions for the perpendicularity of two elements. It is convenient to have a name for the circular-function involved in any distance, and I call it the "*arc*" of that distance, thus from (3), etc.

$$\text{arc}\ \overline{TT'}=\cos^{-1}\left(\frac{\Omega_{12}}{\sqrt{\Omega_{11}\Omega_{22}}}\right),\ \text{etc.}$$

Writing also co. = complement, it is easily seen that if E and E' are any two elements (of the same or different species) and if P is the polar of E (namely, P is plane, point, or straight line according as E is point, plane, or straight line),

(24) $$\text{arc}\ \overline{EE'}=\frac{\eth}{2}-\text{arc}\ \overline{E'P}=\text{co. arc}\ \overline{E'P},$$

provided the proper signs are given to the radicals involved.

We may also speak of the anharmonic ratio involved in the measure of a distance as the "*argument*" of that distance (abbreviated arg.), thus

$$\text{arg.}\ \overline{TT'}=\frac{\Omega_{12}+\sqrt{\Omega_{12}^2-\Omega_{11}\Omega_{22}}}{\Omega_{12}-\sqrt{\Omega_{12}^2-\Omega_{11}\Omega_{22}}};$$

then (24) may also be written

(25) $$\text{arg.}\ \overline{EE'}=-\frac{1}{\text{arg.}\ \overline{E'P}}.$$

It is evident that the two elements E, E' are parallel when

(26) $$\text{arc}\ \overline{EE'}=0\ \text{or}\ \eth,\ i.\,e.\ \text{arg.}\ \overline{EE'}=1.$$

I call two elements E, E' perpendicular when

(27) $$\text{arc}\ \overline{EE'}=\frac{\eth}{2},\ i.\,e.\ \text{arg.}\ \overline{EE'}=-1.$$

The condition that two points $T(x_1, y_1, z_1)$ and $T'(x_2, y_2, z_2)$ shall be perpendicular is then

$$\Omega_{12} = 0, \tag{28}$$

i. e. either lies on the polar plane of the other with respect to the absolute.

Two planes $M(u_1, v_1, w_1)$ and $M'(u_2, v_2, w_2)$ are perpendicular when

$$\mho_{12} = 0, \tag{29}$$

i. e. either passes through the pole of the other with respect to the absolute.

A point $T(x_1, y_1, z_1)$ and a plane $M(u_1, v_1, w_1)$ are perpendicular when, using the notation of (5),

$$\Omega_{11}\mho_{11} - M_1^2 = 0, \tag{30}$$

i. e. the point T is parallel to (at a zero-distance from) the pole of M, and M is parallel to the polar plane of T.

A point T and a line L are perpendicular when, with the notation of (6),

$$\left\{\begin{array}{l} \Omega_{12}^2\Omega_{33} - 2\Omega_{12}\Omega_{13}\Omega_{23} + \Omega_{13}^2\Omega_{22} = 0, \text{ or} \\ R'^2\Omega_{12}^2 - 2N_1'N_2'\Omega_{12} + N_1'^2\Omega_{22} = 0, \text{ or} \\ \begin{vmatrix} \Omega_{11} & T_2 & N_T'' \\ T_2 & \mho_{22} & N_2'' \\ N_T'' & N_2'' & R''^2 \end{vmatrix} = 0, \text{ or } \begin{vmatrix} \Omega_{11} & T_2 & T_3 \\ T_2 & \mho_{22} & \mho_{23} \\ T_3 & \mho_{32} & \mho_{33} \end{vmatrix} = 0; \end{array}\right. \tag{31}$$

or, if (u_2, v_2, w_2) is the plane of T and L,

$$\left\{\begin{array}{l} \Omega_{11}(\mho_{22}\mho_{33} - \mho_{23}^2) - T_3^2\mho_{22} = 0, \text{ or} \\ \Omega_{11}(R''^2\mho_{22} - N_2''^2) - N_T''^2\mho_{22} = 0; \end{array}\right. \tag{32}$$

i. e. L is parallel to the polar plane of T, and T is parallel to the polar line of L.

A plane M and a line L are perpendicular when, with the notation of (9),

$$\left\{\begin{array}{l} \mho_{12}^2\mho_{33} - 2\mho_{12}\mho_{13}\mho_{23} + \mho_{13}^2\mho_{22} = 0, \text{ or} \\ R''^2\mho_{12}^2 - 2N_1''N_2''\mho_{12} + N_1''^2\mho_{22} = 0, \text{ or} \\ \begin{vmatrix} \mho_{11} & M_2 & N_M' \\ M_2 & \Omega_{22} & N_2' \\ N_M' & N_2' & R'^2 \end{vmatrix} = 0, \text{ or } \begin{vmatrix} \mho_{11} & M_2 & M_3 \\ M_2 & \Omega_{22} & \Omega_{23} \\ M_3 & \Omega_{32} & \Omega_{33} \end{vmatrix} = 0; \end{array}\right. \tag{33}$$

or, if (x_2, y_2, z_2) is the intersection of M and L,

$$\left\{\begin{array}{l} \mho_{11}(\Omega_{22}\Omega_{33} - \Omega_{23}^2) - M_3^2\Omega_{22} = 0, \text{ or} \\ \mho_{11}(R'^2\Omega_{22} - N_2'^2) - N_M'^2\Omega_{22} = 0; \end{array}\right. \tag{34}$$

i. e. M is parallel to the polar line of L, and L is parallel to the pole of M with respect to the absolute.

In the case of two lines L, L', with the notation of (11), one value of arc $\overline{LL'}$ is $\frac{\partial}{2}$ when $B = 0$, and both values are $\frac{\partial}{2}$ when $B = 0$ and $AC + \Delta^2 = 0$. In

the former case I call the lines simply "*perpendicular,*" and in the latter case they may be called "*doubly perpendicular.*" The condition for the perpendicularity of two lines is then

(35) $$B=0 \; [\text{or } \mathfrak{g}=0, \text{ or } \mathfrak{D}=0].$$

This condition is equivalent to

(36) $$\Omega_{13}+\lambda\Omega_{23}=0 \text{ and } \Omega_{14}+\lambda\Omega_{24}=0, \text{ or}$$

(37) $$\Omega_{13}+\mu\Omega_{14}=0 \text{ and } \Omega_{23}+\mu\Omega_{24}=0,$$

for some values of λ and μ. The equations (36) imply that a certain point $\left(\frac{x_1+\lambda x_2}{1+\lambda}, \frac{y_1+\lambda y_2}{1+\lambda}, \frac{z_1+\lambda z_2}{1+\lambda}\right)$ of L lies on the polar of L', and equations (37) imply that a certain point $\left(\frac{x_3+\mu x_4}{1+\mu}, \frac{y_3+\mu y_4}{1+\mu}, \frac{z_3+\mu z_4}{1+\mu}\right)$ of L' lies on the polar of L, *i. e.* L intersects the polar line of L', and L' intersects the polar line of L.

[The condition for double perpendicularity of L and L' is

(38) $$B=0, \text{ and } AC+\Delta^2=0;$$

now, by (10), these two conditions give

$$\alpha\alpha'+\beta\beta'+\gamma\gamma'+\delta\delta'=0,$$

and by (36) and (37) and the former definitions of α, β, γ, δ, α', β', γ', δ',

$$\begin{aligned}
\alpha &= (\Omega_{11}+\lambda\Omega_{12})\Omega_{23}, \quad \alpha'=\Omega_{23}\Omega_{44}-\Omega_{24}\Omega_{34}=-(\Omega_{34}+\mu\Omega_{44})\Omega_{24},\\
\beta &= (\Omega_{11}+\lambda\Omega_{12})\Omega_{24}, \quad \beta'=\Omega_{24}\Omega_{33}-\Omega_{23}\Omega_{34}=(\Omega_{33}+\mu\Omega_{34})\Omega_{24},\\
\gamma &= (\Omega_{12}+\lambda\Omega_{22})\Omega_{23}, \quad \gamma'=-\lambda(\Omega_{24}\Omega_{34}-\Omega_{23}\Omega_{44})=\lambda\alpha',\\
\delta &= (\Omega_{12}+\lambda\Omega_{22})\Omega_{24}, \quad \delta'=-\lambda(\Omega_{23}\Omega_{34}-\Omega_{24}\Omega_{33})=\lambda\beta',\\
\alpha\alpha'+\beta\beta'+\gamma\gamma'+\delta\delta' &= (\Omega_{11}+2\lambda\Omega_{12}+\lambda^2\Omega_{22})(\Omega_{23}\alpha'+\Omega_{24}\beta')\\
&= \Omega_{24}^2(\Omega_{11}+2\lambda\Omega_{12}+\lambda^2\Omega_{22})(\Omega_{33}+2\mu\Omega_{34}+\mu^2\Omega_{44}),
\end{aligned}$$

i. e. either $\left(\frac{x_1+\lambda x_2}{1+\lambda}, \frac{y_1+\lambda y_2}{1+\lambda}, \frac{z_1+\lambda z_2}{1+\lambda}\right)$ or $\left(\frac{x_3+\mu x_4}{1+\mu}, \frac{y_3+\mu y_4}{1+\mu}, \frac{z_3+\mu z_4}{1+\mu}\right)$ lies on the absolute, and hence at least one of the lines L, L' is parallel to the polar of the other.]

If L and L' are simply perpendicular and intersect, $B=0$ and $\Delta=0$, one value of arc $\overline{LL'}$ is $\frac{1}{2}\,\Theta$, while the other is 0.

From these definitions of perpendicularity it follows that *the arc of the distance between two given planes is equal to the arc of the (non-evanescent) distance between the two straight lines drawn from any point of their intersection, one in each plane perpendicular to their intersection.* This is easily proved; let $M(u_1, v_1, w_1)$ and $M'(u_2, v_2, w_2)$ be the two planes, and let $M''(u_3, v_3, w_3)$ be the plane of the two perpendiculars to which reference is made in the theorem; then, since the

intersections of M and M'' and of M' and M'' are perpendicular to the intersection of M and M',

$$\mho_{11}\mho_{23} - \mho_{12}\mho_{13} = 0;$$
$$\mho_{12}\mho_{23} - \mho_{22}\mho_{13} = 0,$$

by (35), but in general $\mho_{11}\mho_{22} - \mho_{12}^2$ does not vanish, hence

$$\mho_{13} = 0, \quad \mho_{23} = 0,$$

and the angle between MM'' and $M'M''$ is, by (12),

$$2ik^{\text{v}} \cos^{-1} \frac{\mho_{12}}{\sqrt{\mho_{11}\mho_{22}}},$$

which proves the theorem.

This proof also shows that *if two planes are perpendicular to the same third plane, their intersection is perpendicular to that plane and to every straight line in it.* Similarly *if two points are perpendicular to the same third point, their junction is perpendicular to that point and to every straight line through it.* With these definitions it is not true that every straight line perpendicular to a given plane is perpendicular to every straight line in that plane, or even to every intersecting line in that plane; through any given point there pass an infinite number of straight lines perpendicular to a given plane, namely all which pass through the point and lie in either of the planes tangent to the absolute through the junction of the point with the pole of the given plane. One of these lines, namely the junction of the given point with the pole of the given plane is perpendicular to every straight line in the plane. [It is possible that the definition of perpendicularity of a point and a plane, a point and a line, and of a plane and a line should be narrower than that here given; namely, it may be that a point should be considered perpendicular to a given plane only when it *coincides* with the pole of the plane, instead of being simply at a zero-distance from it; that a straight line should be considered perpendicular to a point only when it lies in the polar plane of the point; and that a straight line should be considered perpendicular to a plane only when it passes through the pole of the plane. This narrower definition of perpendicularity will not, however, affect the further results of this paper.] If two straight lines are perpendicular to the same plane, they will not necessarily be parallel to each other; and two planes which are perpendicular to the same straight line will not necessarily be parallel to each other. It is also evident that two planes parallel to the same plane are not in general parallel to each other; and two straight lines parallel to the same straight line are not necessarily parallel to each other (the odds are equal that they will be so).

If $T(x, y, z)$, $T'(x+dx_1, y+dy_1, z+dz_1)$ and $T''(x+dx_2, y+dy_2, z+dz_2)$ are three infinitely near points, and on TT' and TT'' a plane quadrilateral is constructed whose other two sides are parallel to these, it will be seen, from (13), (12) and (6), that

$$\overline{TT'}=\frac{2ik}{\Omega}\sqrt{\Omega\delta_1^2\Omega-(\delta_1\Omega)^2}=\frac{2ik}{\Omega}\sqrt{A},\quad \overline{TT''}=\frac{2ik}{\Omega}\sqrt{\Omega\delta_2^2\Omega-(\delta_2\Omega)^2}=\frac{2ik}{\Omega}\sqrt{C},$$

$$\sin\frac{\overline{(TT',TT'')}}{2ik^{\mathrm{V}}}=\sqrt{\frac{AC-B^2}{AC}},\quad \overline{(TT',T'')}=2ik'''\sqrt{\frac{\begin{vmatrix}\Omega, & \delta_1\Omega, & \delta_2\Omega\\ \delta_1\Omega, & \delta_1^2\Omega, & \delta_1\delta_2\Omega\\ \delta_2\Omega, & \delta_2\delta_1\Omega, & \delta_2^2\Omega\end{vmatrix}}{\Omega[\Omega\delta_1^2\Omega-(\delta_1\Omega)^2]}}$$

(dx_1, &c., being infinitesimal), where $\overline{TT'}$ and $\overline{TT''}$ denote the lines TT' and TT'', $\overline{(TT',TT'')}$ is the angle between them, $\overline{(TT',T'')}$ is the distance between the line TT' and the point T'', $\delta_1\Omega$, $\delta_2\Omega$, $\delta_1^2\Omega$, $\delta_2^2\Omega$ are what $\delta\Omega$ and $\delta^2\Omega$ already defined become on the substitution of dx_1, dy_1, dz_1 and dx_2, dy_2, dz_2 for dx, dy, dz, and

$$\begin{aligned}\delta_1\delta_2\Omega=\delta_2\delta_1\Omega=\frac{1}{2}\Big[&\frac{\partial^2\Omega}{\partial x^2}dx_1dx_2+\frac{\partial^2\Omega}{\partial y^2}dy_1dy_2+\frac{\partial^2\Omega}{\partial z^2}dz_1dz_2\\ &+\frac{\partial^2\Omega}{\partial y\partial z}(dy_1dz_2+dz_1dy_2)+\frac{\partial^2\Omega}{\partial z\partial x}(dz_1dx_2+dx_1dz_2)\\ &+\frac{\partial^2\Omega}{\partial x\partial y}(dx_1dy_2+dy_1dx_2)\Big],\end{aligned}$$

$$A=\Omega\delta_1^2\Omega-(\delta_1\Omega)^2,\quad C=\Omega\delta_2^2\Omega-(\delta_2\Omega)^2,\quad B=\Omega\delta_1\delta_2\Omega-\delta_1\Omega\delta_2\Omega,$$

$$\begin{aligned}B^2-AC&=\begin{vmatrix}\Omega\delta_1^2\Omega-(\delta_1\Omega)^2, & \Omega\delta_1\delta_2\Omega-\delta_1\Omega\delta_2\Omega\\ \Omega\delta_2\delta_1\Omega-\delta_2\Omega\delta_1\Omega, & \Omega\delta_2^2\Omega-(\delta_2\Omega)^2\end{vmatrix}\\ &=\Omega\begin{vmatrix}\Omega & \delta_1\Omega & \delta_2\Omega\\ \delta_1\Omega & \delta_1^2\Omega & \delta_1\delta_2\Omega\\ \delta_2\Omega & \delta_2\delta_1\Omega & \delta_2^2\Omega\end{vmatrix};\end{aligned}$$

hence

$$\overline{TT'}\,\overline{TT''}\sin\frac{\overline{(TT',TT'')}}{2ik^{\mathrm{V}}}=\frac{(2ik)^2}{\Omega^{\frac{3}{2}}}\sqrt{\begin{vmatrix}\Omega & \delta_1\Omega & \delta_2\Omega\\ \delta_1\Omega & \delta_1^2\Omega & \delta_1\delta_2\Omega\\ \delta_2\Omega & \delta_2\delta_1\Omega & \delta_2^2\Omega\end{vmatrix}},$$

$$\overline{TT''}\,\overline{(TT'',T')}=\overline{TT'}\,\overline{(TT',T'')}=\frac{(2ik)(2ik''')}{\Omega^{\frac{3}{2}}}\sqrt{\begin{vmatrix}\Omega & \delta_1\Omega & \delta_2\Omega\\ \delta_1\Omega & \delta_1^2\Omega & \delta_1\delta_2\Omega\\ \delta_2\Omega & \delta_2\delta_1\Omega & \delta_2^2\Omega\end{vmatrix}};$$

I therefore take as the measure of the area of the infinitesimal parallelogram, k^{VI} being a new constant,

$$(39)\qquad \overline{TT'T''} = d\bar{\sigma} = \frac{(2ik^{\mathrm{VI}})^2}{\Omega^{\frac{3}{2}}}\sqrt{\begin{vmatrix} \Omega & \delta_1\Omega & \delta_2\Omega \\ \delta_1\Omega & \delta_1^2\Omega & \delta_1\delta_2\Omega \\ \delta_2\Omega & \delta_2\delta_1\Omega & \delta_2^2\Omega \end{vmatrix}} = \frac{1}{K_2\Omega^{\frac{3}{2}}}\sqrt{\begin{vmatrix} \Omega & \delta_1\Omega & \delta_2\Omega \\ \delta_1\Omega & \delta_1^2\Omega & \delta_1\delta_2\Omega \\ \delta_2\Omega & \delta_2\delta_1\Omega & \delta_2^2\Omega \end{vmatrix}};$$

where $K_2 = \frac{1}{(2ik^{\mathrm{VI}})^2}$, and this expression for $d\bar{\sigma}$ is not altered if T'' be moved to any other position on the side opposite TT', provided its distance from T remain infinitesimal; *the areas of infinitesimal parallelograms are therefore proportional to the products of their bases by their altitudes*, either side being taken as the base. The area of any figure is then measured by the integral of (39) over the whole figure; namely, the figure is to be considered as divided into infinitesimal parallelograms, and its area is the sum of the areas of these parallelograms. It is to be noticed that the expression in the right-hand member of (39) remains unaltered if T'' be infinitely little displaced on any straight line, through its former position, intersecting TT' at a distance from T which is not infinitesimal; hence for this purpose an infinitesimal parallelogram may be defined as an infinitesimal quadrilateral whose opposite sides intersect at distances from its vertices which are not infinitesimal.

Similarly if $T'''(x+dx_3,\ y+dy_3,\ z+dz_3)$ be a fourth point infinitely near T,

$$\overline{TT'''} = \frac{2ik}{\Omega}\sqrt{\Omega\delta_3^2\Omega - (\delta_3\Omega)^2},$$

where the notation is sufficiently explained by the above definitions of δ and δ^2; the equation of the plane $TT'T''$ (ξ, η, ζ being current coordinates) is

$$\begin{vmatrix} \xi & \eta & \zeta & 1 \\ x & y & z & 1 \\ dx_1 & dy_1 & dz_1 & 0 \\ dx_2 & dy_2 & dz_2 & 0 \end{vmatrix} = 0,$$

or

$$u\xi + v\eta + w\zeta + 1 = 0,$$

where

$$\begin{vmatrix} x & y & z \\ dx_1 & dy_1 & dz_1 \\ dx_2 & dy_2 & dz_2 \end{vmatrix} u = -\begin{vmatrix} dy_1 & dz_1 \\ dy_2 & dz_2 \end{vmatrix},\quad \begin{vmatrix} x & y & z \\ dx_1 & dy_1 & dz_1 \\ dx_2 & dy_2 & dz_2 \end{vmatrix} v = -\begin{vmatrix} dz_1 & dx_1 \\ dz_2 & dx_2 \end{vmatrix},$$

$$\begin{vmatrix} x & y & z \\ dx_1 & dy_1 & dz_1 \\ dx_2 & dy_2 & dz_2 \end{vmatrix} w = -\begin{vmatrix} dx_1 & dy_1 \\ dx_2 & dy_2 \end{vmatrix},$$

whence

$$\begin{vmatrix} x & y & z \\ dx_1 & dy_1 & dz_1 \\ dx_2 & dy_2 & dz_2 \end{vmatrix}^2 \mho = \begin{vmatrix} dy_1 & dz_1 \\ dy_2 & dz_2 \end{vmatrix}^2 + \begin{vmatrix} dz_1 & dx_1 \\ dz_2 & dx_2 \end{vmatrix}^2 + \begin{vmatrix} dx_1 & dy_1 \\ dx_2 & dy_2 \end{vmatrix}^2 - \begin{vmatrix} x & y & z \\ dx_1 & dy_1 & dz_1 \\ dx_2 & dy_2 & dz_2 \end{vmatrix}^2$$

$$= \delta_1^2\Omega\,\delta_2^2\Omega - (\delta_1\delta_2\Omega)^2 - \begin{vmatrix} \Omega+1 & \delta_1\Omega & \delta_2\Omega \\ \delta_1\Omega & \delta_1^2\Omega & \delta_1\delta_2\Omega \\ \delta_2\Omega & \delta_2\delta_1\Omega & \delta_2^2\Omega \end{vmatrix}$$

$$= - \begin{vmatrix} \Omega & \delta_1\Omega & \delta_2\Omega \\ \delta_1\Omega & \delta_1^2\Omega & \delta_1\delta_2\Omega \\ \delta_2\Omega & \delta_2\delta_1\Omega & \delta_2^2\Omega \end{vmatrix};$$

then, with the notation of (9),

$$\begin{vmatrix} x & y & z \\ dx_1 & dy_1 & dz_1 \\ dx_2 & dy_2 & dz_2 \end{vmatrix} M_3 = - \begin{vmatrix} dx_1 & dy_1 & dz_1 \\ dx_2 & dy_2 & dz_2 \\ dx_3 & dy_3 & dz_3 \end{vmatrix},$$

$$\sin\frac{\overline{(TT'T'', TT''')}}{2ik^{\mathrm{IV}}} = \frac{\begin{vmatrix} dx_1 & dy_1 & dz_1 \\ dx_2 & dy_2 & dz_2 \\ dx_3 & dy_3 & dz_3 \end{vmatrix}\sqrt{\Omega}}{\sqrt{-\begin{vmatrix} \Omega & \delta_1\Omega & \delta_2\Omega \\ \delta_1\Omega & \delta_1^2\Omega & \delta_1\delta_2\Omega \\ \delta_2\Omega & \delta_2\delta_1\Omega & \delta_2^2\Omega \end{vmatrix}[\Omega\delta_3^2\Omega - (\delta_3\Omega)^2]}},$$

where $TT'T''$ means the plane of T, T' and T'', and by (5), replacing the infinitesimal angle by its sine,

$$\overline{TT'T'', T'''} = 2ik'' \frac{\begin{vmatrix} dx_1 & dy_1 & dz_1 \\ dx_2 & dy_2 & dz_2 \\ dx_3 & dy_3 & dz_3 \end{vmatrix}}{\sqrt{-\begin{vmatrix} \Omega & \delta_1\Omega & \delta_2\Omega \\ \delta_1\Omega & \delta_1^2\Omega & \delta_1\delta_2\Omega \\ \delta_2\Omega & \delta_2\delta_1\Omega & \delta_2^2\Omega \end{vmatrix}\Omega}};$$

hence

$$\overline{TT'T''}\;\overline{TT'''}\sin\frac{\overline{(TT'T'', TT''')}}{2ik^{\mathrm{IV}}} = \frac{(2ik)(2ik^{\mathrm{VI}})^2}{i\Omega^2}\begin{vmatrix} dx_1 & dy_1 & dz_1 \\ dx_2 & dy_2 & dz_2 \\ dx_3 & dy_3 & dz_3 \end{vmatrix},$$

$$\overline{TT'T''}\;\overline{TT'T'', T'''} = \frac{(2ik'')(2ik^{\mathrm{VI}})^2}{i\Omega^2}\begin{vmatrix} dx_1 & dy_1 & dz_1 \\ dx_2 & dy_2 & dz_2 \\ dx_3 & dy_3 & dz_3 \end{vmatrix};$$

I take therefore as the measure of the volume of the infinitesimal parallelopiped on the three edges LL', LL'' and LL''', k^{VI} being a new constant,

$$(40)\qquad \overline{TT'\,T''\,T'''} = d\bar{v} = \frac{(2ik^{\text{VII}})^3}{i\Omega^2}\begin{vmatrix} dx_1 & dy_1 & dz_1 \\ dx_2 & dy_2 & dz_2 \\ dx_3 & dy_3 & dz_3 \end{vmatrix} = \frac{1}{iK_3\,\Omega^2}\begin{vmatrix} dx_1 & dy_1 & dz_1 \\ dx_2 & dy_2 & dz_2 \\ dx_3 & dy_3 & dz_3 \end{vmatrix},$$

where $K_3 = \frac{1}{(2ik^{\text{VII}})^3}$, and an infinitesimal parallelopiped need merely be defined as an infinitesimal hexahedron whose opposite faces meet at distances from its vertices which are not infinitesimal. It then follows that *the volumes of infinitesimal parallelopipeds are proportional to the products of their bases by their altitudes*, either face of each being taken as the base. The volume of any solid may be found by integrating (40) over the whole solid; and for this purpose we may suppose the solid divided into infinitesimal parallelopipeds by planes whose equations are $x = \text{const.}$, $y = \text{const.}$, $z = \text{const.}$, so that dy_1, dz_1, dx_2, dz_2, dx_3, dy_3 vanish, and

$$(41)\qquad d\bar{v} = \frac{1}{iK_3}\frac{dx_1 dy_2 dz_3}{\Omega^2}.$$

The constants K_2 and K_3 correspond to the constant curvatures of space of two and three dimensions in Gauss' theory (see p. 210).

A convenient transformation is

$$(42)\qquad x = \rho\sin\vartheta\cos\omega,\quad y = \rho\sin\vartheta\sin\omega,\quad z = \rho\cos\vartheta,$$

whence

$$(43)\qquad \begin{cases} dx = d\rho\sin\vartheta\cos\omega + \rho\cos\vartheta\cos\omega\, d\vartheta - \rho\sin\vartheta\sin\omega\, d\omega, \\ dy = d\rho\sin\vartheta\sin\omega + \rho\cos\vartheta\sin\omega\, d\vartheta + \rho\sin\vartheta\cos\omega\, d\omega, \\ dz = d\rho\cos\vartheta \qquad\qquad - \rho\sin\vartheta\, d\vartheta; \end{cases}$$

then, with the value of Ω given by (1),

$$\delta\Omega = \rho\, d\rho,\ \delta^2\Omega = d\rho^2 + \rho^2 d\vartheta^2 + \rho^2\sin^2\vartheta . d\omega^2,$$
$$\delta_1\delta_2\Omega = d\rho_1 d\rho_2 + \rho^2 d\vartheta_1 d\vartheta_2 + \rho^2\sin^2\vartheta . d\omega_1 d\omega_2,$$

and (13), (39) and (40) become

$$(44)\qquad d\bar{s} = 2ik\frac{\sqrt{-d\rho^2 - \rho^2(1-\rho^2)(d\vartheta^2 + \sin^2\vartheta d\omega^2)}}{1-\rho^2},$$

$$(45)\ d\bar{\sigma} = \frac{1}{K_2}\frac{\rho}{1-\rho^2}\sqrt{\frac{\rho^2(1-\rho^2)\sin^2\vartheta(d\vartheta_1 d\omega_2 - d\vartheta_2 d\omega_1)^2 + (d\rho_1 d\vartheta_2 - d\rho_2 d\vartheta_1)^2 + \sin^2\vartheta(d\rho_1 d\omega_2 - d\rho_2 d\omega_1)^2}{1-\rho^2}},$$

$$(46)\qquad d\bar{v} = \frac{1}{iK_3}\frac{\rho^2\sin\vartheta}{(1-\rho^2)^2}\begin{vmatrix} d\rho_1 & d\vartheta_1 & d\omega_1 \\ d\rho_2 & d\vartheta_2 & d\omega_2 \\ d\rho_3 & d\vartheta_3 & d\omega_3 \end{vmatrix},$$

which expressions may be taken either positive or negative; and, if a solid is divided into infinitesimal parallelopipeds by surfaces of the systems $\rho = \text{const.}$, $\vartheta = \text{const.}$, $\omega = \text{const.}$, the volume of any one of these parallelopipeds is

(47) $$d\bar{v} = \frac{1}{iK_3}\frac{\rho^2 \sin\vartheta\, d\rho_1\, d\vartheta_2\, d\omega_3}{(1-\rho^2)^2},$$
corresponding to (41).

It will be seen that the new definitions here given are extensions of Professor Cayley's definitions of "projective" measurement (Sixth Memoir upon Quantics, *Phil. Trans.*, vol. 149, 1859) as generalized by Professor Klein (*Math. Ann.*, Vol. IV).

Application to the Sphere.

A sphere is defined as the locus of points at a given distance (radius of the sphere) from a given point (centre of the sphere); the radius being $\bar{r}$ and centre (x_1, y_1, z_1), the condition that (x_2, y_2, z_2) should be a point of the sphere is, by (3),

(48) $$\Omega_{12}^2 = \Omega_{11}\Omega_{22}\cos^2\left(\frac{\bar{r}}{2ik}\right),$$
or, say,

(49) $$\lambda\Omega_{11}\Omega_{22} - \mu\Omega_{12}^2 = 0,$$
where Ω_{11}, Ω_{22}, Ω_{12} are defined by (1).

The sphere (49) will be a cone for certain values of λ, μ determined by the equation

$$0 = \begin{vmatrix} \lambda\Omega_{11} - \mu x_1^2\,, & -\mu x_1 y_1, & -\mu x_1 z_1, & \mu x_1 \\ -\mu x_1 y_1, & \lambda\Omega_{11} - \mu y_1^2\,, & -\mu y_1 z_1, & \mu y_1 \\ -\mu x_1 z_1\,, & -\mu y_1 z_1, & \lambda\Omega_{11} - \mu z_1^2\,, & \mu z_1 \\ \mu x_1\,, & \mu y_1\,, & \mu z_1\,, & -\lambda\Omega_{11} - \mu \end{vmatrix}$$
$$= \lambda^3(\mu - \lambda)\Omega_{11}^4,$$
i. e. in general for $\lambda = 0$ and $\mu = \lambda$.

For $\lambda = 0$ the equation of the sphere is

(50) $$\Omega_{12}^2 = 0,$$
i. e. the sphere consists of two planes coincident with the polar plane of the centre with respect to the absolute. In this case

$$\cos^2\left(\frac{\bar{r}}{2ik}\right) = \frac{\lambda}{\mu} = 0,$$
i. e.

(51) $$\bar{r} = ik\,\eth.$$

A great circle on such a sphere consists of two coincident straight lines. It is then evident that a plane may be considered as *half* a sphere of radius $ik\eth$, and a straight line as *half* a circle of the same radius. The outside of such a

sphere is the portion of space included between the coincident planes, so that the sphere *includes* all space, and the outside of such a circle is the portion of its plane between the coincident lines, so that the circle *includes* the whole plane. I shall presently find, by this means, the length of the whole straight line, the circumference and area of the whole plane, the area of the boundary of all space, and the volume of all space.

For $\mu = \lambda$ the equation of the sphere is

(52) $$\Omega_{11}\Omega_{22} - \Omega_{12}^2 = 0,$$

i. e. the sphere is the tangent cone to the absolute from the centre. In this case

$$\cos^2\left(\frac{\bar{r}}{2ik}\right) = \frac{\lambda}{\mu} = 1,$$

i. e.

(53) $$\bar{r} = 0.$$

In the general consideration of the sphere, I take, for simplicity, the centre as origin, and write the variables (x, y, z) instead of (x_2, y_2, z_2); equation (48) then becomes

$$\Omega = -\sec^2\left(\frac{\bar{r}}{2ik}\right),$$

i. e.

(54) $$x^2 + y^2 + z^2 = 1 - \sec^2\left(\frac{\bar{r}}{2ik}\right) = -\tan^2\left(\frac{\bar{r}}{2ik}\right) = \tau^2, \text{ say.}$$

With the polar system (42), the equation of the sphere becomes

(55) $$\rho = \tau.$$

The form (1) of Ω allows perfect freedom in the choice of the origin, of one coördinate axis through the origin, and of one coordinate plane through this axis, but when the origin, axis and plane have been chosen, the whole system is determined. The circumference and area of any great circle of the sphere will be most conveniently determined by taking this plane as the plane $z = 0$, *i. e.* the plane $\vartheta = \frac{1}{2}\Theta$. Then the circumference of the circle is, by (44) (putting $\rho = \tau$, $\vartheta = \frac{1}{2}\Theta$, $d\rho = 0$, $d\vartheta = 0$),

(56) $$C_{\bar{r}} = \pm 2ik \frac{\tau i}{\sqrt{1-\tau^2}} \int_0^{2\Theta} d\omega = \pm 4ik\Theta \frac{\tau i}{\sqrt{1-\tau^2}} = \pm 4ik\Theta \sin\left(\frac{\bar{r}}{2ik}\right)$$
$$= \pm 2k\Theta\left(\mathfrak{S}^{\frac{\bar{r}}{2k}} - \mathfrak{S}^{-\frac{\bar{r}}{2k}}\right);$$

the latter of which expressions was given by Gauss; and the area of the circle is, by (45) (putting $\vartheta = \frac{1}{2}\Theta$, $d\rho_1 = d\rho$, $d\vartheta_1 = 0$, $d\omega_1 = 0$, $d\rho_2 = 0$, $d\vartheta_2 = 0$, $d\omega_2 = d\omega$),

$$(57) \quad A_{\bar{r}} = \pm \frac{1}{K_2} \int_0^\tau \int_0^{2\circleddash} \frac{\rho\, d\rho\, d\omega}{(1-\rho^2)^{\frac{3}{2}}} = \pm \frac{2\circleddash}{K_2}\left(\frac{1}{\sqrt{1-\tau^2}} - 1\right)$$
$$= \mp \frac{2\circleddash}{K_2}\left[1 - \cos\left(\frac{\bar{r}}{2ik}\right)\right] = \mp \frac{4\circleddash}{K_2} \sin^2\left(\frac{\bar{r}}{4ik}\right) = \pm \frac{\circleddash}{K_2}\left(\mathfrak{S}^{\frac{\bar{r}}{4k}} - \mathfrak{S}^{-\frac{\bar{r}}{4k}}\right)^2.$$

The area of the surface of the sphere of radius $\bar{r}$ is, by (45) (putting $\rho = \tau$, $d\rho_1 = 0$, $d\vartheta_1 = d\vartheta$, $d\omega_1 = 0$, $d\rho_2 = 0$, $d\vartheta_2 = 0$, $d\omega_2 = d\omega$)

$$(58) \quad S_{\bar{r}} = \pm \frac{1}{K_2} \frac{\tau^2}{1-\tau^2} \int_0^{\circleddash} \int_0^{2\circleddash} \sin\vartheta\, d\vartheta\, d\omega = \mp \frac{4\circleddash}{K_2} \sin^2\left(\frac{\bar{r}}{2ik}\right)$$
$$= \pm \frac{\circleddash}{K_2}\left(\mathfrak{S}^{\frac{\bar{r}}{2k}} - \mathfrak{S}^{-\frac{\bar{r}}{2k}}\right)^2;$$

and the volume of the sphere is, by (47) (putting $d\rho_1 = d\rho$, $d\vartheta_2 = d\vartheta$, $d\omega_3 = d\omega$),

$$(59) \quad V_{\bar{r}} = \pm \frac{1}{iK_3} \int_0^\tau \int_0^{\circleddash} \int_0^{2\circleddash} \frac{\rho^2 \sin\vartheta\, d\rho\, d\vartheta\, d\omega}{(1-\rho^2)^2} = \pm \frac{\circleddash}{K_3}\left[\frac{\bar{r}}{ik} - \sin\left(\frac{\bar{r}}{ik}\right)\right]$$
$$= \mp \frac{\circleddash}{2iK_3}\left(\mathfrak{S}^{\frac{\bar{r}}{k}} - \mathfrak{S}^{-\frac{\bar{r}}{k}} - 2\frac{\bar{r}}{k}\right).$$

From (57) and (58) follows

$$(60) \quad S_{\bar{r}} = A_{2\bar{r}},$$

i. e. the area of the surface of a sphere equals the area of a circle of twice the radius, which corresponds to the theorem of the Euclidean geometry, commonly stated thus: *the area of the surface of a sphere equals four times the area of a circle of the same radius.*

Formulæ (56), (57), (58) and (59) applied to the case in which $\bar{r} = ik\circleddash$ give

$$(61) \quad C_{ik\circleddash} = 4ik\circleddash, \; A_{ik\circleddash} = \frac{2\circleddash}{K_2}, \; S_{ik\circleddash} = \frac{4\circleddash}{K_2}, \; V_{ik\circleddash} = \frac{\circleddash^2}{K_3},$$

whence *the length of the whole straight line is* $2ik\circleddash$, *the area of the whole plane is* $\frac{2\circleddash}{K_2}$, *the area of the surface of all space is* $\frac{4\circleddash}{K_2}$, *and the volume of all space is* $\frac{\circleddash^2}{K_3}$.

In the Euclidean geometry $K_2 = -\frac{1}{4k^2}$, $K_3 = \frac{i}{8k^3}$, and k is infinite (see Professor Klein's paper, *Math. Annalen*, Vol. IV, p. 592, lines 3, 4, where c has the same meaning as k in the present paper), hence $\frac{\bar{r}}{ik}$ is infinitesimal,

$$\sin\left(\frac{\bar{r}}{2ik}\right) = \frac{\bar{r}}{2ik}, \; \sin\left(\frac{\bar{r}}{4ik}\right) = \frac{\bar{r}}{4ik}, \; \frac{\bar{r}}{ik} - \sin\left(\frac{\bar{r}}{ik}\right) = \frac{1}{6}\left(\frac{\bar{r}}{ik}\right)^3, \text{ and}$$

$$(62) \quad C_{\bar{r}} = 2\circleddash\bar{r}, \; A_{\bar{r}} = \circleddash\bar{r}^2, \; S_{\bar{r}} = 4\circleddash\bar{r}^2, \; V_{\bar{r}} = \frac{4}{3}\circleddash\bar{r}^3,$$

the ordinary formulæ.

Before considering the areas of figures drawn on a sphere, it will be convenient to obtain the formulæ of non-Euclidean spherical trigonometry. A spherical triangle is a figure on the sphere bounded by three arcs of great circles (intersections of the sphere with planes through the centre). The angle between any two such arcs may be measured by the distance (angle) between their planes or by the distance (angle) between the tangents to them at their intersection; the angles of the triangle measured in the second way I call $\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$, then it may easily be proved that the angles measured in the first way will be $\frac{k'}{k^{\mathrm{v}}}\bar{\alpha}$, $\frac{k'}{k^{\mathrm{v}}}\bar{\beta}$, $\frac{k'}{k^{\mathrm{v}}}\bar{\gamma}$; the lengths of the sides or bounding arcs opposite $\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$ I call respectively $\bar{a}$, $\bar{b}$, $\bar{c}$.

I begin with the case of a right triangle (a right angle is the angle between perpendicular lines); let $\bar{\gamma}$ be the right angle, *i. e.* $\bar{\gamma} = ik^{\mathrm{v}}\Theta$, take the centre of the sphere as origin, the radius to the angle $\bar{\gamma}$ as axis of x, the plane of the side $\bar{a}$ as the plane $y = 0$ (*i. e.* $\omega = 0$), then the plane of the side $\bar{b}$ will be the plane $z = 0$ (*i. e.* $\vartheta = \frac{1}{2}\Theta$). For the radius to the angle $\bar{\beta}$, then, $\omega = 0$, $\vartheta = \vartheta_1$, say, and for the radius to the angle $\bar{\alpha}$, $\vartheta = \frac{1}{2}\Theta$, $\omega = \omega_1$, say. Then, by (44)

$$\bar{a} = \mp 2k\frac{\tau}{\sqrt{1-\tau^2}}\int_{\vartheta_1}^{\frac{1}{2}\Theta} d\vartheta = \pm\, 2ik\sin\left(\frac{\bar{r}}{2ik}\right).(\tfrac{1}{2}\Theta - \vartheta_1),$$

$$\bar{b} = \mp 2k\frac{\tau}{\sqrt{1-\tau^2}}\int_0^{\omega_1} d\omega = \pm\, 2ik\sin\left(\frac{\bar{r}}{2ik}\right).\omega_1;$$

the plane of $\bar{c}$, the hypothenuse of the triangle, has the equation

$$\begin{vmatrix} \sin\vartheta\cos\omega & \sin\vartheta\sin\omega & \cos\vartheta \\ \cos\omega_1 & \sin\omega_1 & 0 \\ \sin\vartheta_1 & 0 & \cos\vartheta_1 \end{vmatrix} = 0,$$

which may also be written

$$\sin(\omega_1 - \omega) = \tan\vartheta_1 \sin\omega_1 \cot\vartheta,$$

from which follows

$$\sqrt{d\vartheta^2 + \sin^2\vartheta.d\omega^2} = \frac{\sin\vartheta.d\vartheta\sqrt{\tan^2\vartheta_1\sin^2\omega_1 + 1}}{\sqrt{1-(\tan^2\vartheta_1\sin^2\omega_1+1)\cos^2\vartheta}},$$

and then, by (44),

$$\bar{c} = \mp 2k\frac{\tau}{\sqrt{1-\tau^2}}\int_{\vartheta_1}^{\frac{1}{2}\Theta}\sqrt{d\vartheta^2 + \sin^2\vartheta.d\omega^2} = \mp 2k\frac{\tau}{\sqrt{1-\tau^2}}\int_{\vartheta_1}^{\frac{1}{2}\Theta}\frac{\sin\vartheta\, d\vartheta\sqrt{\tan^2\vartheta_1\sin^2\omega_1+1}}{\sqrt{1-(\tan^2\vartheta_1\sin^2\omega_1+1)\cos^2\vartheta}}$$

$$= \pm\, 2ik\sin\left(\frac{\bar{r}}{2ik}\right)\cos^{-1}(\sin\vartheta_1\cos\omega_1);$$

the planes of $\bar{a}$, $\bar{b}$, $\bar{c}$ are

$$y=0\,,\ z=0\,,\ -x\cos\vartheta_1\sin\omega_1+y\cos\vartheta_1\cos\omega_1+z\sin\vartheta_1\sin\omega_1=0\,,$$

respectively, hence by (4),

$$\frac{k'}{k^{\mathrm{V}}}\bar{\alpha}=2ik'\cos^{-1}\left(\frac{\sin\vartheta_1\sin\omega_1}{\sqrt{\cos^2\vartheta_1+\sin^2\vartheta_1\sin^2\omega_1}}\right)=2ik'\tan^{-1}(\cot\vartheta_1\operatorname{cosec}\omega_1),$$

$$\frac{k'}{k^{\mathrm{V}}}\bar{\beta}=2ik'\cos^{-1}\left(\frac{\cos\vartheta_1\cos\omega_1}{\sqrt{\cos^2\vartheta_1+\sin^2\vartheta_1\sin^2\omega_1}}\right)=2ik'\tan^{-1}(\tan\vartheta_1\tan\omega_1),$$

i. e. $\qquad \bar{\alpha}=2ik^{\mathrm{V}}\tan^{-1}(\cot\vartheta_1\operatorname{cosec}\omega_1),\ \bar{\beta}=2ik^{\mathrm{V}}\tan^{-1}(\sec\vartheta_1\tan\omega_1).$

From these values of $\bar{a}$, $\bar{b}$, $\bar{c}$, $\bar{\alpha}$, $\bar{\beta}$ follow

$$\cos\left(\frac{\bar{c}}{2ik\sin\left(\frac{\bar{r}}{2ik}\right)}\right)=\cos\left(\frac{\bar{a}}{2ik\sin\left(\frac{\bar{r}}{2ik}\right)}\right)\cos\left(\frac{\bar{b}}{2ik\sin\left(\frac{\bar{r}}{2ik}\right)}\right),$$

$$\tan\left(\frac{\bar{a}}{2ik\sin\left(\frac{\bar{r}}{2ik}\right)}\right)=\sin\left(\frac{\bar{b}}{2ik\sin\left(\frac{\bar{r}}{2ik}\right)}\right)\tan\left(\frac{\bar{\alpha}}{2ik^{\mathrm{V}}}\right),$$

$$\tan\left(\frac{\bar{b}}{2ik\sin\left(\frac{\bar{r}}{2ik}\right)}\right)=\sin\left(\frac{\bar{a}}{2ik\sin\left(\frac{\bar{r}}{2ik}\right)}\right)\tan\left(\frac{\bar{\beta}}{2ik^{\mathrm{V}}}\right),$$

where the ambiguous signs in the right-hand members have been so determined that for small values of $\bar{a}$ and $\bar{b}$, $\bar{c}$ shall be small, and for a small positive value of $\bar{b}$ and values of $\bar{\alpha}$ and $\bar{\beta}$ less than $ik^{\mathrm{V}}\Theta$, $\bar{a}$ shall have a small positive value. Now these three formulæ are what the corresponding formulæ of the Euclidean trigonometry for the sphere of radius unity become when for each side is substituted that side divided by $2ik\sin\left(\frac{\bar{r}}{2ik}\right)$ and for each angle is substituted that angle divided by $2ik^{\mathrm{V}}$; and it is possible to obtain the formulæ for any spherical triangle whatever, by dividing it into two right triangles, from the formulæ just written; hence *the formulæ for the non-Euclidean trigonometry on a sphere of radius* $\bar{r}$ *are obtained from the formulæ of the ordinary spherical trigonometry by substituting for each side that side divided by* $2ik\sin\left(\frac{\bar{r}}{2ik}\right)$ *and for each angle that angle divided by* $2ik^{\mathrm{V}}$.

The area of the right triangle above considered is, by (45) (putting $\rho=\tau$, $d\vartheta_1=d\vartheta$, $d\omega_1=0$, $d\vartheta_2=0$, $d\omega_2=d\omega$)

$$\bar{\sigma}=\pm\frac{1}{K_2}\frac{\tau^2}{1-\tau^2}\int_0^{\omega_1}\int_{\vartheta}^{\frac{1}{2}\Theta}\sin\vartheta\,d\vartheta\,d\omega$$

where the lower limit of ϑ is

$$\cot^{-1}\left(\frac{\sin(\omega_1-\omega)}{\tan\vartheta_1\sin\omega_1}\right)=\cos^{-1}\left(\frac{\sin(\omega_1-\omega)}{\sqrt{1+\tan^2\vartheta_1\sin^2\omega_1-\cos^2(\omega_1-\omega)}}\right),$$

i. e.

$$\begin{aligned}\bar{\sigma}&=\mp\frac{1}{K_2}\sin^2\left(\frac{\bar{r}}{2ik}\right)\int_0^{\omega_1}\frac{\sin(\omega_1-\omega)\,d\omega}{\sqrt{1+\tan^2\vartheta_1\sin^2\omega_1-\cos^2(\omega_1-\omega)}}\\
&=\mp\frac{1}{K_2}\sin^2\left(\frac{\bar{r}}{2ik}\right)\left[\tan^{-1}\left(\frac{\cos(\omega_1-\omega)}{\sqrt{1+\tan^2\vartheta_1\sin^2\omega_1-\cos^2(\omega_1-\omega)}}\right)\right]_0^{\omega_1}\\
&=\mp\frac{1}{K_2}\sin^2\left(\frac{\bar{r}}{2ik}\right)\left[\tan^{-1}(\cot\vartheta_1\operatorname{cosec}\omega_1)-\tan^{-1}(\cot\vartheta_1\cot\omega_1)\right]\\
&=\mp\frac{1}{K_2}\sin^2\left(\frac{\bar{r}}{2ik}\right)\left(\frac{\bar{\alpha}}{2ik^{\text{v}}}+\frac{\bar{\beta}}{2ik^{\text{v}}}-\frac{\eth}{2}\right)\\
&=\mp\frac{1}{K_2}\sin^2\left(\frac{\bar{r}}{2ik}\right)\left(\frac{\bar{\alpha}+\bar{\beta}+\bar{\gamma}}{2ik^{\text{v}}}-\eth\right).\end{aligned}$$

Now any spherical oblique triangle whose angles are $\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$ can be divided into two spherical right triangles the sum of whose angles is $\bar{\alpha}+\bar{\beta}+\bar{\gamma}+2ik^{\text{v}}\eth$, and the sum of the areas of these right triangles, as determined by the formula just written, *i. e.* the area of the oblique triangle, is

$$\frac{1}{K_2}\sin^2\left(\frac{\bar{r}}{2ik}\right)\left(\frac{\bar{\alpha}+\bar{\beta}+\bar{\gamma}}{2ik^{\text{v}}}-\eth\right), \tag{63}$$

i. e. the area of any spherical triangle on a sphere of radius $\bar{r}$ *is* $\frac{1}{K_2}\sin^2\left(\frac{\bar{r}}{2ik}\right)$ *times the excess of the sum of its angles divided by* $2ik^{\text{v}}$ *over* $\eth$. For the plane $\bar{r}=ik\eth$, and the area is

$$\frac{1}{K_2}\left(\frac{\bar{\alpha}+\bar{\beta}+\bar{\gamma}}{2ik^{\text{v}}}-\eth\right). \tag{64}$$

This expression shows that the Euclidean area of any triangle on a sphere of radius unity bears the constant ratio K_2 to the non-Euclidean area of the plane triangle having the same angles (when $k^{\text{v}}=-\frac{1}{2}i$), which is a proof, different from that given by Professor Klein (*Math. Ann.*, Vol. IV, p. 619), that the plane non-Euclidean geometry is identical with the Euclidean geometry on a surface of constant curvature K_2; and this proof proceeds directly from Gauss' definition of curvature.

These formulæ can evidently be extended to a spherical polygon of any number of sides; the area of a spherical polygon of n sides is $\frac{1}{K_2}\sin^2\left(\frac{\bar{r}}{2ik}\right)$ times the excess of the sum of its angles divided by $2ik^{\text{v}}$ over $(n-2)\eth$.

The area of a conic may be found in a similar manner, namely, taking the plane of the conic as the plane $z = 0$, and any two sides of that triangle which is self-conjugate with respect to the conic and also with respect to the section of the absolute by the plane $z = 0$ as the axes of x and y, $\Omega = x^2 + y^2 - 1$ and the equation of the conic is

$$\frac{x^2}{p^2} + \frac{y^2}{q^2} - 1 = 0,$$

where $(\pm p,\, 0)$ and $(0,\, \pm q)$ are the intersections of the curve with the axes of x and y. If now $\pm \bar{a}$ and $\pm \bar{b}$ are the distances of these intersections from the origin, by (3),

$$\cos^2\left(\frac{\bar{a}}{2ik}\right) = \frac{1}{1-p^2}, \text{ i. e. } p = \pm i \tan\left(\frac{\bar{a}}{2ik}\right),$$

$$\cos^2\left(\frac{\bar{b}}{2ik}\right) = \frac{1}{1-q^2}, \text{ i. e. } q = \pm i \tan\left(\frac{\bar{b}}{2ik}\right),$$

and the equation of the conic is

$$\frac{x^2}{\tan^2\left(\frac{\bar{a}}{2ik}\right)} + \frac{y^2}{\tan^2\left(\frac{\bar{b}}{2ik}\right)} + 1 = 0,$$

or the coordinates of any point of it may be written

$$x = p\cos\psi = i\tan\left(\frac{\bar{a}}{2ik}\right)\cos\psi,\; y = q\sin\psi = i\tan\left(\frac{\bar{b}}{2ik}\right)\sin\psi,$$

and the area of the conic is, by (39) (putting $z = 0$, $dx_1 = dx$, $dy_1 = 0$, $dx_2 = 0$, $dy_2 = dy$)

$$\frac{1}{K_2}\int\!\!\int \frac{dx\,dy}{(1-x^2-y^2)^{\frac{3}{2}}} = \frac{4}{K_2}\int_0^p \left[\frac{y}{(1-x^2)\sqrt{1-x^2-y^2}}\right]_{y=0}^{y=\frac{q}{p}\sqrt{p^2-x^2}} dx$$

$$= \frac{4}{K_2}\sin\left(\frac{\bar{a}}{2ik}\right)\cos^2\left(\frac{\bar{a}}{2ik}\right)\tan\left(\frac{\bar{b}}{2ik}\right)\int_0^{\frac{1}{2}\odot} \frac{\sin^2\psi\,.\,d\psi}{\left[1-\sin^2\left(\frac{\bar{a}}{2ik}\right)\sin^2\psi\right]\sqrt{1-\left(1-\frac{\cos^2\left(\frac{\bar{a}}{2ik}\right)}{\cos^2\left(\frac{\bar{b}}{2ik}\right)}\right)\sin^2\psi}}$$

$$= \frac{4i}{K_2}\int_0^{F_1} \frac{\varkappa^2 \operatorname{sn} a \operatorname{cn} a \operatorname{dn} a \operatorname{sn}^2 u\, du}{1-\varkappa^2\operatorname{sn}^2 a \operatorname{sn}^2 u} = \frac{4i}{K_2}\Pi_1(u,\, a),$$

where am u, modulus $\varkappa$, and parameter a are defined thus:

$$\operatorname{am} u = \psi,\; \varkappa' = \sqrt{1-\varkappa^2} = \frac{\cos\left(\frac{\bar{a}}{2ik}\right)}{\cos\left(\frac{\bar{b}}{2ik}\right)},\; -\varkappa^2\operatorname{sn}^2 a = -\sin^2\left(\frac{\bar{a}}{2ik}\right),$$

F_1 is the complete elliptic integral of the first kind, and $\Pi_1(u,\, a)$ is the complete elliptic integral of the third kind in Jacobi's notation.

BALTIMORE, *December*, 1882.

On Cubic Curves.

By F. Franklin.

The following discussion of some points in the theory of cubics is given on account of the method by which the results are deduced; most, if not all, of the theorems obtained are known. I take the system of cubics $x^3+y^3+z^3+6lxyz=0$, and obtain easily and consecutively a number of results, some relating to single cubics and some to the system, from the consideration that the mixed polars of a pair of points with respect to all the cubics pass through a fixed point, and that the polar lines and polar conics of a point likewise pass through one and four fixed points respectively. These facts are obvious, since the equations of these polar derivatives contain linearly a single indeterminate, l.

The system of cubics, the system of mixed polars of a given pair of points, 1 and 2, the system of polar conics of a given point, the system of polar lines of a given point, are given by the equations

(1) $$x^3+y^3+z^3+6lxyz=0$$

(2) $$x_1x_2x+y_1y_2y+z_1z_2z+l\{(y_1z_2+y_2z_1)x+(z_1x_2+z_2x_1)y+(x_1y_2+x_2y_1)z\}=0$$

(3) $$x_1x^2+y_1y^2+z_1z^2+2l(x_1yz+y_1zx+z_1xy)=0$$

(4) $$x_1^2x+y_1^2y+z_1^2z+2l(y_1z_1x+z_1x_1y+x_1y_1z)=0.$$

Through every point in the plane (not a common point of the system in question) there passes one and only one member of each of the above systems. It is to be remembered that the mixed polar of two points is the polar line of either with respect to the polar conic of the other; and it is plain that if the mixed polar of A and B with respect to any cubic passes through C, the mixed polar of A and C passes through B, and in particular that if C is the fixed point on the mixed polars of A and B, B is the fixed point on the mixed polars of A and C.

First let us inquire whether the fixed point on the mixed polars of a pair of points, 1 and 2, lies upon the line joining 1 and 2. The necessary and sufficient condition that it should, is that when the mixed polar is made to pass through 1, it shall also pass through 2.

Putting in equation (2) for x, y, z successively x_1, y_1, z_1 and x_2, y_2, z_2, we obtain

$$(5)\quad \begin{aligned} x_1^2x_2+y_1^2y_2+z_1^2z_2+2l(y_1z_1x_2+z_1x_1y_2+x_1y_1z_2)=0 \\ x_2^2x_1+y_2^2y_1+z_2^2z_1+2l(y_2z_2x_1+z_2x_2y_1+x_2y_2z_1)=0\,, \end{aligned}$$

which are to be satisfied by the same value of l. Hence the required condition is

$$0=\begin{vmatrix} x_1^2x_2+y_1^2y_2+z_1^2z_2, & y_1z_1x_2+z_1x_1y_2+x_1y_1z_2 \\ x_2^2x_1+y_2^2y_1+z_2^2z_1, & y_2z_2x_1+z_2x_2y_1+x_2y_2z_1 \end{vmatrix}=(x_1^3+y_1^3+z_1^3)x_2y_2z_2-(x_2^3+y_2^3+z_2^3)x_1y_1z_1.$$

But this is obviously also the condition that the two points shall lie on the same cubic of the system. Hence, in order that the fixed point on the mixed polar of 1 and 2 should be on their junction, it is necessary and sufficient that the points 1 and 2 lie on the same cubic of the system. But if 3 is the fixed point on the mixed polars of 1 and 2, 2 is the fixed point on the mixed polars of 1 and 3; hence, by what has just been proved, 3 also is on the same cubic as 1; therefore 3 is the connective of 1 and 2.

It follows from the above, that the connective of two given points, 1 and 2, is the common intersection of the lines

$$x_1x_2x+y_1y_2y+z_1z_2z=0 \qquad (a)$$

$$(y_1z_2+y_2z_1)x+(z_1x_2+z_2x_1)y+(x_1y_2+x_2y_1)z=0 \qquad (b)$$

$$(y_1z_2-y_2z_1)x+(z_1x_2-z_2x_1)y+(x_1y_2-x_2y_1)z=0\,, \qquad (c)$$

the last of these being the junction of 1 and 2. Or, taking the sum and difference of (b) and (c), the connective is the common intersection of the lines

$$x_1x_2x+y_1y_2y+z_1z_2z=0 \qquad (a')$$

$$y_1z_2x+z_1x_2y+x_1y_2z=0 \qquad (b')$$

$$y_2z_1x+z_2x_1y+x_2y_1z=0\,. \qquad (c')$$

From (b') and (c') we obtain for the coordinates of the connective

$$x:y:z=\begin{Vmatrix} y_1z_2 & z_1x_2 & x_1y_2 \\ y_2z_1 & z_2x_1 & x_2y_1 \end{Vmatrix}=x_1^2y_2z_2-x_2^2y_1z_1:y_1^2z_2x_2-y_2^2z_1x_1:z_1^2x_2y_2-z_2^2x_1y_1\,,$$

which are the values given by Professor Sylvester, *Amer. Jour. Math.* Vol. III, p. 62; while from (a) and (c) we obtain the values given by Cauchy, viz:

$$x:y:z=\begin{Vmatrix} x_1x_2 & y_1y_2 & z_1z_2 \\ (y_1z_2) & (z_1x_2) & (x_1y_2) \end{Vmatrix}.$$

When 1 and 2 coincide, their connective becomes the tangential, and equations (a) and (b) become

$$x_1^2x+y_1^2y+z_1^2z=0$$

$$y_1z_1x+z_1x_1y+x_1y_1z=0$$

whence

$$x:y:z = x_1(y_1^3 - z_1^3):y_1(z_1^3 - x_1^3):z_1(x_1^3 - y_1^3).$$

We have found that if two points lie on the same cubic, the fixed point on their mixed polars is their connective; and thus (since when the two points coincide, the mixed polar becomes the polar line) that the fixed point on the polar lines of a given point is the tangential of that point. The four fixed points on the polar conics of a given point are evidently points which have the given point for the fixed point on their polar lines; therefore by what precedes, they are on the cubic on which the given point lies, and have the given point for their tangential. In particular, it may be noted that the degenerate polar conics of a point are the three pairs of lines obtained by joining the four anti-tangentials of the point.

The condition that two points shall have the same tangential (which includes the condition that they shall lie on the same cubic) is the simultaneous satisfiability of the four equations

$$x_1^2x + y_1^2y + z_1^2z = 0, \quad y_1z_1x + z_1x_1y + x_1y_1z = 0;$$
$$x_2^2x + y_2^2y + z_2^2z = 0, \quad y_2z_2x + z_2x_2y + x_2y_2z = 0.$$

Combining the first equations of these pairs, we find

$$x:y:z = (y_1z_2)(y_1z_2 + y_2z_1):(z_1x_2)(z_1x_2 + z_2x_1):(x_1y_2)(x_1y_2 + x_2y_1);$$

and combining the second equations

$$x:y:z = (y_1z_2)x_1x_2:(z_1x_2)y_1y_2:(x_1y_2)z_1z_2;$$

and thus the condition that two points shall have a common tangential, *i. e.* that they shall be "corresponding" points in Maclaurin's sense, is

$$x_1x_2:y_1y_2:z_1z_2 = y_1z_2 + y_2z_1:z_1x_2 + z_2x_1:x_1y_2 + x_2y_1.$$

But this is also the condition that the mixed polar of the two points with respect to every cubic of the system shall be a fixed line; *i. e.* that the polar of either with respect to the system of polar conics of the other shall be a fixed line; so that each is one of the three double points in the system of polar conics of the other. Hence we have the theorem that if two points, 1 and 2, have the same tangential, each is the double point of a polar conic of the other, and conversely. Moreover, if 3 and 4 are the other two anti-tangentials of the tangential of 1, they are the other two double points of the system of polar conics of 1, and the polar of 2 with respect to any of these conics is $\overline{34}$; but we have seen that this polar, being the mixed polar of 1 and 2, passes through their connective: whence the theorem that if 1234 be the anti-tangentials of any point on the cubic, $\overline{12}$ and $\overline{34}$ intersect on the cubic. Of the four points, any

two are a pair of corresponding points, and the other two may be called the complementary pair of corresponding points; and we have found that the junction of a pair of corresponding points is the mixed polar of the complementary pair, and that the junctions of two complementary pairs constitute the polar conic, with respect to some cubic, of the common tangential of the four points. Hence any centre of the quadrangle 1234 is the double point of a polar conic of the common tangential, so that the three centres and the common tangential form again a system of corresponding points.—It is to be observed that we have now a special geometrical interpretation of the equations

$$x_1x_2x + y_1y_2y + z_1z_2z = 0, \; (y_1z_2 + y_2z_1)x + (z_1x_2 + z_2x_1)y + (x_1y_2 + x_2y_1)z = 0$$

in the case when 1 and 2 are corresponding points (and when consequently these two lines coincide); namely, if 3 and 4 are the complementary pair of corresponding points, the above is the equation of $\overline{34}$, the line which, with $\overline{12}$, forms one of the degenerate polar conics of the common tangential.

We have found the necessary and sufficient condition that the mixed polar of a pair of points with respect to some cubic of the system should be the line joining the points to be that the two points lie on the same cubic; but the algebraical expression of this condition, viz. the coexistence of equations (5), is the same as that of the condition that each point should be, with respect to some cubic of the system, an intersection of the polar line and polar conic of the other; hence we see that the intersections of the polar line and polar conic of a point with respect to *any* cubic of the system lie on the cubic passing through the point. Thus, then, denoting by A the cubic passing through 1 and by X a variable cubic of the system, we have found that of the six intersections with A of the polar conic of 1 with respect to X, four are fixed and two variable; of the three intersections with A of the polar line of 1 with respect to X, one is fixed and two variable; and the two variable intersections of the polar conic coincide with the two variable intersections of the polar line.

We have just found that the intersections of the polar line and polar conic of a point with respect to *any* cubic of the system lie on the cubic passing through the point; or in other words that the contacts of tangents drawn from a point to any of its polar conics lie on the cubic passing through the point. But the system of polar conics of a point with respect to the cubics is simply a system of conics through four fixed points; starting from such a system, then, instead of from the system of cubics, we may state some of the foregoing

results as follows: given a system of conics through four fixed points, the locus of the points of contact of tangents to these conics from any point P is a cubic which passes through P, through the four fixed points, and through the three centres of the complete quadrangle formed by these four points; the tangents to this cubic at the four fixed points pass through P, and the tangents at the three centres and at P meet in a point on the cubic.

To find the locus directly, take for the triangle of reference that formed by the three centres of the quadrangle; then the equation of the system may be taken to be

$$x^2-y^2+\lambda(x^2-z^2)=0,$$

and the polars of any point α,β,γ are

$$\alpha x-\beta y+\lambda(\alpha x-\gamma z)=0.$$

It may be noted in passing that the fixed point on these lines—which we have seen, is the tangential of α,β,γ with respect to the cubic locus sought—is $\frac{1}{\alpha},\frac{1}{\beta},\frac{1}{\gamma}$.

Eliminating λ between the above two equations, we find for the locus of the points of contact of tangents from α,β,γ to the conics,

$$\begin{vmatrix} x^2-y^2 & x^2-z^2 \\ \alpha x-\beta y & \alpha x-\gamma z \end{vmatrix}=0,$$

i. e. the cubic

$$x^2(\beta y-\gamma z)+y^2(\gamma z-\alpha x)+z^2(\alpha x-\beta y)=0.$$

For this cubic, we have

$$S=-(\alpha^4+\beta^4+\gamma^4)+\beta^2\gamma^2+\gamma^2\alpha^2+\alpha^2\beta^2$$

$$T=-4\{2(\alpha^6+\beta^6+\gamma^6)+12\alpha^2\beta^2\gamma^2-3(\beta^4\gamma^2+\beta^2\gamma^4+\gamma^4\alpha^2+\gamma^2\alpha^4+\alpha^4\beta^2+\alpha^2\beta^4)\}.$$

The discriminant is T^2+64S^3. Writing $\alpha^2,\beta^2,\gamma^2=f,g,h$ and observing that

$$f^2+g^2+h^2-gh-hf-fg=(f+\rho g+\rho^2h)(f+\rho^2g+\rho h),$$

where ρ is one of the imaginary cube roots of unity, we have

$$\begin{aligned}-S^3&=(f+\rho g+\rho^2h)^3(f+\rho^2g+\rho h)^3\\ &=\{f^3+g^3+h^3+6fgh+3\rho(g^2h+h^2f+f^2g)-3\rho^2(gh^2+hf^2+fg^2)\}\\ &\quad\times\{f^3+g^3+h^3+6fgh+3\rho^2(g^2h+h^2f+f^2g)-3\rho(gh^2+hf^2+fg^2)\}\end{aligned}$$

whence

$$\begin{aligned}-4S^3&=\{2(f^3+g^3+h^3+6fgh)-3(g^2h+gh^2+\dots)-3\sqrt{-3}(g^2h-gh^2+\dots)\}\\ &\quad\times\{2(f^3+g^3+h^3+6fgh)-3(g^2h+gh^2+\dots)-3\sqrt{-3}(g^2h-gh^2+\dots)\}\\ &=\{2(f^3+g^3+h^3)+12fgh-3(g^2h+gh^2+\dots)\}^2+27(g^2h-gh^2+\dots)^2.\end{aligned}$$

Also,

$$T^2 = 16\{2(f^3+g^3+h^3)+12fgh-3(g^2h+gh^2+\ldots)\}^2.$$

Hence

$$T^2+64S^3 = -16.27(g^2h-gh^2+h^2f-hf^2+f^2g-fg^2)^2 = -16.27\begin{vmatrix} 1 & 1 & 1 \\ f & g & h \\ f^2 & g^2 & h^2 \end{vmatrix}^2$$

$$= -16.27(g-h)^2(h-f)^2(f-g)^2 = -16.27(\beta^2-\gamma^2)^2(\gamma^2-\alpha^2)^2(\alpha^2-\beta^2)^2.$$

Hence, in order that the cubic shall have a double point, it is necessary and sufficient that $\beta^2=\gamma^2$ or $\gamma^2=\alpha^2$ or $\alpha^2=\beta^2$, *i. e.* that the point α, β, γ lie on one of the lines $y \pm z$, $z \pm x$, $x \pm y$; that is, on one of the degenerate conics of the system. But when $\beta=\gamma$, the equation of the cubic becomes

$$(y-z)\{\beta(x^2+yz)-\alpha x(y+z)\}=0,$$

and similarly for the other cases; hence the cubic cannot have a double point without breaking up into a straight line and a conic.

The discriminant of $\beta(x^2+yz)-\alpha x(y+z)$ is $\frac{1}{4}\beta(\alpha^2-\beta^2)$, which vanishes when $\beta=0$ or $\beta=\pm\alpha$; in the latter case P is one of the four base-points, in the former case it is one of the three centres: these, then, are the only positions of P which give rise to a cubic consisting of three straight lines.

It was obvious geometrically that if P were taken on one of the degenerate conics, that branch of the degenerate conic on which it lay would be part of the locus, which would thus consist of a straight line and a conic; and that if P were one of the four base-points or one of the three centres the locus would consist of three straight lines; the algebraic investigation has served to show that these are the only cases of singularity in the cubic.

On the Solution of the Differential Equation of Sources.

BY J. HAMMOND.

---

1. Let

$$c_{2m+1} = a_0^2 a_{2m+1} - u_1 a_0 a_1 a_{2m} + a_{2m-1}(u_2 a_0 a_2 + v_2 a_1^2) - a_{2m-2}(u_3 a_0 a_3 + v_3 a_1 a_2) + \ldots \quad (1)$$

where the general term is $a_{2m+1-\lambda}(u_\lambda a_0 a_\lambda + v_\lambda a_1 a_{\lambda-1})$,

and the last term is $\{a_{m+1}(u_m a_0 a_m + v_m a_1 a_{m-1}) - w_m a_1 a_m^2\}$,

be a particular solution of the differential equation

$$\left(a_0 \frac{d}{da_1} + 2a_1 \frac{d}{da_2} + 3a_2 \frac{d}{da_3} + \ldots + na_{n-1} \frac{d}{da_n}\right) z = 0. \quad (2)$$

Then the coefficients of (1) are found from the equations

$$\left.\begin{array}{l} 2m+1 = u_1 \\ 2mu_1 = 2u_2 + 2v_2 \\ (2m-1)u_2 = 3u_3 + v_3 \\ (2m-2)u_3 = 4u_4 + v_4 \\ \ldots\ldots\ldots\ldots \\ (2m+1-\lambda)u_\lambda = (\lambda+1)u_{\lambda+1} + v_{\lambda+1} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ (m+2)u_{m-1} = mu_m + v_m \\ (m+1)u_m = w_m \end{array}\right\} \quad (3)$$

$$\left.\begin{array}{l} (2m-1)v_2 = 2v_3 \\ (2m-2)v_3 = 3v_4 \\ \ldots\ldots\ldots \\ (2m+1-\lambda)v_\lambda = \lambda v_{\lambda+1} \\ \ldots\ldots\ldots\ldots \\ (m+2)v_{m-1} = (m-1)v_m \\ (m+1)v_m = 2mw_m \end{array}\right\} \quad (4)$$

The general solution of (4) is

$$v_\lambda = \frac{(2m+2-\lambda)(2m+3-\lambda)\ldots 2m}{(\lambda-1)!} c \quad (5)$$

where c is independent of λ.

Assume in (3)

$$u_\lambda = \frac{(2m+2-\lambda)(2m+3-\lambda)\ldots(2m+1)}{\lambda!} \alpha_\lambda$$

then $$(2m+1)\alpha_\lambda = (2m+1)\alpha_{\lambda+1} + c$$

or $$\alpha_\lambda = c' - \frac{c\lambda}{2m+1}.$$

To determine c and c' we have, since

$$u_1 = 2m+1,\ u_2 + v_2 = \frac{(2m+1)2m}{1.2} = 2mc + \frac{2m(2m+1)}{1.2}\left(c' - \frac{2c}{2m+1}\right)$$

$c' = 1$.

The other constant c must be determined by means of the equations

$$(m+1)u_m = w_m,\ (m+1)v_m = 2mw_m,\ \text{which give } v_m = 2mu_m$$

or $$\frac{(m+2)\dots 2m}{(m-1)!}c = 2m\left(1 - \frac{cm}{2m+1}\right)\frac{(m+2)\dots(2m+1)}{m!}$$

reducing to $$c = 2(2m+1-cm) \text{ or } c = 2.$$

Hence the coefficients of c_{2m+1} are

$$u_1 = 2m+1 \qquad\qquad v_2 = 2.2m$$

$$u_2 = \frac{(2m+1)2m}{1.2}\left(1 - \frac{4}{2m+1}\right) \qquad\qquad v_3 = 2.\frac{2m(2m-1)}{1.2}$$

$$u_3 = \frac{(2m+1)2m(2m-1)}{1.2.3}\left(1 - \frac{6}{2m+1}\right) \qquad\qquad v_4 = 2.\frac{2m(2m-1)(2m-2)}{1.2.3.}$$

$$u_4 = \frac{(2m+1)2m(2m-1)(2m-2)}{1.2.3.4}\left(1 - \frac{8}{2m+1}\right) \qquad\qquad \dots\dots\dots$$

$$\dots\dots\dots \qquad\qquad w_m = \frac{2m(2m-1)\dots(m+1)}{m!}.$$

Now since $v_2, v_3, v_4, \dots v_m$ are double the coefficients of $x, x^2, x^3, \dots x^{m-1}$ respectively in $(1+x)^{2m}$, and $w_m =$ the coefficient of the middle term, if Q_{2m} denote the quadrinvariant of the $2m^{\text{ic}}$, we have

$$C_{2m+1} = a_0 V_{2m+1} - 2a_1 Q_{2m} \tag{6}$$

where V_{2m+1} satisfies a similar differential equation to (2).

For if we call the operator of (2) δ, we have $\delta C_{2m+1} = 0$, $\delta Q_{2m} = 0$, and the result of operating on (6) with δ is

$$\delta V_{2m+1} = 2Q_{2m}.$$

2. The system of simultaneous equations

$$\frac{dz}{0} = \frac{da_1}{a_0} = \frac{da_2}{2a_1} = \frac{da_3}{3a_2} = \dots \frac{da_n}{na_{n-1}} \tag{7}$$

are satisfied by $z =$ const.

$$H = a_0 a_2 - a_1^2 = \text{const.}$$

$$C_3 = a_0^2 a_3 - 3a_0 a_1 a_2 + 2a_1^3 = \text{const.}$$

$$\dots\dots\dots\dots$$

$$Q_{2m} = \text{const.},\ C_{2m+1} = \text{const.}, \dots$$

For if we multiply both numerator and denominator of the first two fractions by $\frac{dH}{da_1}$, $\frac{dH}{da_2}$ respectively and add, each fraction $= \frac{dH}{0}$; the first three fractions giving in like manner $\frac{dC_3}{0}$; and so on for Q_{2m}, C_{2m+1}, . . .

Thus the general solution of (2) is

$$z = \phi(H, C_3, Q_4, C_5, \ldots) \tag{8}$$

the last letter being Q_n or C_n according as n is even or odd.

In proceeding from (7) to (8) any other set of $n-1$ particular solutions of (2) might have been used instead of H, C_3, Q_4, C_5, . . .; and the result would have been a perfectly general solution, provided all the letters $a_0, a_1, a_2, a_3, \ldots a_n$ were involved in the set chosen. But since H, C_3, Q_4, C_5, . . . are ground-sources, and ground-sources too of lower degree than any other set, (8) is the simplest possible form of the general solution of (2).

For all weights, up to 10 inclusive, we have

$$U = a_0$$
$$H = a_0a_2 - a_1^2$$
$$C_3 = a_0^2a_3 - 3a_0a_1a_2 + 2a_1^3$$
$$Q_4 = a_0a_4 - 4a_1a_3 + 3a_2^2$$
$$C_5 = a_0^2a_5 - 5a_0a_1a_4 + 2a_0a_2a_3 + 8a_1^2a_3 - 6a_1a_2^2$$
$$Q_6 = a_0a_6 - 6a_1a_5 + 15a_2a_4 - 10a_3^2$$
$$C_7 = a_0^2a_7 - 7a_0a_1a_6 + 9a_0a_2a_5 + 12a_1^2a_5 - 5a_0a_3a_4 - 30a_1a_2a_4 + 20a_1a_3^2$$
$$Q_8 = a_0a_8 - 8a_1a_7 + 28a_2a_6 - 56a_3a_5 + 35a_4^2$$
$$C_9 = a_0^2a_9 - 9a_0a_1a_8 + 20a_0a_2a_7 + 16a_1^2a_7 - 28a_0a_3a_6 - 56a_1a_2a_6 + 14a_0a_4a_5 + 112a_1a_3a_5 - 70a_1a_4^2$$
$$Q_{10} = a_0a_{10} - 10a_1a_9 + 45a_2a_8 - 120a_3a_7 + 210a_4a_6 - 126a_5^2,$$

where U is the source of the Quantic $(a_0, a_1, a_2, a_3, \ldots a_n)(x, y)^n$, H that of its Hessian, C_3 a ground-source for the Cubic and all higher Quantics, Q_4 for the Quartic and all higher Quantics; and generally Q_{2m} is a ground-source (the Quadrinvariant) for the $2m^{\text{ic}}$ and for all higher Quantics, the source of a covariant, C_{2m+1} is a ground-source for the $(2m+1)^{\text{ic}}$ and all higher Quantics.

3. A glance at the first terms of U, H, C_3, Q_4, C_5, . . . as written above, will suffice to show that none of the powers or products of these quantities is a linear function of any of the rest, and that the number of linearly independent compounds, of degree = weight $= w$, that can be formed of them is equal to the number of terms, of degree = weight $= w$, that can be formed of the letters $a_0, a_2, a_3, \ldots a_n$ alone. But these terms are equal in number to the linearly

independent sources of degree $=$ weight $= w$; for the general function of degree $=$ weight $= w$ may be formed by multiplying the general function of degree $=$ weight $= w-1$ by a_1 and adding on those terms that can be formed, of degree $=$ weight $= w$, with all the letters except a_1.

Hence all the sources of degree $=$ weight $= w$ are linear functions of the compounds of $U, H, C_3, Q_4, C_5, \ldots$; and if $w > 3$ there can be no ground-source whose degree and weight are equal.

Or if S denote a source of degree μ and weight w,

$$SU^{w-\mu} = \phi(U, H, C_3, Q_4, C_5, \ldots) \tag{9}$$

where ϕ is rational, integral, and homogeneous, of degree $=$ weight $= w$.

Thus in the case $w = 9$,

$$SU^{9-\mu} = \alpha_1 H^3 C_3 + \alpha_2 C_3^3 + U^2(\alpha_3 HC_3 Q_4 + \alpha_4 H^2 C_5) + U_4(\alpha_5 Q_4 C_5 + \alpha_6 C_3 Q_6 + \alpha_7 HC_7) + \alpha_8 U^6 C_9 \tag{10}$$

where $\alpha_1, \alpha_2, \alpha_3, \ldots$ are arbitrary constants.

When the order of the Quantic (n) is not less than w, the last suffix in (9) is w; but if $n < w$, every letter whose suffix is greater than n must be struck out. Thus in the case of the Octavic the term containing C_9 disappears from (10), and in the case of the Quartic (10) becomes $SU^{9-\mu} = (\alpha_1 H^3 + \alpha_2 C_3^2 + \alpha_3 U^2 HQ_4) C_3$, and every source of weight 9 for either the Quartic or Cubic contains C_3 as a factor.

When the arbitrary constants that occur in (9) are so chosen that S, or one or more of its factors, is a ground-source distinct from $H, C_3, Q_4, C_5, \ldots$ the result is a syzygant. The total number of such syzygants is the total number of ground-sources of weight w + the total number of compounds formed by multiplying together ground-sources of inferior weight, rejecting those that are made up of $H, C_3, Q_4, C_5, \ldots$ only; or it is the total number of compounds of degree $=$ weight $= w$ minus the number of linearly independent sources. In all cases where syzygants of other forms occur they are deducible from syzygants of this form and the same weight. No syzygant of weight $2m$ contains either Q_{2m} or C_{2m-1}, and no syzygant of weight $2m+1$ contains either C_{2m+1} or Q_{2m}.

The first example of a syzygant occurs when $w = 6$, in which case

$$TU^3 = U^2 HQ_4 - 4H^3 - C_3^2 \tag{11}$$

where T is the cubinvariant of the Quartic and a ground-source for all higher Quantics. The corresponding syzygant for the Cubic, viz.

$$U^2 \Delta - 4H^3 - C_3^2 = 0$$

is deduced from this as follows: we retain only the last two terms of (11), as

being the only terms corresponding to the Cubic, and noticing that the rejected terms are divisible by U^2 we replace them by a single term $U^2\Delta$. Hence Δ, the Discriminant of the Cubic is a *Special Form; i. e.* it is a groundform for the Cubic, but not for any higher Quantic. By this method we are enabled to detect Special Forms whenever they exist.

When $w=7$ there is only one syzygant, viz. after reduction by dividing out a factor U^2, we have

$$UP = HC_5 - C_3 Q_4 \tag{12}$$

where P is of degree 4, and a ground-source for the Quintic and all higher Quantics. There are obviously no Special Forms of weight 7. For the Quintic and Sextic there is only one ground-source of weight 7, viz. P (source of the covariant 4.6 for the Quintic, 4.10 for the Sextic, and $4.4n-14$ for the n^{ic}); the Septimic and all higher Quantics have also the ground-source C_7 of weight 7, (source of $3.3n-14$ for the n^{ic}). It may also be noticed that the order of the groundform P is higher than that of any other groundform of the fourth degree, and that H and C_3 are the highest ordered quadratic and cubic forms respectively; for example H, C_3 and P are the sources of 2.16, 3.24, and 4.26 respectively for the Decimic.

4. Those syzygants in which S is a compound are formed immediately from the syzygants of inferior weight. Thus in the case $w=9$ the only compounds, formed by multiplying together ground-sources of inferior weight, are, besides the powers and products of H, C_3, Q_4, ... $C_3 T$ and HP; and from (11) and (12) we obtain immediately

$$\left.\begin{aligned} C_3 TU^3 &= U^2 HC_3 Q_4 - 4H^3 C_3 - C_3^3 \\ HPU &= H^2 C_5 - HC_3 Q_4 \end{aligned}\right\} \tag{13}$$

To find the other syzygants we must determine the arbitrary constants in (10) so that S may be a ground-source, distinct from C_9. Hence $\alpha_8 = 0$ and

$$SU^{9-\mu} = \alpha_1 H^3 C_3 + \alpha_2 C_3^3 + U^2(\alpha_3 HC_3 Q_4 + \alpha_4 H^2 C_5) + U^4(\alpha_5 Q_4 C_5 + \alpha_6 C_3 Q_6 + \alpha_7 HC_7).$$

Now since S is a ground-source $\mu < 9$, and the above relation can only be identical when both sides are divisible by U, which can only happen when $\alpha_1 = 4\alpha_2$ and

$$SU^{9-\mu} = \alpha_2(4H^3 + C_3^2)C_3 + U^2(\alpha_3 HC_3 Q_4 + \alpha_4 H^2 C_5) + U^4(\alpha_5 Q_4 C_5 + \alpha_6 C_3 Q_6 + \alpha_7 HC_7).$$

The right-hand side of this when divided by U^2 consists of compound terms, so that $\mu < 7$ and both sides must be divisible by U^3 if S is to be a ground-source.

For this it is necessary that $\alpha_2 + \alpha_3 + \alpha_4 = 0$, whence

$$SU^{9-\mu} = \alpha_2(4H^3 + C_3^2 - U^2 HQ_4)C_3 + \alpha_4 U^2 H(HC_5 - C_3 Q_4) + U^4(\alpha_5 Q_4 C_5 + \alpha_6 C_3 Q_6 + \alpha_7 HC_7).$$

Or reducing by means of (13)

$$SU^{6-\mu} = \alpha_4 HP - \alpha_2 C_3 T + U(\alpha_5 Q_4 C_5 + \alpha_6 C_3 Q_6 + \alpha_7 HC_7) \qquad (14)$$

With the values

$T = a_0(a_2a_4 - a_3^2) - (a_1^2 a_4 - 2a_1a_2a_3 + a_2^3)$

$P = a_0^2(a_2a_5 - a_3a_4) - a_0(a_1^2a_5 + 2a_1a_2a_4 - 4a_1a_3^2 + a_2^2a_3) + 3a_1(a_1^2a_4 - 2a_1a_2a_3 + a_2^3)$

and those of H, C_3, ... we are able to determine the arbitrary constants of (14) so that the right-hand side is divisible by a_0^2; when

$$\left.\begin{array}{l} 3\alpha_4 = 2\alpha_2 = 6\beta \\ \beta = -6\alpha_5 = 6(\alpha_6 + \alpha_7) \end{array}\right\}$$

or if $\beta = 6\gamma$, $\alpha_5 = -\gamma$, $\alpha_6 = \gamma - \alpha_7$, (14) becomes

$$SU^{6-\mu} = \gamma\{U(C_3Q_6 - Q_4C_5) + 12HP - 18C_3T\} + \alpha_7 U(HC_7 - C_3Q_6).$$

Now suppose

$$SU^2 = U(C_3Q_6 - Q_4C_5) + 12HP - 18C_3T$$
$$S'U = HC_7 - C_3Q_6$$

where by actual calculation

$$S = a_0^2(a_3a_6 - a_4a_5) - a_0(3a_1a_2a_6 + 2a_1a_3a_5 - 9a_2^2a_5 - 5a_1a_4^2 + 17a_2a_3a_4 - 8a_3^3)$$
$$+ 2a_1(a_1^2a_6 - 3a_1a_2a_5 + a_1a_3a_4 + 3a_2^2a_4 - 2a_2a_3^2)$$

$$S' = a_0^2(a_2a_7 - a_3a_6) - a_0(a_1^2a_7 + 4a_1a_2a_6 - 6a_1a_3a_5 - 9a_2^2a_5 + 20a_2a_3a_4 - 10a_3^3)$$
$$+ 5a_1(a_1^2a_6 - 3a_1a_2a_5 + a_1a_3a_4 + 3a_2^2a_4 - 2a_2a_3^2).$$

Hence

$$2S' - 5S = UR,$$

where

$R = a_0(2a_2a_7 - 7a_3a_6 + 5a_4a_5)$

$\qquad - (2a_1^2a_7 - 7a_1a_2a_6 - 22a_1a_3a_5 + 27a_2^2a_5 + 25a_1a_4^2 - 45a_2a_3a_4 + 20a_3^3)$

And finally

$$U^2R = 2(HC_7 - C_3Q_6) - 5US,$$

showing that R and S are ground-sources.

When written in their standard form, the syzygants which give R and S are

$$\left.\begin{array}{l} SU^5 = U^4(C_3Q_6 - Q_4C_5) + 6U^2H(2HC_5 - 5C_3Q_4) + 18C_3(C_3^2 + 4H^3) \\ RU^6 = U^4(2HC_7 - 7C_3Q_6 + 5Q_4C_5) - 30U^2H(2HC_5 - 5C_3Q_4) - 90C_3(C_3^2 + 4H^3) \end{array}\right\} (15)$$

and, by the method of the preceding article, either of these will give the same Special Form Λ for the Quintic, which is the source of the covariant 5.7 and is given by the syzygant

$$U^4\Lambda = U^2H(2HC_5 - 5C_3Q_4) + 3C_3(C_3^2 + 4H^3).$$

The second equation of (15) gives no Special Form for the Sextic, for if we write

$U^4(A - 7C_3Q_6 + 5Q_4C_5) -$ &c. $= 0$ we have $U^4(A - 2C_3Q_6) = 5SU^5$, or $A = 2C_3Q_6 + 5SU$ which is a compound form.

It need hardly be noticed that the value of Λ given above is there given in its simplest form, whereas the rejection of the terms containing S and Q_6 in the

first equation of (15) and those containing R, Q_6, and C_7 in the second, would give two sources which though different are not really distinct from Λ.

The complete list of ground-sources of weight 9 is: for the Quintic Λ, source of the covariant 5.7; for the Sextic and all higher Quantics S, source of 4.6 for the Sextic, $4.4n-18$ for the n^{ic}; for the Septimic and all higher Quantics R, source of 3.3 for the Septimic, $3.3n-18$ for the n^{ic}; for the Nonic and all higher Quantics C_9, source of 3.9 for the Nonic, $3.3n-18$ for the n^{ic}. These results agree with Prof. Sylvester's Tables of Groundforms (American Journal, Vol. II, p. 223–251).

The syzygants (15) when reduced to their lowest terms become

$$\left.\begin{aligned} SU^2 &= U(C_3Q_6 - Q_4C_5) + 12HP - 18C_3T \\ RU^2 + 5SU &= 2(HC_7 - C_3Q_6) \end{aligned}\right\} \tag{16}$$

of which the first is a ground-syzygant for the Sextic and all higher Quantics, and the second is a ground-syzygant for the Septimic and all higher Quantics. These with the special ground-syzygant for the Quintic

$$U\Lambda = 2HP - 3C_3T$$

form a complete set of ground-syzygants of weight 9 for all Quantics.

5. If in equation (6) we write for an instant $a_0 = 0$ we have

$$\begin{aligned} C_{2m+1} &= -2a_1Q_{2m} \\ C_{2p+1} &= -2a_1Q_{2p} \\ C_{2q+1} &= -2a_1Q_{2q} \\ &\cdots\cdots\cdots\cdots \end{aligned}$$

whence it is easy to deduce

$$\begin{aligned} &C_{2m+1}^2 + 4HQ_{2m}^2 = U(*) \\ &C_{2m+1}Q_{2p} - C_{2p+1}Q_{2m} = U(*) \\ &C_{2m+1}C_{2p+1} + 4HQ_{2m}Q_{2p} = U(*) \\ &C_{2m+1}^3 + 4C_3Q_{2m}^3 = U(*) \\ &C_{2m+1}^2Q_{2p}Q_{2q} - Q_{2m}^2C_{2p+1}C_{2q+1} = U(*) \\ &C_{2m+1}C_{2p+1}C_{2q+1} + 4C_3Q_{2m}Q_{2p}Q_{2q} = U(*) \end{aligned}$$

with many other relations of like nature.

The expressions on the left-hand are of frequent occurrence in syzygants; the particular case $m=1$ gives $C_3^2 + 4H^3 = U^2\Delta$, for weight 8 there is a syzygant of the form $C_3C_5 + 4H^2Q_4 = U(*)$, and for weight 10 one of the form $C_5^2 + 4HQ_4^2 = U(*)$; other examples are easily found.

In the actual calculation of ground-sources equation (6) is sometimes useful; thus the relations $C_{2m+1} = a_0V_{2m+1} - 2a_1Q_{2m}$ and $C_{2p+1} = a_0V_{2p+1} - 2a_1Q_{2p}$ give

$$C_{2m+1}Q_{2p} - C_{2p+1}Q_{2m} = U(V_{2m+1}Q_{2p} - V_{2p+1}Q_{2m}),$$

and the advantage consists in this that V_{2m+1} has only $m+1$ terms, whilst C_{2m+1} has $2m+1$ terms.

Thus in calculating the ground-source P of weight 7 from the syzygant

$$PU = HC_5 - C_3 Q_4$$

we have

$$P = HV_5 - V_3 Q_4 = (a_0a_2 - a_1^2)(a_0a_5 - 3a_1a_4 + 2a_2a_3) - (a_0a_3 - a_1a_2)(a_0a_4 - 4a_1a_3 + 3a_2^2)$$

the expanded value is given already.

It may be noticed that Q_{2m} and V_{2m+1} may be expressed as the sum of determinants of the form $(a_p a_q - a_{p+1} a_{q-1})$; thus

$$H = a_0 a_2 - a_1^2$$
$$Q_4 = (a_0 a_4 - a_1 a_3) - 3(a_1 a_3 - a_2^2)$$
$$Q_6 = (a_0 a_6 - a_1 a_5) - 5(a_1 a_5 - a_2 a_4) + 10(a_2 a_4 - a_3^2)$$
$$Q_8 = (a_0 a_8 - a_1 a_7) - 7(a_1 a_7 - a_2 a_6) + 21(a_2 a_6 - a_3 a_5) - 35(a_3 a_5 - a_4^2)$$
$$Q_{10} = (a_0a_{10} - a_1a_9) - 8(a_1a_9 - a_2a_8) + 36(a_2a_8 - a_3a_7) - 84(a_3a_7 - a_4a_6) + 126(a_4a_6 - a_5^2)$$

. . . .

the coefficients of the determinants in Q_{2m} being those of the binomial expansion of $(1+x)^{2m-1}$.

To express V_{2m+1} in this way, observe that if $(a_p a_q - a_{p+1} a_{q-1}) = (p, q)$

$$\begin{aligned}\delta(p, q) &= \left\{ pa_{p-1}\frac{d}{da_p} + qa_{q-1}\frac{d}{da_q} + (p+1)a_p\frac{d}{da_{p+1}} + (q-1)a_{q-2}\frac{d}{da_{q-1}} \right\}(p, q) \\ &= p(p-1, q) + (q-1)(p, q-1)\end{aligned}$$

and the coefficients of V_{2m+1} are found from the differential equation

$$\delta V_{2m+1} = 2Q_{2m}.$$

The values obtained are

$$V_3 = a_0 a_3 - a_1 a_2$$
$$V_5 = (a_0 a_5 - a_1 a_4) - 2(a_1 a_4 - a_2 a_3)$$
$$V_7 = (a_0 a_7 - a_1 a_6) - 4(a_1 a_6 - a_2 a_5) + 5(a_2 a_5 - a_3 a_4)$$
$$V_9 = (a_0 a_9 - a_1 a_8) - 6(a_1 a_8 - a_2 a_7) + 14(a_2 a_7 - a_3 a_6) - 14(a_3 a_6 - a_4 a_5)$$

. . . .

and the coefficients are not Binomial Coefficients.

DISPROOF OF PROF. SYLVESTER'S FUNDAMENTAL POSTULATE.

The development of the G. F. for the Binary Septimic is

$$1 + ax^7 + a^2(x^2 + x^6 + x^{10} + x^{14}) + a^3(x^3 + x^5 + x^7 + 2x^9 + x^{11} + \ldots)$$
$$+ a^4(\ldots + x^6 + \ldots) + a^5(\ldots + 4x^{13} + \ldots) + \ldots$$

whence we see that there are four linearly independent covariants of deg-order (5.13), and exactly 4 compound covariants of the same deg-order, viz.

$$(1.7)(4.6),\ (2.2)(3.11),\ (2.6)(3.7),\ (2.10)(3.3).$$

Or with the notation of the present paper, using F to denote the covariant (4.6)

$$UF, \quad Q_6 C_5, \quad Q_4 C_7, \quad HR.$$

If then the Fundamental Postulate is universally true, the Binary Septimic has neither a groundform nor a syzygant of deg-order (5.13).

But among other relations it has been proved in the preceding article that

$$C_{2m+1}Q_{2p} - C_{2p+1}Q_{2m} = U(*)\,;$$

where obviously $(*)$ is a source of degree 4 and weight $2m+2p+1$, or what is the same thing a covariant of deg-order $(4.4n-4m-4p-2)$ for the n^{ic}.

In particular if $n=7$, $m=3$, $p=2$; $(*)$ is of deg-order (4.6), and since there is only one such covariant, viz. F, we have

$$Q_4 C_7 - C_5 Q_6 = UF.$$

Thus the number of linearly independent compounds is reduced to 3, and a groundform of deg-order (5.13) is necessary to make up the required number of independent forms, *i. e.* the Binary Septimic has both a groundform and a syzygant of deg-order (5.13).

Let Θ denote the source of the new groundform (5.13), then the values of F and Θ are found to be

$$\begin{aligned} F = {} & a_0^2(a_4a_7 - a_5a_6) \\ & - a_0(4a_1a_3a_7 - 3a_2^2a_7 + 2a_1a_4a_6 + 2a_2a_3a_6 - 6a_1a_5^2 + 6a_2a_4a_5 - 10a_3^2a_5 + 5a_3a_4^2) \\ & + 20a_1^2a_3a_6 - 15a_1a_2^2a_6 - 18a_1^2a_4a_5 - 24a_1a_2a_3a_5 + 27a_2^3a_5 + 45a_1a_2a_4^2 \\ & \qquad - 10a_1a_3^2a_4 - 45a_2^2a_3a_4 + 20a_2a_3^3 \\ \Theta = {} & a_0^2(a_2a_4a_5 - 3a_3^2a_5 + 2a_3a_4^2) \\ & - a_0(a_1^2a_4a_5 - 14a_1a_2a_3a_5 + 9a_2^3a_5 + 11a_1a_2a_4^2 + a_1a_3^2a_4 - 14a_2^2a_3a_4 + 6a_2a_3^3) \\ & - a_1(8a_1^2a_3a_5 - 6a_1a_2^2a_5 - 9a_1^2a_4^2 + 16a_1a_2a_3a_4 - 3a_2^3a_4 + 2a_2^2a_3^2 - 8a_1a_3^3). \end{aligned}$$

The compound sources of the same degree-order are

$$\begin{aligned} UF = {} & a_0^3(a_4a_7 - a_5a_6) - a_0^2(4a_1a_3a_7 - 3a_2^2a_7 + \&c.) + a_0(20a_1^2a_3a_6\ldots) \\ HR = {} & a_0^2(2a_2^2a_7 - 7a_2a_3a_6 + 5a_2a_4a_5) \\ & - a_0(4a_1^2a_2a_7 - 7a_1^2a_3a_6 - 7a_1a_2^2a_6 + 5a_1^2a_4a_5 - 22a_1a_2a_3a_5 + 27a_2^3a_5 \\ & \qquad + 25a_1a_2a_4^2 - 45a_2^2a_3a_4 + 20a_2a_3^3) \\ & + a_1^2(2a_1^2a_7 - 7a_1a_2a_6 - 22a_1a_3a_5 + 27a_2^2a_5 + 25a_1a_4^2 - 45a_2a_3a_4 + 20a_3^3) \\ C_7Q_4 = {} & a_0^3a_4a_7 - a_0^2(4a_1a_3a_7 - 3a_2^2a_7 + 7a_1a_4a_6 - 9a_2a_4a_5 + 5a_3a_4^2) \\ & + a_0(28a_1^2a_3a_6 - 21a_1a_2^2a_6 - 36a_1a_2a_3a_5 + 12a_1^2a_4a_5 + 27a_2^3a_5 - 30a_1a_2a_4^2 \\ & \qquad + 40a_1a_3^2a_4 - 15a_2^2a_3a_4) \\ & - 48a_1^3a_3a_5 + 36a_1^2a_2^2a_5 + 120a_1^2a_2a_3a_4 - 90a_1a_2^3a_4 - 80a_1^2a_3^3 + 60a_1a_2^2a_3^2 \\ C_5Q_6 = {} & a_0^3a_5a_6 - a_0^2(5a_1a_4a_6 - 2a_2a_3a_6 + 6a_1a_5^2 - 15a_2a_4a_5 + 10a_3^2a_5) \\ & + a_0(8a_1^2a_3a_6 - 6a_1a_2^2a_6 + 30a_1^2a_4a_5 - 12a_1a_2a_3a_5 - 75a_1a_2a_4^2 + 30a_2^2a_3a_4 \\ & \qquad + 50a_1a_3^2a_4 - 20a_2a_3^3) \\ & - 48a_1^3a_3a_5 + 36a_1^2a_2^2a_5 + 120a_1^2a_2a_3a_4 - 90a_1a_2^3a_4 - 80a_1^2a_3^3 + 60a_1a_2^2a_3^2. \end{aligned}$$

Whence it is easy to verify the relation

$$C_7Q_4 - C_5Q_6 = UF$$

and to see that Θ, HR, $C_7 Q_4$, $C_5 Q_6$, are not connected by any syzygy, or that Θ which is a ground-source for the Quintic and Sextic is also a ground-source for the Septimic.

SPECIMEN TABLE OF GROUNDFORMS FOR ALL BINARY QUANTICS.

2	3	4	5	6	7	8	9	10	n	Weight	Name	SYZYGANTS.
1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	1.10	$1.n$	0	U	
2.0	2.2	2.4	2.6	2.8	2.10	2.12	2.14	2.16	$2.2n-4$	2	H	
	3.3	3.6	3.9	3.12	3.15	3.18	3.21	3.24	$3.3n-6$	3	C_3	
		2.0	2.2	2.4	2.6	2.8	2.10	2.12	$2.2n-8$	4	Q_4	
			3.5	3.8	3.11	3.14	3.17	3.20	$3.3n-10$	5	C_5	
	4.0									6	Δ	$U^2\Delta = C_3^2 + 4H^3$
		3.0	3.3	3.6	3.9	3.12	3.15	3.18	$3.3n-12$	6	T	$U^3 T = U^2 HQ_4 - C_3^2 - 4H^3$
				2.0	2.2	2.4	2.6	2.8	$2.2n-12$	6	Q_6	
			4.6	4.10	4.14	4.18	4.22	4.26	$4.4n-14$	7	P	$UP = HC_5 - C_3 Q_4$
					3.7	3.10	3.13	3.16	$3.3n-14$	7	C_7	
			4.4							8		
				3.2	3.5	3.8	3.11	3.14	$3.3n-16$	8		
						2.0	2.2	2.4	$2.2n-16$	8	Q_8	
			5.7							9	Λ	$U\Lambda = 2HP - 3C_3 T$
				4.6	4.10	4.14	4.18	4.22	$4.4n-18$	9	S	$U^2 S = U(C_3 Q_6 - Q_4 C_5) + 12HP - 18C_3 T$
					3.3	3.6	3.9	3.12	$3.3n-18$	9	R	$U^2 R + 5US = 2(HC_7 - C_3 Q_6)$
							3.9	3.12	$3.3n-18$	9	C_9	
			4.0	4.4	4.8	4.12	4.16	4.20	$4.4n-20$	10		
					4.8					10		
						3.4	3.7	3.10	$3.3n-20$	10		
								2.0	$2.2n-20$	10	Q_{10}	

The arrangement of this table is by the weight of sources, given in the weight column; the table gives a complete list of groundforms of all weights up to 10, those with names and the syzygants are given in the present paper, the rest are copied from Prof. Sylvester's tables.

To embrace all Prof. Sylvester's results it would be necessary to continue the above table to weight 85.

Invariants are enclosed in dark lines to catch the eye.

Bibliography of Bernoulli's Numbers.

By G. S. Ely, *Fellow in Mathematics, Johns Hopkins University.*

The numbers of Bernoulli were first discovered by James Bernoulli, and the earliest mention he makes of them is in the *Ars Conjectandi* in connection with expressions for the sums of the powers of the natural numbers. Since then other uses of the numbers have been found, of which may be mentioned the expansion of various trigonometrical functions, Stirling's formula for the approximate value of the continued product of the numbers from 1 to n, and the Maclaurin sum-formula. It may not be out of place to mention the special points about which the most has been written; they are the Staudt or Clausen theorem concerning the divisors of Bernoulli's numbers, expressions for the sums of the powers of the natural numbers, various expansions by means of Bernoulli's numbers, expressions for the numbers in terms of definite integrals, relations existing between the numbers and the legitimacy of the use of the Maclaurin sum-formula, the latter point being one which Professor Sylvester characterizes as much the most important in the whole subject of Finite Differences. In the following bibliography of Bernoulli's numbers, which the author does not by any means regard as complete, will be found titles of such of the papers on the kindred subjects of Euler's numbers and the $\Delta^m 0^n$ class of numbers as have come under his eye and seemed to him to have sufficient bearing on the subject to warrant their introduction. Additions to the list may be madé in some future number of the Journal.

Adams, J. C. On some properties of Bernoulli's numbers, in particular on Clausen's Theorem respecting the fractional parts of these numbers: *Proc. Cam. Phil. Soc.*, ii, 269. 1872.

——Table of the first sixty-two numbers of Bernoulli: *Crelle*, lxxxv, 269. 1878.

——On the calculation of Bernoulli's numbers up to B_{62} by means of Staudt's Theorem: *Rep. Brit. Ass.*, 1877.

ARNDT. Entwickelung der Summen der n-ten Potenzen der natürlichen Zahlen nach den Potenzen Index: *Crelle*, xxxi, 249. 1846.

BAUER, G. Von den Gamma-functionen und einer besonderen Art unendlicher Product: *Crelle*, xxi, 270. 1840.

——Von einigen Summen- und Differenzen-Formeln und den Bernoulli'schen Zahlen: *Crelle*, lviii, 292. 1860.

BELLAVITIS, GIUSTO. Sulle serie di numeri che comprendono i Bernoulliani: *Tortolini Annali*, iv, 108–127. 1853.

BERNOULLI, JAMES. Ars Conjectandi. 1713.

BERTRAND. Cal. Diff. et Int., i, 148, 306, 346–7. 1870.

BESSEL. Ueber die Summation der Progression: *Astronomische Nachrichten*, xvi, No. 361. 1839.

BINET. Mémoires sur les Intégrales Définies Euleriennes: *Journal de l'École Polytechnique*, xxvii, 240 *et seq.* 1839.

BJÖRLING, E. G. Om de Bernoulli'ska talen: *Öfversigt Stockholm*, xiv, 107. 1857.

BLISSARD, J. On Generic Equations: *Quart. Jour.*, iv *et seq.* 1861.

—— On the properties of the $\Delta^m 0^n$ class of numbers: *Quart. Jour.*, viii, 85. 1867.

—— On the sums of reciprocals: *Quart. Jour.*, vi, 242. 1864.

BOOLE, G. Finite Differences, chaps. v–viii. 1872.

BRINKLEY, J. An investigation on the general term of an important series in the inverse method of Finite Differences: *Phil. Trans.*, 1807.

BROCARD. Solutions des questions proposées: *Nouv. Cor. Math.*, v, 282. 1879.

—— Sur les dénominateurs des nombres de Bernoulli: *Nouv. Cor. Math.*, v, 282. 1879.

—— Forme linéaire des diviseurs premiers des nombres de Bernoulli: *Nouv. Cor. Math.*, v, 284. 1879.

CALLANDREAU, O. Sur la formule sommatoire de Maclaurin: *Com. Ren.*, lxxxvi, 589. 1878.

CATALAN, E. Sur le calcul des nombres de Bernoulli: *Com. Ren.*, lviii, 1105. 1864.

—— Remarques sur une note de B. Le Besgue (relative aux nombres de Bernoulli): *Com. Ren.*, lvii, 902. 1863.

—— Sur les nombres de Bernoulli et d'Euler et sur quelques intégrales définies: *Brux. Acad. Sci. Mém.*, xxxvii. 1869.

Catalan, E. Sur les differences de 1^p et sur le calcul des nombres de Bernoulli: *Tortolini*, ii, 195. 1859.

——Sur les nombres de Bernoulli et sur quelques formules qui en dependent: *Com. Ren.*, liv, 1030 and 1089. 1862.

——Note sur la communication de M. C. Le Paige: *Bulletin de Belg.* (2) xli, 1018. 1876.

——Extraits d'une lettre à M. Hermite: *Nouv. Cor. Math.*, vi, 320. 1880.

——Note sur les nombres de Bernoulli: *Com. Ren.*, lxxxi, 441. 1875.

——Sur les nombres de Bernoulli: *Nouv. Cor. Math.*, iv, 119. 1878.

——Sur les nombres d'Euler: *Com. Ren.*, lxvi, 415. 1868.

——Recherches sur le développement de la function Γ: *Bull. de l'Acad. de Brux.*, xxxvi, 4. 1873.

——Sur les manières contradictoires de définir les nombres de Bernoulli: *Nouv. Cor. Math.*, v, 196. 1879.

Cayley, A. A dissertation on Bernoulli's numbers: *Messenger* (2), v, 157. 1875.

Cesaro, E. Quelques formules: *Nouv. Cor. Math.*, vi, 450. 1880.

Darboux. Sur les développements en série des functions d'une seule variable: *Liouville* (3), ii, 291. 1876.

De Morgan, A. Diff. and Int. Cal., chaps. xiii and xx. 1842.

Eisenlohe, O. Entwickelung der Functionsweise der Bernoulli'schen Zahlen: *Crelle*, xxxv, 193. 1847.

Ettingshausen. Vorlesungen über die höhere Mathematik: Bd. i.

Euler. De summis serierum numeros Bernoullianos involventium: *Academia Petropolitana*, xiv, 129. 1769.

——Institutiones Calculi Differentialis. 1755.

Eytelwein. Grundlehren der höhern Analysis: Bd. ii.

——Ueber die Vergleichung der Differenz-Coefficienten mit den Bernoulli'schen Zahlen: *Berlin Abhandl.* 1816, 28.

Fergola, Emmanuele. Sopra lo sviluppo della funzione $\frac{1}{ce^x-1}$ e sopra una nuova expressione die numeri di Bernoulli: *Napola Accad. Sci. Mem.*, ii, 1855–7.

Gilbert. Observations sur deux notes de M. Genocchi, relatives au développements de la function log $\Gamma(x)$: *Bull. de l'Acad. de Brux.*, xxxvi, 541. 1873.

Glaisher, J. W. L. On a deduction from Von Staudt's property of Bernoulli's numbers: *Proc. Lon. Math. Soc.*, iv, 212. 1872.

GLAISHER, J. W. L. On the function which stands in the same relation to Bernoulli's numbers that the gamma function does to factorials: *Brit. Assoc. Rep.*, 1872, 17.

——Table of the first 250 Bernoulli's numbers (to nine figures) and their logarithms (to ten figures): *Cam. Phil. Trans.*, xii, 384. 1871.

——Simple proof of a known property of Bernoulli's numbers: *Messenger*, ii, 190. 1872.

——On the constants that occur in certain summations by Bernoulli's numbers: *Proc. Lon. Math. Soc.*, iv, 48. 1872.

——Arithmetical proof of Clausen's Identity: *Messenger* (2), vi, 83. 1876.

——On Semi-convergent series: *Quart. Jour.*, xii, 52. 1873.

GENOCCHI, A. Sur la formule sommatoire de Maclaurin et les fonctions interpolaires: *Com. Ren.*, lxxxvi, 466. 1878.

——Sulla formula sommatoria di Eulero: *Tortolini*, iii, 406. 1852.

——Intorno all' espressione generali di numeri Bernoulliani: *Tortolini*, iii, 395. 1852.

GÖPEL. Einige Bemerkungen über Recursionsformeln für die Bernoulli'schen Zahlen: *Grunert*, iii, 64. 1843.

GRUNERT. Supplement to Klugel, Bernoulli'schen Zahlen.

HALL, T. G. Encyclopædia of Pure Mathematics: Art. *Calculus of Finite Differences*, 261–270. 1847.

HAMILTON, SIR WM. On an expression for the numbers of Bernoulli by means of a definite integral: *Phil. Mag.* xxiii, 360. 1843.

HAMMOND, J. On the relation between Bernoulli's numbers and the binomial coefficients: *Proc. Lon. Math. Soc.*, vii, 7. 1875.

HERMITE, CH. Sur la formule de Maclaurin: *Crelle*, lxxxiv, 64. 1878.

——Extrait d'une lettre à M. Borchardt: *Crelle*, lxxxi, 93. 1876.

——Extrait d'une lettre: *Nouv. Cor. Math.*, vi, 121. 1880.

HERSCHEL, J. F. W. On the development of exponential functions: *Phil. Trans.*, 1816, 25.

HILL, C. J. D. När äro de n forsta termerno af Bernoullis serie gifven funktion af den i den sista ingående derivatan: *Öfversigt*, xiv, 259. 1857.

HORNER, JOSEPH. On the forms of $\Delta^n 0^x$ and their congeners: *Quart. Jour.*, iv, 111 and 204. 1861.

JACOBI, C. J. G. De usu legitimo formulae summatoriae Maclaurinianae: *Crelle*, xii, 263. 1834.

JEFFERY, H. M. On Staudt's proposition relating to the Bernoullian numbers: *Quart. Jour.*, vi, 179. 1864.

JUNG, W. Bemerkungen über die Bernoulli'schen Zahlen: *Casopis*, ix, 103. 1880.

KNAR. Entwickelung den vorzüglichsten Eigenschaften einiger mit den goniometrischen zunachst verwandten Functionen: *Grunert*, xxvii, 365. 1856.

KÓRTEWEG, D. J. Over benaderings formelen voor de som von reeksen welke uit een groot aantal termen festaan: *Nieuw Archief voor Wiskunde*, ii, 161. 1876.

KRONECKER. Sur quelques fonctions symétriques et sur les nombres de Bernoulli: *Liouville*, i, 385. 1856.

KUMMER, E. F. Erganzungssätze der Reciprocitätsgesetze: *Crelle*, lvi, 270. 1859.

—— Ueber eine allgemeine Eigenschaft der rationalen Entwickelung coëfficienten einer bestimmten Gattung analytischer Functionen: *Crelle*, xli, 368. 1851.

KÜTTNER, W. Zur Theorie der Bernoulli'schen Zahlen: *Zeitschrift*, xxiv, 250. 1879.

LACROIX. Cal. Diff. et Int., iii, Arts. 951, 952, 985, 1001 *et seq.* 1819.

LAMPE. Auszug eines Schreibens an Herrn Stern über die Verallgemeinerung einer Jacobi'schen Formeln: *Crelle*, lxxxiv, 270. 1878.

LE BESGUE. Sur les nombres de Bernoulli: *Com. Ren.*, lviii, 853. 1864.

LE PAIGE, C. Note sur les nombres de Bernoulli: *Com. Ren.*, lxxxi, 966. 1875.

—— Sur les nombres de Bernoulli et sur quelques fonctions qui s'y rattachent: *Ann. Soc. Scient. Brux.*, i, B, 43. 1876.

—— Relation nouvelle entre les nombres de Bernoulli: *Bull. de Belg.* (2), xli, 1017. 1876.

—— Sur le développement de cot x: *Com. Ren.*, lxxxviii, 1075. 1879.

—— Sur une formule de Scherk: *Nouv. Cor. Math.*, iii, 159. 1877.

LIGOWSKI, W. Die Bestimmung der Summe Σx^r: *Grunert*, lxv, 329. 1880.

LIPSCHITZ, R. Ueber die Darstellung gewisser Functionen durch die Eulersche Summen-formel: *Crelle*, lvi, 11. 1859.

LONGCHAMPS, G. Sur les nombres de Bernoulli: *Annales de l'École Normale* (2), viii, 55. 1879.

LUCAS ET CATALAN. Sur le calcul symbolique des nombres de Bernoulli: *Nouv. Cor. Math.*, ii, 328. 1876.

LUCAS, E. Théorie nouvelle des nombres de Bernoulli: *Com. Ren.*, lxxxiii, 539. 1876.

—— Sur les rapports qui existent entre le triangle arithmetique de Pascal et les nombres de Bernoulli: *Nouvelles Annales* (2), xv, 497. 1876.

—— Sur les développements en séries: *Bull. de la Soc. Math.*, vi, 57. 1878.

—— Théorie nouvelle des nombres de Bernoulli et d'Euler: *Brioschi Annali* (2), viii, 56. 1877.

—— Sur les théorèmes de Binet et de Staudt: *Nouvelles Annales* (2), xvi, 157. 1877.

—— Sur la généralisation de deux théorèmes dus à MM. Hermite et Catalan: *Nouv. Cor. Math.*, iii, 69. 1877.

—— Sur les nouvelles formules de MM. Seidel et Stern concernant les nombres de Bernoulli: *Bull. Soc. Math.*, viii, 169. 1880.

—— On Eulerian numbers: *Messenger*, vii, 139. 1877.

—— On the development of $\left(\frac{z}{1-e^{-z}}\right)^{a}$ in a series: *Messenger* (2), vii, 82. 1877.

—— On the successive summations of $1^m + 2^m + \ldots + x^m$: *Messenger* (2), vii, 84. 1877.

—— On development in series: *Messenger* (2), vii, 116. 1877.

MALMSTÉN, C. J. Sur la formule

$$hu'_x = \Delta u_x - \frac{h}{2}\Delta u'_x + B_1\frac{h^2}{2!}\Delta u''_x - B_3\frac{h^4}{4!}\Delta u^{\text{IV}}_x + \text{etc.}$$

Crelle, xxxv, 55. 1847.

—— Note sur l'Intégrale finie $\Sigma e^x y$: *Grunert*, vi, 41. 1845.

MEYER, G. F. Vorlesungen über die Theorie der Bestimmten Integrale, §§ 53, 54. 1871.

—— Verschiedene arithmetische Sätze: *Grunert*, xxxviii, 241. 1862.

—— Einige Beitrage zur Theorie der Bernoulli'schen Zahlen und der Secanten-coefficienten: *Grunert*, xxxv, 449. 1860.

NAEGELSBACH. Zur independente Darstellung der Bernoulli'schen Zahlen: *Zeitschrift*, xix, 219. 1874.

OHM, M. Etwas über die Bernoulli'schen Zahlen: *Crelle*, xx, 11. 1840.

PENNY CYCLOPÆDIA: Arts. Numbers of Bernoulli and Series. 1833.

PLANA. Note sur une nouvelle expression analytique des nombres Bernoulliens propre à exprimer en termes finis la forme générale pour la sommation: *Mém. de l'Acad. de Turin*, 1820.

POISSON. Sur le calcul numerique des Intégrales définies: *Acad. des Sciences*, vi, 571. 1823.

RAABE. Zurückführung einiger Summen und bestimmten Integrale auf die Jacob-Bernoulli'sche Function: *Crelle*, xlii, 348. 1851.

—— Diff. und Int. Rech., i, 412 *et seq.*

—— Augenäherte Bestimmung der Factorenfolge $1.2.3\ldots n = \Gamma(1+n) = \int x^n e^{-x} dx$, wenn n eine sehr grosse Zahl ist: *Crelle*, xxv, 146, and xxviii, 10. 1843–44.

RADICKE, A. Die Recursions-formeln für die Berechnung die Bernoulli'schen und Eulerschen Zahlen: *Halle a. s. Nebert.* 1880.

—— Démonstration du théorème de Staudt: *Nouv. Cor. Math.*, vi, 503. 1880.

—— Démonstration d'un théorème de Stern: *Nouv. Cor. Math.*, vi, 507. 1880.

—— Solutions des questions proposées: *Nouv. Cor. Math.*, vi, 69. 1880.

—— Solutions des questions proposées: *Nouv. Cor. Math.*, v, 333. 1879.

—— Extrait d'une lettre: *Nouv. Cor. Math.*, v, 196. 1879.

SCHÄFLI. On Staudt's theorem relating to the Bernoulli's numbers: *Quart. Jour.*, vi, 75. 1864.

SCHENDEL, L. Die Bernoulli'schen Functionen u. s. w.: *Jena, Costenoble.* 1876.

—— Zur Theorie der Reihen: *Zeitschrift*, xvi, 211. 1871.

SCHERK. Ueber einen allgemein die Bernoulli'schen Zahlen betreffend: *Crelle*, xxxii, 299. 1846.

—— Ueber einen allgemein die Bernoulli'schen Zahlen und die Coëfficienten der Secant-reihe zugleich darstellenden Ausdruck: *Crelle*, iv, 299. 1829.

SCHLÖMILCH, O. Ueber Bernoulli'schen Zahlen und die Coëfficienten der Secant-reihen: *Grunert*, i, 360. 1841.

—— Developpement d'une formule qui donne en même temps les nombres de Bernoulli et les coefficients la serie qui exprime la sécante: *Crelle*, xxxii, 360. 1846.

—— Ueber die Bernoulli'schen Functionen und deren Gebrauch bei der Entwickelung halber-convergenter Reihen: *Zeitschrift*, i, 193. 1856.

—— Neue Formeln zur independente Bestimmung der Sekanten- und Tangenten-koeffizienten: *Grunert*, xvi, 411. 1851.

—— Ueber die Summe der Reihe $1^n + 2^n + \ldots + r^n$: *Grunert*, x, 342. 1847.

SCHLÖMILCH, O. Neue Methode zur Summirung endlicher und unendlicher Reihen: *Grunert*, xii, 130. 1849.

——Ueber die independente Bestimmung der Coefficienten unendlicher Reihen u. s. w.: *Grunert*, xviii, 306. 1852.

——Ueber die Lambert'sche Reihe: *Zeitschrift*, vi, 407. 1861.

——Ueber die rekurrirende Bestimmung der Bernoulli'schen Zahlen: *Grunert*, iii, 9. 1843.

SERRET. Sur l'évaluation approchée du produit $1.2.3\ldots x$, lorsque x est un très-grand nombre et sur la formule de Stirling: *Com. Ren.*, l, 662. 1860.

SPITZER, SIMON. Note über die Summen-formel

$$\Sigma x^m = C + \frac{x^{m+1}}{(m+1)h} - \frac{1}{2}x^m + \text{etc.: } \textit{Grunert}, \text{ xxiii, 457. 1854.}$$

STAUDT. Beweis eines Lehrsatzes die Bernoulli'schen Zahlen betreffend: *Crelle*, xxi, 372. 1840.

——De Numeris Bernoullianis: Erlangen, 1845.

STERN. Ueber eine Eigenschaft der Bernoulli'schen Zahlen: *Crelle*, lxxxii, 290. 1877.

——Zur Theorie der Bernoulli'schen Zahlen: *Crelle*, xcii, 349. 1882.

——Zur Theorie der Bernoulli'schen Zahlen: *Crelle*, lxxxiv, 267. 1878.

——Zur Theorie der Bernoulli'schen Zahlen: *Crelle*, lxxxviii, 85. 1879.

——Zur Theorie der Eulerschen Zahlen: *Crelle*, lxxix, 67. 1875.

SYLVESTER. Note on the numbers of Bernoulli and Euler, etc.: *Phil. Mag.*, xxi, 127. 1861.

——Sur une propriété de nombres premiers qui se rattache au dernier théorème de Fermat: *Com. Ren.*, lii, 161. 1861.

——Addition à la précédente note: *Com. Ren.*, lii, 212. 1861.

——Extrait d'une lettre adressée à M. Serret: *Com. Ren.*, lii, 307. 1861.

THACKER. Ein Beitrag zur Zahlen-theorie: *Crelle*, xl, 89. 1850.

THOMAN. Développement des séries alternativement positifs et negatifs à l'aide des nombres de Bernoulli: *Com. Ren.*, lxiv, 665. 1867.

WORONTZOFF. Sur les nombres de Bernoulli: *Nouvelles Annales*, xv, 12. 1876.

On Division of Series.

By Revd. John Hagen, S. J., *College of the Sacred Heart, Prairie Du Chien, Wis.*

If the series $\Sigma\alpha_\rho x^\rho$ is divided by the series $\Sigma\beta_\sigma x^\sigma$, the quotient will be a series whose coefficients may be expressed in determinant form in the following manner:

$$\sum_{\rho=0}^{\rho=r}\alpha_\rho x^\rho \div \sum_{\sigma=0}^{\sigma=s}\beta_\sigma x^\sigma = \sum_{\tau=0}^{\tau=\infty}\frac{(-1)^\tau}{\beta_0^{\tau+1}}\Delta_\tau,\ \Delta_\tau = \begin{vmatrix} \alpha_0 & \beta_0 & 0 & \dots & 0 \\ \alpha_1 & \beta_1 & \beta_0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \alpha_\tau & \beta_\tau & \beta_{\tau-1} & \dots & \beta_1 \end{vmatrix}.$$

This proposition may be proved either directly, by performing the indicated division, or indirectly by applying the method of indeterminate coefficients. In the first case the following formula will be found which may serve as a definition of the above determinant:

$$\Delta_{\tau+1} = \beta_1\Delta_\tau - \beta_0\Delta^\tau_{\tau+1} \tag{1}$$

where $\Delta^\tau_{\tau+1}$ denotes a determinant of the above form whose last line begins with $\alpha_{\tau+1}$, but in which the line beginning with α_τ is wanting; in the second case we assume the quotient to be $\Sigma\gamma_\tau x^\tau$, and multiplying it by the divisor find the identical equation $\Sigma\alpha_\rho x^\rho = \Sigma\Sigma\beta_\sigma\gamma_\tau x^{\sigma+\tau}$. Then changing the index τ by the formula $\sigma+\tau=\omega$ we have the equation

$$\sum_{\rho=0}^{\rho=r}\alpha_\rho x^\rho = \sum_{\omega=0}^{\omega=\infty}x^\omega\sum_{\sigma=0}^{\sigma=s}\beta_\sigma\gamma_{\omega-\sigma},$$

from which may be deduced the following two

$$\alpha_\omega = \sum_{\sigma=0}^{\sigma=s}\beta_\sigma\gamma_{\omega-\sigma},\ (r \geqq \omega \geqq \sigma);\ 0 = \sum_{\sigma=0}^{\sigma=s}\beta_\sigma\gamma_{\omega-\sigma},\ (\omega > r). \tag{2}$$

Writing these systems in the following way:

$$\begin{aligned} \alpha_0 &= \beta_0\gamma_0 \\ \alpha_1 &= \beta_1\gamma_0 + \beta_0\gamma_1 \\ &\dots\dots\dots\dots \\ \alpha_\omega &= \beta_\omega\gamma_0 + \beta_{\omega-1}\gamma_1 + \dots + \beta_0\gamma_\omega \\ 0 &= \beta_{\omega+1}\gamma_0 + \beta_\omega\gamma_1 + \dots + \beta_1\gamma_\omega + \beta_0\gamma_{\omega+1}\ \text{etc.,} \end{aligned}$$

and remembering that for the solution with regard to γ_τ we need only the first τ equations, the following equation results:

$$\gamma_\tau \begin{vmatrix} \beta_0 & 0 & \dots 0 \\ \beta_1 & \beta_0 & \dots 0 \\ \dots & \dots & \dots \\ \beta_{\tau-1} & \beta_{\tau-2} & \dots 0 \\ \beta_\tau & \beta_{\tau-1} & \dots \beta_0 \end{vmatrix} = \begin{vmatrix} \beta_0 & 0 & \dots \alpha_0 \\ \beta_1 & \beta_0 & \dots \alpha_1 \\ \dots & \dots & \dots \\ \beta_{\tau-1} & \beta_{\tau-2} & \dots \alpha_{\tau-1} \\ \beta_\tau & \beta_{\tau-1} & \dots \alpha_\tau \end{vmatrix}, \text{ or } \gamma_\tau = \frac{(-1)^\tau}{\beta_0^{\tau+1}} \begin{vmatrix} \alpha_0 & \beta_0 & \dots 0 \\ \alpha_1 & \beta_1 & \dots 0 \\ \dots & \dots & \dots \\ \alpha_{\tau-1} & \beta_{\tau-1} & \dots \beta_0 \\ \alpha_\tau & \beta_\tau & \dots \beta_1 \end{vmatrix}$$

as stated in the proposition. When $\omega > r$, we have of course $\alpha_\omega = 0$.

In case the two given series are finite, the quotient will be a Recurring Series, *i. e.* beginning from a certain index each determinant may be expressed as a linear function of a constant number of preceding determinants. This linear function is represented in the second formula of (2), which, after substituting the value of γ, reads as follows:

$$\sum_{\sigma=0}^{\sigma=s}(-1)^\sigma \beta_\sigma \beta_0^\sigma \Delta_{\omega-\sigma} = 0, \ (\omega > r).$$

The preceding results may be applied also to multiple series. If the dividend is a multiple sum with the general term $a_{\rho,\kappa\dots\lambda}$, the division may be performed upon any of the indices, say the second, κ; and we shall have

$$\sum_\rho\sum_\kappa \dots \sum_\lambda a_{\rho,\kappa\dots\lambda} \div \sum_\sigma b_\sigma = \sum_\rho\sum_\tau \dots \sum_\lambda \frac{(-1)^\tau}{b_0^{\tau+1}} \begin{vmatrix} a_{\rho,0\dots\lambda} & b_0 & 0 \\ a_{\rho,1\dots\lambda} & b_1 & 0 \\ a_{\rho,\tau\dots\lambda} & b_\tau & b_1 \end{vmatrix}.$$

If the divisor is a multiple sum $\sum_\sigma\sum_\phi \dots \sum_\chi b_{\sigma,\phi\dots\chi}$, it may be reduced to a simple sum by putting $\sum_\phi \dots \sum_\chi b_{\sigma,\phi\dots\chi} = \beta_\sigma$, whereupon it admits at once of being operated upon by the formula above obtained.

Sur le Développement des Fonctions Rationelles.

By the Revd. Faà de Bruno.

Les belles transformations données à ce sujet par M. Sylvester dans N°. 20 des *Johns Hopkins University Circulars* ont éveillé en moi l'idée de donner explicitement le coefficient de x^p dans le développement d'une fonction rationelle $\phi(x)$ selon les puissances ascendantes de x, travail qui dans la formule de M. Sylvester resterait encore à faire car ces les racines seulement qui y figurent. Soit

$$\phi(x) = \frac{1}{a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n}; \tag{1}$$

le coefficient de x^p dans le développement de $\phi(x)$ selon les puissances ascendantes de x sera le déterminant

$$\frac{1}{a_0^{p+1}} \begin{vmatrix} a_1 & a_0 & 0 & 0 & \ldots 0 \\ a_2 & a_1 & a_0 & 0 & \ldots 0 \\ a_3 & a_2 & a_1 & a_0 & \ldots 0 \\ a_4 & a_3 & a_2 & a_1 & \ldots 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{p-1}, & a_{p-2}, & a_{p-3}, & a_{p-4} & \ldots a_0 \\ a_p & a_{p-1} & a_{p-2} & a_{p-3} & \ldots a_1 \end{vmatrix} \tag{2}$$

Soient en effet $\alpha, \beta, \gamma \ldots$ les racines de

$$a_n x^n + a_{n-1} x^{n-1} + \ldots + a_0 = 0, \tag{3}$$

on aura

$$\phi(x) = \frac{(-1)^n}{a_n(\alpha - x)(\beta - x)(\gamma - x) \ldots}, \tag{4}$$

et en prennant le logarithme

$$\log \phi(x) = n\pi i - n \log a_n - \log \alpha\beta\gamma \ldots + \Sigma \frac{x^p}{p\alpha^p}.$$

De là on tire

$$\phi(x) = \frac{1}{a_0} e^{\Sigma \frac{s_{-p}}{p} x^p}.$$

Pour arriver maintenant au développement du second membre nous aurons recours à un théorème trés utile que nous avons donné pour la première fois en 1855 dans les *Annales de Tortolini* et qui se trouve reproduit dans notre *Théorie des formes binaires*. Voici le théorème.

Soit une fonction

$$\theta(y), \quad \text{où } y = \omega(z)$$

on aura

$$D_z^p \theta = \Sigma \frac{(p)}{(k_1)(k_2)\ldots(k_p)} D_x^g \theta \left(\frac{\omega'}{1}\right)^{k_1} \left(\frac{\omega''}{1.2}\right)^{k_2} \cdots \left(\frac{\omega^{(p)}}{1.2\ldots p}\right)^{k_p}$$

sous les conditions

$$g = k_1 + k_2 + \ldots + k_p$$
$$p = k_1 + 2k_2 + \ldots + pk_p$$
$$(i) = 1.2.3\ldots i.$$

Supposons pour rentrer dans notre cas que

$$f(y) = e^y, \quad y = \Sigma \frac{s_{-p}}{p} x^p,$$

alors on aura

$$D_x^p e^{\Sigma \frac{s_{-p}}{p} x^p} = \Sigma \frac{(p)}{(k_1)(k_2)\ldots(k_p)} \left(\frac{s_{-1}}{1}\right)^{k_1} \left(\frac{s_{-2}}{2}\right)^{k_2} \left(\frac{s_{-3}}{3}\right)^{k_3} \cdots$$

Ainsi le coefficient de x^p en $\phi(x)$ sera

$$\frac{1}{1.2.3\ldots p} D_x^p \frac{1}{a_0} e^{\Sigma \frac{s_{-p}}{p} x^p} = \frac{1}{a_0} \Sigma \frac{1}{(k_1)(k_2)\ldots(k_p)} \left(\frac{s_{-1}}{1}\right)^{k_1} \left(\frac{s_{-2}}{2}\right)^{k_2} \left(\frac{s_{-3}}{3}\right)^{k_3} \cdots \left(\frac{s_{-p}}{p}\right)^{k_p}.$$

Or sans répéter ici ce que nous avons dit dans notre *Théorie des formes binaires* à page 157 le second membre peut être réduit à un déterminant et l'on a

$$\text{coefficient de } x^p = \frac{1}{a_0} \begin{vmatrix} s_{-1} & -1 & 0 & 0 \ldots\ldots 0 \\ s_{-2} & s_{-1} & -2 & 0 \ldots\ldots 0 \\ s_{-3} & s_{-2} & s_{-1} & -3 \ldots\ldots 0 \\ s_{-4} & s_{-3} & s_{-2} & s_{-1} \ldots 0 \\ \ldots & \ldots & \ldots & \ldots\ldots\ldots \\ s_{-p} & s_{-(p-1)}, & s_{-(p-2)} & \ldots\ldots s^{-1} \end{vmatrix}$$

Cela seules constitue déjà une belle propriété de ce coefficient. Mais ensuite par une analyse semblable à celle employée page 65 du même ouvrage on arriverait à trouver que ce déterminant n'est autre chose que

$$\frac{1}{a_0^{p+1}} \begin{vmatrix} a_1 & a_0 & 0 & 0 \ldots\ldots 0 \\ a_2 & a_1 & a_0 & 0 \ldots\ldots 0 \\ a_3 & a_2 & a_1 & a_0 \ldots\ldots 0 \\ \ldots & \ldots & \ldots & \ldots\ldots\ldots \\ a_{p-1} & a_{p-2} & a_{p-3} & a_{p-4} \ldots a_0 \\ a_p & a_{p-1} & a_{p-2} & a_{p-3} \ldots a_1 \end{vmatrix}$$

Je crois que c'est là un résultat simple et élégant, sur lequel on s'arrêtera désormais dans le développement des fonctions.

Si la fonction rationelle avait la forme $\frac{\psi x}{\varphi x}$ il est évident qu'il suffirait de multiplier les coefficients de $x^0 x^1 x^2 \ldots x^p$ par ceux respectivement de $\frac{1}{\varphi(x)}$ en x^p, x^{p-1}, $x^{p-2} \ldots x^0$ que nous avons appris à calculer tout à l'heure pour trouver le coefficient générale de x^p correspondant au développement de $\frac{\psi(x)}{\varphi(x)}$. C'est pour cela que nous sommes restreints a considérer seulement la fonction

$$\phi(x) = \frac{1}{a_0 + a_1 x + a_2 x^2 + \ldots}.$$

Turin, 10 *Janvier*, 1883.

Tables of Generating Functions, Reduced and Representative for certain Ternary Systems of Binary Forms.

BY J. J. SYLVESTER.

The annexed tables have been calculated under my directions by Messrs. Durfee and Ely, out of the fund placed at my disposition by the British Association for the Advancement of Science in the year 1881. Subsequent investigation will be necessary in order to ascertain whether there exist or not extra tabular groundforms which escape the operation of tamisage.

G. F. it will be understood stands for the words Generating Function.

SYSTEM OF TWO QUADRATICS AND ONE QUARTIC.

G. F. for invariants, reduced form.

Denominator: $(1-b^2)(1-\beta^2)(1-d^2)(1-d^3)(1-b\beta)(1-bd)$
$(1-\beta d)(1-b^2d)(1-\beta^2 d)$.

Numerator:

		d^0	d^1	d^2	d^3	d^4			d^0	d^1	d^2	d^3	d^4
	β^0	1						β^1			$\bar{1}$		
b^0	β^1		$\bar{1}$				b^3	β^2				1	
	β^2			1				β^3					$\bar{1}$
	β^0		$\bar{1}$					β^0			1		
b^1	β^1		1	2			b^2	β^1		1		$\bar{1}$	
	β^2		1		$\bar{1}$			β^2			$\bar{2}$	$\bar{1}$	
	β^3			$\bar{1}$				β^3				1	

G. F. for invariants, representative form.

Denominator: $(1-b^2)(1-\beta^2)(1-d^2)(1-d^3)(1-b\beta)(1-b^2d^2)$
$(1-\beta^2d^2)(1-b^2d)(1-\beta^2d)$.

Numerator:

		d^0	d^1	d^2	d^3	d^4	d^5
	β^0	1					
b^0	β^1						
	β^2						
	β^3				1		
	β^4						
	β^0						
	β^1		1	1			
b^1	β^2		1	1	1		
	β^3						
	β^4				$\bar{1}$		
	β^0						
b^2	β^1		1	1	1		
	β^2						
	β^3				$\bar{1}$	$\bar{1}$	$\bar{1}$
	β^4						

		d^0	d^1	d^2	d^3	d^4	d^5	d^6
	β^0							
b^4	β^1				$\bar{1}$			
	β^2							
	β^3							
	β^4							$\bar{1}$
	β^0				1			
	β^1							
b^3	β^2				$\bar{1}$	$\bar{1}$	$\bar{1}$	
	β^3					$\bar{1}$	$\bar{1}$	
	β^4							

TABLE OF GROUNDFORMS.

Deg. in coeff's of quadratic	Deg. in coeff's of quadratic	Deg. in coeff's of quartic. 0	1	2	3
0	0			1	1
	1				
	2	1	1	1	
	3				1
1	0				
	1	1	1	1	
	2		1	1	1
2	0	1	1	1	
	1		1	1	1
3	0				1

SYSTEM OF QUADRATIC, CUBIC, AND QUARTIC.

G. F. for invariants, reduced form.

Denominator: $(1-b^2)(1-c^4)(1-d^2)(1-d^3)(1-bc^2)(1-b^3c^2)$
$(1-bd)(1-b^2d)(1-c^2d)(1-c^2d^3)(1-c^4d)$
$(1-c^4d^3)$.

Numerator:

		d^0	d^1	d^2	d^3	d^4	d^5	d^6	d^7	d^8	d^9	d^{10}
	c^0	1										
	c^2		$\bar{1}$									
b^0	c^4			2	2	2	1					
	c^6			1	1		$\bar{1}$	$\bar{1}$				
	c^8				$\bar{1}$	$\bar{2}$	$\bar{2}$	$\bar{2}$				
	c^{10}								1			
	c^{12}									$\bar{1}$		
	d^0		$\bar{1}$									
	c^2		2	4	2	1						
	c^4		2	2		$\bar{1}$	$\bar{2}$	$\bar{1}$				
b^1	c^6			$\bar{2}$	$\bar{3}$	$\bar{3}$	$\bar{1}$	1	1			
	c^8			$\bar{1}$	$\bar{1}$		1	1	1			
	c^{10}				1	1		$\bar{1}$	$\bar{2}$	$\bar{2}$		
	c^{12}								$\bar{1}$		1	
	c^{14}									1		
	c^0			1								
	c^2		2	1	$\bar{1}$	$\bar{1}$	$\bar{1}$					
	c^4		1	$\bar{1}$	$\bar{1}$	$\bar{1}$	$\bar{1}$					
	c^6			$\bar{1}$			1	1				
b^2	c^8			$\bar{1}$	$\bar{1}$	$\bar{2}$	$\bar{2}$	$\bar{1}$				
	c^{10}					$\bar{1}$	$\bar{1}$			1	1	
	c^{12}					1	2	2	1	2	1	
	c^{14}										$\bar{1}$	
	c^2		1	$\bar{1}$	$\bar{1}$	$\bar{1}$						
	c^4	1	1		1			$\bar{1}$				
	c^6		$\bar{2}$	$\bar{3}$	$\bar{2}$	$\bar{2}$	$\bar{1}$	$\bar{2}$	$\bar{1}$	$\bar{1}$		
b^3	c^8		$\bar{1}$	$\bar{1}$	$\bar{2}$	$\bar{1}$		1	2	1	1	
	c^{10}			1	1	2	1	2	2	3	2	
	c^{12}					1			$\bar{1}$		$\bar{1}$	$\bar{1}$
	c^{14}							1	1	1	$\bar{1}$	

		d^0	d^1	d^2	d^3	d^4	d^5	d^6	d^7	d^8	d^9	d^{10}
	c^4			1								
	c^6				$\bar{1}$							
b^6	c^8					2	2	2	1			
	c^{10}					1	1		$\bar{1}$	$\bar{1}$		
	c^{12}						$\bar{1}$	$\bar{2}$	$\bar{2}$	$\bar{2}$		
	c^{14}										1	
	c^{16}											$\bar{1}$
	c^2			$\bar{1}$								
	c^4		$\bar{1}$		1							
	c^6			2	2	1		$\bar{1}$	$\bar{1}$			
b^5	c^3				$\bar{1}$	$\bar{1}$	$\bar{1}$		1	1		
	c^{10}				$\bar{1}$	$\bar{1}$	1	3	3	2		
	c^{12}					1	2	1		$\bar{2}$	$\bar{2}$	
	c^{14}							$\bar{1}$	$\bar{2}$	$\bar{4}$	$\bar{2}$	
	c^{16}										1	
	c^2		1									
	c^4		$\bar{1}$	$\bar{2}$	$\bar{1}$	$\bar{2}$	$\bar{2}$	$\bar{1}$				
	c^5		$\bar{1}$	$\bar{1}$			1	1				
	c^8					1	2	2	1	1		
b^4	c^{10}					$\bar{1}$	$\bar{1}$			1		
	c^{12}						1	1	1	1	$\bar{1}$	
	c^{14}						1	1	1	$\bar{1}$	$\bar{2}$	
	c^{16}									$\bar{1}$		

G. F. for invariants, representative form.

Denominator: $(1-b^2)(1-c^4)(1-d^2)(1-d^3)(1-bc^2)(1-b^3c^2)(1-b^2d^2)$
$(1-b^2d)(1-c^4d^2)(1-c^2d^3)(1-c^4d)(1-c^4d^3)$

Numerator:

		d^0	d^1	d^2	d^3	d^4	d^5	d^6	d^7	d^8	d^9	d^{10}	d^{11}
	c^0	1											
	c^2												
	c^4			1	2	2	1						
	c^6			1	3	2	1						
	c^8					$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{1}$				
	c^{10}					$\bar{1}$	$\bar{2}$	$\bar{2}$	$\bar{1}$				
	c^{12}												
	c^{14}										$\bar{1}$		
	c^2		2	3	2	1							
	c^4		2	4	5	3	1						
	c^6												
	c^8			$\bar{1}$	$\bar{3}$	$\bar{3}$	$\bar{3}$	$\bar{2}$	$\bar{1}$				
	c^{10}						$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{2}$			
	c^{12}					1	1		$\bar{2}$	$\bar{2}$	$\bar{1}$		
	c^{14}												
	c^{16}										1		
	c^2		2	3	3	1							
	c^4		1	3	4	2	$\bar{1}$	$\bar{2}$	$\bar{1}$				
	c^6				$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{2}$	$\bar{1}$				
	c^8			$\bar{1}$	$\bar{3}$	$\bar{5}$	$\bar{5}$	$\bar{2}$	1	2	1		
	c^{10}				$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{2}$	$\bar{1}$				
	c^{12}					1	2	2	1				
	c^{14}						1	2	2	1	1	1	1
	c^0				1								
	c^2		1	1		$\bar{1}$	$\bar{1}$	$\bar{1}$					
	c^4	1	1	2	1	$\bar{1}$	$\bar{3}$	$\bar{3}$	$\bar{1}$				
	c^6		$\bar{1}$	$\bar{2}$	$\bar{2}$	$\bar{2}$	$\bar{2}$	$\bar{2}$	$\bar{2}$	$\bar{1}$			
	c^8		$\bar{1}$	$\bar{3}$	$\bar{6}$	$\bar{5}$	$\bar{4}$	$\bar{2}$		1			
	c^{10}					$\bar{1}$	$\bar{2}$	$\bar{1}$	1	4	4	2	
	c^{12}				1	2	3	2	2	3	4	3	1
	c^{14}						1	2	3	2			
	c^{16}											[illegible]	[illegible]

		d^1	d^2	d^3	d^4	d^5	d^6	d^7	d^8	d^9	d^{10}	d^{11}	d^{12}
	c^4			1									
	c^6												
	c^8					1	2	2	1				
	c^{10}					1	3	2	1				
b^7	c^{12}							$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{1}$		
	c^{14}							$\bar{1}$	$\bar{2}$	$\bar{2}$	$\bar{1}$		
	c^{16}												
	c^{18}												$\bar{1}$
	c^2			$\bar{1}$									
	c^4												
	c^6			1	2	2		$\bar{1}$	$\bar{1}$				
b^6	c^8				2	3	2	1					
	c^{10}					1	2	3	3	3	1		
	c^{12}												
	c^{14}							$\bar{1}$	$\bar{3}$	$\bar{5}$	$\bar{4}$	$\bar{2}$	
	c^{16}								$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{2}$	
	c^4	$\bar{1}$	$\bar{1}$	$\bar{1}$	$\bar{1}$	$\bar{2}$	$\bar{2}$	$\bar{1}$					
	c^6					$\bar{1}$	$\bar{2}$	$\bar{2}$	$\bar{1}$				
	c^8					1	2	3	2	1			
b^5	c^{10}			$\bar{1}$	$\bar{2}$	$\bar{1}$	2	5	5	3	1		
	c^{12}					1	2	3	2	1			
	c^{14}					1	2	1	$\bar{2}$	$\bar{4}$	$\bar{3}$	$\bar{1}$	
	c^{16}								$\bar{1}$	$\bar{3}$	$\bar{3}$	$\bar{2}$	
	c^2	1	1	$\bar{1}$	$\bar{1}$	$\bar{1}$							
	c^4				$\bar{2}$	$\bar{3}$	$\bar{2}$	$\bar{1}$					
	c^6	$\bar{1}$	$\bar{3}$	$\bar{4}$	$\bar{3}$	$\bar{2}$	$\bar{2}$	$\bar{3}$	$\bar{2}$	$\bar{1}$			
b^4	c^8		$\bar{2}$	$\bar{4}$	$\bar{4}$	$\bar{1}$	1	2	1				
	c^{10}				$\bar{1}$		2	4	5	6	3	1	
	c^{12}				1	2	2	2	2	2	2	1	
	c^{14}					1	3	3	1	$\bar{1}$	$\bar{2}$	$\bar{1}$	$\bar{1}$
	c^{16}						1	1	1		$\bar{1}$	$\bar{1}$	
	c^{18}									[illegible]			

TABLE OF GROUNDFORMS.

Deg. in coeff's of Quadratic	Deg. in coeff's of Cubic.	Degree in coeff's of Quartic. 0	1	2	3	4	5
0	0			1	1		
	2				1		
	4	1	1	2	3	2	1
	6			1	3	2	1
1	2	1	2	3	2	1	
	4		2	4	5	3	1
2	0	1	1	1			
	2		2	3	3	1	
	4		1				
3	0				1		
	2	1	1	1			
	4	1	1				
4	2		1	1			

SYSTEM OF ONE QUADRATIC AND TWO QUARTICS.

G. F. for invariants, reduced form.

Denominator: $(1-b^2)(1-\delta^2)(1-\delta^3)(1-d^2)(1-d^3)(1-b\delta)(1-b^2\delta)$
$(1-bd)(1-b^2d)(1-\delta d)(1-\delta^2 d)(1-\delta d^2)$

Numerator:

		d^0	d^1	d^2	d^3	d^4	d^5	d^6			d^0	d^1	d^2	d^3	d^4	d^5	d^6
	δ^0	1								δ^2			1				
	δ^1									δ^3							
b^0	δ^2			1					b^5	δ^4					1		
	δ^3									δ^5							
	δ^4					1				δ^6							1
	δ^0		$\bar{1}$							δ^1			$\bar{1}$				
	δ^1	$\bar{1}$	1	1	1					δ^2		$\bar{1}$					
b^1	δ^2		1	1					b^4	δ^3				1		1	
	δ^3		1		1					δ^4					1	1	
	δ^4						$\bar{1}$			δ^5				1	1	1	$\bar{1}$
	δ^5					$\bar{1}$				δ^6						$\bar{1}$	
	δ^0			1						δ^1		1	$\bar{1}$				
	δ^1		2		$\bar{1}$	$\bar{1}$				δ^2		$\bar{1}$				$\bar{1}$	
b^2	δ^2	1		$\bar{1}$	$\bar{2}$				b^3	δ^3					$\bar{2}$	$\bar{1}$	
	δ^3		$\bar{1}$	$\bar{2}$						δ^4				$\bar{2}$	$\bar{1}$		1
	δ^4		$\bar{1}$				$\bar{1}$			δ^5			$\bar{1}$	$\bar{1}$		2	
	δ^5					$\bar{1}$	1			δ^6					1		

G. F. for invariants, representative form.

Denominator: $(1-b^2)(1-\delta^2)(1-\delta^3)(1-d^2)(1-d^3)(1-b^2\delta^2)(1-b^2\delta)$
$(1-b^2d^2)(1-b^2d)(1-\delta d)(1-\delta^2 d)(1-d^2\delta)$.

Numerator:

		d^0	d^1	d^2	d^3	d^4	d^5	d^6
	δ^0	1						
	δ^1							
b^0	δ^2			1				
	δ^3							
	δ^4					1		
	δ^1		1	1	1			
b^1	δ^2		1	1	1			
	δ^3		1	1	1			
	δ^1		1	1				
	δ^2		1	1				
b^2	δ^3				1	1		
	δ^4				1		$\bar{1}$	$\bar{1}$
	δ^5					$\bar{1}$		
	δ^6					$\bar{1}$		
	δ^0				1			
	δ^1		1	1		$\bar{1}$	$\bar{1}$	
b^3	δ^2		1	1	$\bar{1}$	$\bar{2}$	$\bar{1}$	
	δ^3	1		$\bar{1}$	$\bar{3}$	$\bar{2}$	$\bar{1}$	
	δ^4		$\bar{1}$	$\bar{2}$	$\bar{2}$			
	δ^5		$\bar{1}$	$\bar{1}$	$\bar{1}$			

		d^1	d^2	d^3	d^4	d^5	d^6	d^7
	δ^3			1				
	δ^4							
b^7	δ^5					1		
	δ^6							
	δ^7							1
	δ^4				1	1	1	
b^6	δ^5				1	1	1	
	δ^6				1	1	1	
	δ^1			$\bar{1}$				
	δ^2			$\bar{1}$				
b^5	δ^3	$\bar{1}$	$\bar{1}$		1			
	δ^4			1	1			
	δ^5					1	1	
	δ^6					1	1	
	δ^2				$\bar{1}$	$\bar{1}$	$\bar{1}$	
	δ^3				$\bar{2}$	$\bar{2}$	$\bar{1}$	
b^4	δ^4		$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{1}$		1
	δ^5		$\bar{1}$	$\bar{2}$	$\bar{1}$	1	1	
	δ^6		$\bar{1}$	$\bar{1}$		1	1	
	δ^7				1			

TABLE OF GROUNDFORMS.

Deg. in coeff's of quadratic.	Deg. in coeff's of quartic.	Deg. in coeff's of quartic. 0	1	2	3
0	0			1	1
	1		1	1	
	2	1	1	1	
	3	1			
1	0				
	1		1	1	1
	2		1	1	1
	3		1	1	
2	0	1	1	1	
	1	1	1	1	
	2	1	1		
3	0				1
	1		1	1	
	2		1		
	3	1			

SYSTEM OF THREE QUARTICS.

G. F. for invariants, reduced form.

$$\text{Denominator: } (1-\partial^2)(1-\partial^3)(1-\delta^2)(1-\delta^3)(1-d^2)(1-d^3)$$
$$(1-\partial\delta)(1-\partial d)(1-\delta d)(1-\partial^2 d)(1-\partial d^2)$$
$$(1-\partial^2\delta)(1-\partial\delta^2)(1-\delta^2 d)(1-\delta d^2).$$

Numerator:

		d^0	d^1	d^2	d^3	d^4	d^5	d^6	d^7	d^8
	δ^0	1								
	δ^1									
	δ^2			1						
∂^0	δ^3									
	δ^4					1				
	δ^1		1	1						
	δ^2		1	1						
	δ^3				1	1				
∂^1	δ^4				1		$\bar{1}$	$\bar{1}$		
	δ^5					$\bar{1}$				
	δ^6					$\bar{1}$				
	δ^0			1						
	δ^1		1	1						
	δ^2	1	1	1			$\bar{1}$	$\bar{1}$		
∂^2	δ^3				1	$\bar{1}$	$\bar{2}$	$\bar{1}$		
	δ^4				$\bar{1}$	$\bar{3}$	$\bar{2}$		1	
	δ^5			$\bar{1}$	$\bar{2}$	$\bar{2}$	1	1		
	δ^6			$\bar{1}$	$\bar{1}$		1	1		
	δ^7					1				
	δ^1				1	1				
	δ^2				1	$\bar{1}$	$\bar{2}$	$\bar{1}$		
∂^3	δ^3		1	1		$\bar{4}$	$\bar{3}$	$\bar{1}$		
	δ^4		1	$\bar{1}$	$\bar{4}$	$\bar{5}$		2	1	
	δ^5			$\bar{2}$	$\bar{3}$		3	2		
	δ^6			$\bar{1}$	$\bar{1}$	2	2	1		
	δ^7					1				

		d^1	d^2	d^3	d^4	d^5	d^6	d^7	d^8
	δ^4				$\bar{1}$				
	δ^5								
	δ^6						$\bar{1}$		
∂^8	δ^7								
	δ^8								$\bar{1}$
	δ^2				1				
	δ^3				1				
	δ^4		1	1		$\bar{1}$			
∂^7	δ^5				$\bar{1}$	$\bar{1}$			
	δ^6						$\bar{1}$	$\bar{1}$	
	δ^7						$\bar{1}$	$\bar{1}$	
	δ^1				$\bar{1}$				
	δ^2		$\bar{1}$	$\bar{1}$		-1	1		
	δ^3		$\bar{1}$	$\bar{1}$	2	2	1		
∂^6	δ^4	$\bar{1}$		2	3	1			
	δ^5		1	2	1	$\bar{1}$			
	δ^6		1	1			$\bar{1}$	$\bar{1}$	$\bar{1}$
	δ^7						$\bar{1}$	$\bar{1}$	
	δ^8						$\bar{1}$		
	δ^1				$\bar{1}$				
	δ^2		$\bar{1}$	$\bar{2}$	$\bar{2}$	1	1		
∂^5	δ^3		$\bar{2}$	$\bar{3}$		3	2		
	δ^4	$\bar{1}$	$\bar{2}$		5	4	1	$\bar{1}$	
	δ^5		1	3	$\bar{4}$		$\bar{1}$	$\bar{1}$	
	δ^6		1	2	1	$\bar{1}$			
	δ^7				$\bar{1}$	$\bar{1}$			

Numerator—*Continued:*

		d^0	d^1	d^2	d^3	d^4	d^5	d^6	d^7	d^8
	∂^0					1				
	∂^1				1		$\bar{1}$	$\bar{1}$		
	∂^2				$\bar{1}$	$\bar{3}$	$\bar{2}$		1	
∂^4	∂^3		1	$\bar{1}$	$\bar{4}$	$\bar{5}$		2	1	
	∂^4	1		$\bar{3}$	$\bar{5}$		5	3		$\bar{1}$
	∂^5		$\bar{1}$	$\bar{2}$		5	4	1	$\bar{1}$	
	∂^6		$\bar{1}$		2	3	1			
	∂^7			1	1		$\bar{1}$			
	∂^8					$\bar{1}$				

Representative form same as reduced form.

TABLE OF GROUNDFORMS.

Deg. in coeff's of quartic	Deg. in coeff's of quartic	Deg. in coeff's of quartic. 0	1	2	3
0	0			1	1
	1		1	1	
	2	1	1	1	
	3	1			
1	0		1	1	
	1	1	1	1	
	2	1	1	1	
2	0	1	1	1	
	1	1	1	1	
	2	1	1		
3	0	1			

A Constructive theory of Partitions, arranged in three Acts, an Interact and an Exodion.

BY J. J. SYLVESTER, *with Insertions by* DR. F. FRANKLIN.

ACT I. ON PARTITIONS REGARDED AS ENTITIES.

. . . seeming parted,
But yet a union in partition.
Twelfth-night.

(1) In the new method of partitions it is essential to consider a partition as a *definite thing*, which end is attained by regularization of the succession of its parts according to some prescribed law. The simplest law for the purpose is that the arrangement of the parts shall be according to their order of magnitude. A leading idea of the method is that of correspondence between different complete systems of partitions regularized in the manner aforesaid. The perception of the correspondence is in many cases greatly facilitated by means of a graphical method of representation, which also serves *per se* as an instrument of transformation.

(2) The most obvious mode of graphically representing a partition is by means of a network or web formed by two systems of parallel lines or filaments. Any continuous portion of such web will serve to represent a partition, as *ex gr.* the graph

$$\begin{array}{ccccc}
 & * & * & * & \\
* & * & * & * & * \\
* & * & * & * & * \\
* & * & * & * & \\
 & * & * & * &
\end{array}$$

will represent the partition 3 5 5 4 3 of 20 by reading off the successive numbers of nodes parallel to the horizontal lines of the web. This, however, is not a regularized partition; the partition will be represented in its regularized form by such a graph as the following:

$$\begin{array}{ccccccc}
* & * & * & * & * & & \\
 & * & * & * & * & * & \\
 & * & * & * & * & & \\
 & & & * & * & * & \\
 & & & & & * & *
\end{array}$$

which corresponds to the order 5 5 4 3 3, but it may be represented much more advantageously by the figure

$$\begin{array}{ccccc} \ast & \ast & \ast & \ast & \ast \\ \ast & \ast & \ast & \ast & \ast \\ \ast & \ast & \ast & \ast & \\ \ast & \ast & \ast & & \\ \ast & \ast & \ast & & \end{array}$$

which is a portion of the web bounded by lines of nodes belonging to the two systems of parallel filaments. Any such portion becomes then subject to the important condition that the two transverse parallel readings will each give a regularized partition, one being in the present example 5 5 4 3 3, and the other 5 5 5 3 2. Any such graph as this will be termed a *regular* partition-graph, and the two partitions which it represents will be said to be conjugate to one another. The mere conception of a regular graph serves at once by effecting an interchange (so to say) between the warp and the woof, through the principle of correspondence, to establish a well-known fundamental theorem of reciprocity. In the last figure, the extent* of (meaning the number of nodes contained by) the uppermost horizontal line or filament is the maximum magnitude of any element (or part) of the partition, and the extent of the first vertical line is the number of the parts. Hence, every regularized partition beginning with i and containing j parts is conjugate to another beginning with j and containing i parts. The content of the graph (*i. e.* the sum of the parts) of the partition is the same in both cases (it will sometimes be convenient to speak of the *partible number* as the content of the elements of the partition). From the above correspondence it is clear that if two complete partition-systems be formed with the same content in one of which the largest part is i and the number of parts j, and in the other the largest part is j and the number of the parts i, the order (*i. e.* the number of partitions) of the first system will be identical with the order of the second: so that calling the content n, it follows that $n - i$ may be decomposed in as many ways into $j - 1$ parts as $n - j$ into $i - 1$ parts.

(3) This, however, is not the usual nor the more convenient mode of expressing the reciprocity in question. We may, for the two partition systems spoken of, substitute two others of larger inclusion, taking for the first, all partitions of n in which no one part is greater than i, and the number of parts is not greater than j (*i. e.* is j or fewer), and for the second system, one subject to the same conditions as just stated, but with i and j (as before) interchanged: it is obvious that each regularized partition of one system will be conjugate to

* *Extent* may be used to denote the number of nodes on a line or column or angle of a graph; *content* the number of nodes in the graph itself; but I have by inadvertence in what follows frequently applied *content* alike to designate areal and linear numerosity.

one regularized partition of the other system, and accordingly the order of the two systems will be the same.*

(4) When $i=\infty$ it follows from the general theorem of reciprocity last established, that the number of partitions of n into j parts or fewer will be the same as the number of ways of composing n with the integers $1, 2, \ldots j$, and is therefore the coefficient of x^n in the expansion of $\frac{1}{1-x.1-x^2\ldots 1-x^j}$.

Thus, then, we can at once find the general term in $\frac{1}{(1-a)(1-ax)(1-ax^2)\,ad\,inf.}$ expanded according to ascending powers of a, for, if the above fraction be regarded as the product of an infinite number of infinite series arising from the expansion of the several factors $\frac{1}{1-a}, \frac{1}{1-ax}, \frac{1}{1-ax^2}, \ldots$ it will readily be seen that the coefficient of $x^n a^j$ will be the number of ways in which n can be resolved into j parts or fewer, *i. e.* by what has been just shown is the coefficient of x^n in $\frac{1}{1-x.1-x^2\ldots 1-x^j}$, and this being true for all values of n, it follows that the entire coefficient of a^j is the fraction last written developed in ascending powers of x so that

$$\frac{1}{(1-a)(1-ax)(1-ax^2)\ldots ad\,inf.}=1+\frac{1}{1-x}a+\frac{1}{1-x.1-x^2}a^2+\frac{1}{1-x.1-x^2.1-x^3}a^3\ldots$$

as is well known.

The general term in $\frac{1}{(1-a)(1-ax)\ldots(1-ax^i)}$ is also well known to be $\frac{1-x^{i+1}.1-x^{i+2}\ldots 1-x^{i+j}}{1-x.1-x^2\ldots 1-x^j}a^j$, or in other words, the number of ways of resolving n into j parts none greater than i is the coefficient of x^n in the fraction $\frac{1-x^{i+1}.1-x^{i+2}\ldots 1-x^{i+j}}{1-x.1-x^2\ldots 1-x^j}$, which (denoting $1-x^q$ by (q)) is the same as $\frac{(1)(2)\ldots(i+j)}{(1)(2)\ldots(i).(1)(2)\ldots(j)}$, and furnishes, if I am not mistaken, Euler's proof of the theorem of reciprocity already established by means of the correspondence of conjugate partitions.

(5) [It may be as well to advert here to the practical method of obtaining the conjugate to a given partition. For this purpose it is only necessary to call a_i the number of parts in the given partition not less than i; $a_1, a_2, a_3 \ldots a_i \ldots$ continued to infinity (or which comes to the same thing until i is equal to the maximum part), will be the required conjugate.]

*The above proof of the theorem of reciprocity is due to Dr. Ferrers, the present head of Gonville and Caius' College, Cambridge. It possesses the double merit of having set the first example of graphical construction and of putting into salient relief the principle of correspondence, applied to the theory of partitions. It was never made public by its author, but first promulgated by myself in the *Lond. and Edin. Phil. Mag.* for 1853.

(6) The following very beautiful method of obtaining the general term in question by the constructive method is due to Mr. F. Franklin of the Johns Hopkins University:*

He, as it were, interpolates between the theorem to be established in general and the theorem for $i=\infty$, and attaches a definite meaning to the above fraction regarded as a generating function when the factors in the numerator are limited to the first q of them, q being any number not exceeding i, so that in fact the theorem to be proved, according to this view, is only the extreme case of (the last link in the chain to) a new and more general one with which he has enriched the theory of partitions. The method will be most easily understood by means of an example or two: the proof and use to be made of the construction will be given towards the end of the Act.

Let $n=10$, $i=5$, $j=4$.

Write down the indefinite partitions of 10 into 4 or fewer parts, or say rather into 4 parts, among which zeros are admissible: they will be

(1)	10.0.0.0	5.5.0.0
(1)	9.1.0.0	5.4.1.0
(1)	8.2.0.0	5.3.2.0
(1)	8.1.1.0	5.3.1.1
(2)	7.3.0.0	5.2.2.1
(2)	7.2.1.0	4.4.2.0
(1)	7.1.1.1	4.4.1.1
(2)	6.4.0.0	4.3.3.0
(3)	6.3.1.0	4.3.2.1
(3)	6.2.2.0	4.2.2.2
(4)	6.2.1.1	3.3.3.1
		3.3.2.2

The partitions to which (1) is prefixed are those in which the *first excess, i. e.* the excess of the first (the dominant) part over the next is *too great* (meaning greater than i, here 5); those to which (2) is prefixed are those in which after the batch marked with (1) are removed, the second excess, *i. e.* the excess of the first over the third element is "too great"; those to which (3) is prefixed are those in which after the batches marked (1) and (2) are removed, the third excess is "too great," and lastly those (only one as it happens) marked with j (here 4) are those in which, so to say, the *absolute excess* of the dominant, *i. e.* its actual value is "too great," *i. e.* exceeding i (here 5); the partitions that are left

* For a vindication of the constructive method applied to this and an allied theorem, see p. 268 *et seq.*

over will be the partitions of n (here 10) into 4 parts, none exceeding i (here 5) in magnitude.

It is easy to see from this how to *construct* the partitions which are to be *eliminated* from the indefinite partitions of the n (10) into 4 (j) parts so as to obtain the quaternary partitions in which no part superior to 5 (i) appears. To obtain the first batch we must subtract $i+1$ (6) from n (10) and form the system of indefinite partitions of 4 into four parts, viz:

4.0.0.0
3.1.0.0
2.2.0.0
2.1.1.0
1.1.1.0

and adding to each of these 6.0.0.0 (term-to-term addition) batch (1) will be obtained.

To obtain the second batch, form the quaternary partitions of $n-(i+2)$, *i. e.* 3, viz:

3.0.0.0
2.1.0.0
1.1.1.0

[but omit those in which the first excess is "too great" (greater than i); here there are none such to be omitted] and bring the second element into the first place: thus we shall obtain the system

0 3 0 0
1 2 0 0
1 1 1 0

The *augments* of those obtained by adding 6.1.0.0 to each of them will reproduce batch (2).

Again, form the quaternary partition-system of $2n-(i+3)$, rejecting all those (here there are none such) in which the second excess is "too great." We thus obtain

2 0 0 0
1 1 0 0

and now bringing the third element in each of these into the first place so as to obtain

0 2 0 0
0 1 1 0

The augments of these last partitions obtained by adding 6.1.1.0 to each of them will give the third batch, and finally taking the quaternary partition-system to $n-(i+j)$, *i. e.* 1, rejecting (if there should be any such) those in which the third excess is "too great," we obtain 1.0.0.0, and bringing the fourth element

to the first place so as to get 0.1.0.0, and adding 6.1.1.1, the fourth batch 6.2.1.1 is reconstructed.

As another example take $n = 15$, $i = 3$, $j = 3$.

The indefinite ternary partitions of 15 are

15.0.0	(1)	9.4.2	(1)
14.1.0	(1)	9.3.3	(1)
13.1.1	(1)	8.7.0	(2)
12.3.0	(1)	8.6.1	(2)
12.2.1	(1)	8.5.2	(2)
11.4.0	(1)	8.4.3	(2)
11.3.1	(1)	7.7.1	(2)
10.5.0	(1)	7.6.2	(2)
10.4.1	(1)	7.5.3	(2)
10.3.2	(1)	7.4.4	(3)
9.6.0	(2)	6.6.3	(3)
9.5.1	(1)	6.5.4	(3)
		5.5.5	(3)

There are, of course, no partitions left in which no part exceeds 3, as the maximum content subject to that condition would be only 9.

The partitions marked (1) (2) (3) are those in which the first, second and absolute excess respectively exceed 3.

1°. The indefinite ternary partitions of 15—4 or 11 augmented by 4.0.0 will obviously reproduce the system of partitions marked (1).

2°. Taking the indefinite ternary partitions of 10 in which the *first* excess and those of 9 in which the *second* excess does not exceed 3, we shall obtain

6.4.0	and 5.2.2
6.3.1	4.4.1
6.2.2	4.3.2
5.5.0	3.3.3
5.4.1	
5.3.2	
4.4.2	
4.3.3	

which by *metastasis* become

4.6.0	2.5.2
3.6.1	1.4.4
2.6.2	2.4.3
5.5.0	3.3.3
4.5.1	
3.5.2	
4.4.2	
3.4.3	

and adding to each term of these two groups 4.1.0 and 4.1.1 respectively, the systems of partitions marked (2) and (3) respectively result.

(7) It may, I think, be desirable to give here my own construction for the case of repeated partitions, which, having regard to its features of resemblance to the one preceding, it is proper to state preceded it in the date of its discovery and promulgation. The problem which I propose to myself is to construct a system of portions of a given number into parts, limited in number and magnitude by means of partitions of itself and other numbers into parts limited in number but not in magnitude.

As before, let i be the limit of magnitude, j the number of parts (zeros admissible), and n the partible number; form a square matrix of the j^{th} order in which the diagonals are all $i+1$, the elements below the diagonal all of them unity, and those above the diagonal all of them zeros, say M_1.

From this matrix construct $M_1, M_2, M_3, \ldots M_j$, such that the lines in M_q (q being any integer from 1 to j inclusive) are the sums of those in M_1, added (term-to-term) q and q together.

Let (r, q) be the r^{th} line in M_q and $[r, q]$ the sum of the numbers which it contains.

Form the complete system of the partitions of $n-[r, q]$ into j parts, and to each such add (term-to-term) (r, q).

In this way, by giving r all possible values we shall obtain a system of partitions of n into j parts corresponding to M_q, which may be called P_q. I say that $P_1 - P_2 + P_3 \ldots + (-)^{j-1} P_j$ will be the complete system of partitions of n into j parts in which one element at least exceeds i; where it is to be observed that having regard to the effect of the $-$ and $+$ signs (which are used here to indicate the addition and subtraction, or say rather the ad-duction and sub-duction not of numbers but of things), each such partition will occur once and once only, so that calling P the complete system of indefinite partitions of n into j parts, the complete system of partitions of n into j parts in which no part exceeds i in magnitude will be

$$P - P_1 + P_2 \ldots + (-)^i P_j \,*$$

(8) This construction, which I will illustrate by two examples, proceeds upon the fact which, although confirmed by a multitude of instances, *remains to be proved,*

*It must, however, be understood that the same partition is liable to appear in more than one, and not exclusively in its regularized phase, or as it may be expressed, the regularized partition undergoes *metastasis.*

that if $k_1, k_2, \ldots k_j$ be any partition of n into j parts and the number of *unequal* parts greater than i be μ, then the number of times in which this partition, in its regular or any other phase, appears in P_q is $\frac{\mu.(\mu-1)\ldots(\mu-q+1)}{1.2\ldots q}$ (interpreted to mean 1 when $q=0$), and consequently its total number of appearances in $P-P_1+P_2\ldots$ is $(1-1)^{\mu}$, *i. e.* is 0.

From this it follows that the total number of partitions of n into j parts none exceeding i in magnitude will be $C-C_1+C_2\ldots$ where C_q is the sum of the number of ways in which the various numbers $n_1, n_2, n_3\ldots$ can be decomposed into j parts, the numbers above named being n diminished by the sums of the quantities $i+1, i+2, \ldots, i+j$ added q and q together, and is therefore the coefficient of x^n in $\frac{x^{n_1}+x^{n_2}+x^{n_3}+\ldots}{(1-x)(1-x^2)\ldots(1-x^j)}$ and consequently the number of partitions of n into j parts none exceeding i in magnitude will be the coefficient of x^n in $\frac{1-x^{i+1}.1-x^{i+2}\ldots 1-x^{i+j}}{1-x.1-x^2\ldots 1-x^j}$ as was to be shown.

(9) As a first example let $i=2$, $j=3$, $n=12$, the matrices and the partitions corresponding to their several lines will be as underwritten; the indefinite partitions of the reduced contents are written opposite to the respective matrix lines to which they correspond, and their augments found by adding each such line to the corresponding partition system are written immediately under them. The zeros are omitted for the sake of brevity.

3.0.0	9	8.1	7.2	7.1.1	6.3	6.2.1	5.4	5.3.1	5.2.2	4.4.1	4.3.2	3.3.3
	12	11.1	10.2	10.1.1	9.3	9.2.1	8.4	8.3.1	8.2.2	9.4.1	7.3.2	6.3.3
1.3.0	8	7.1	6.2	6.1.1	5.3	5.2.1	4.4	4.3.1	4.2.2	3.3.2		
	9.3	8.4	7.5	7.4.1	6.6	6.5.1	5.7	5.6.1	5.5.2	4.6.2		
1.1.3	7	6.1	5.2	5.1.1	4.3	4.2.1	3.3.1	3.2.2				
	8.1.3	7.2.3	6.3.3	6.2.4	5.4.3	5.3.4	4.4.4	4.3.5				
—												
4.3.0	5	4.1	3.2	3.1.1	2.2.1							
	9.3	8.4	7.5	7.4.1	6.5.1							
4.1.3	4	3.1	2.2	2.1.1								
	8.1.3	7.2.3	6.3.3	6.2.4								
2.4.3	3	2.1	1.1.1									
	5.4.3	4.5.3	3.5.4									
—												
5.4.3	0											
	5.4.3											

In 6.3.3 there are two unlike elements greater than 2; accordingly 6.3.3 occurs 2 times in P_1 and 1 time in P_2.

In 7.3.2 there are again two unlike elements greater than 2, and 7.3.2, 7.2.3 (the metastatic equivalent to the former) are found in P_1 and 7.2.3 in P_2.

Again, in 5.4.3 there are 3 unlike elements greater than 2, and we find

5.4.3	5.3.4	4.3.5	in P_1
5.4.3	4.5.3	3.5.4	" P_2
5.4.3			" P_3

and such terms as 8.1 7.1.1 4.4.1 3.3.3 in which there is only one distinct element greater than 2 is found 1 time only in P_1 and not at all in P_2 or P_3.

As another example let $n = 12$, $i = 4$, $j = 3$, then a similarly constructed table to the foregoing will be as follows, in which, however, all matrices or lines of matrices which have a sum too large to give rise to partition systems are omitted.

5.0.0.0	7	6.1	5.2	5.1.1	4.3	4.2.1	3.3.1	3.2.2
	12	11.1	10.2	10.1.1	9.3	9.2.1	8.3.1	8.2.2
1.5.0.0	6	5.1	4.2	4.1.1	3.3	3.2.1	2.2.2	
	7.5	6.6	5.7	5.6.1	4.8	4.7.1	3.7.2	
1.1.5.0	5	4.1	3.2	3.1.1	2.2.1			
	6.1.5	5.2.5	4.3.5					
1.1.1.5	4	3.1						
	5.1.1.5	4.2	1.5					
—								
6.5.0.0	1							
	7.5							
6.1.5.0	0							
	6.1.5							

7.5 and 6.5.1 are the only two partitions of 12 into 3 parts in which there are two unlike parts greater than 4; each of these accordingly is found twice (in one or another phase) in P_1 and once in P_2. Every other partition of 12 into 3 parts in which one of them at least is greater than 4 will be found exclusively and only once in P_3.

(10) The two expansions for $(1-ax)(1-ax^2)\ldots(1-ax^i)$ and its reciprocal may readily be obtained from one another by the method of correspondence.

The coefficient of $x^n a^j$ in the former is the number of partitions of n into j *unequal*, and in the latter into j *equal or unequal* parts none greater than i or less than unity. The correspondence to be established has been given by Euler for the case of $i = \infty$ (*Comm. Arith.*, 1849, *Tom.* 1, p. 88), and is probably known for the general case, but as coming strictly within the purview of the essay, seems to deserve mention here.

If $k_1, k_2, k_3, \ldots, k_j$ be a partition of n into j equal or unequal parts written in ascending order, none exceeding i, on adding to it $0, 1, 2 \ldots (j-1)$ it becomes a partition of $n + \frac{j^2-j}{2}$ into j parts none exceeding $i+j-1$, and conversely if $\lambda_1, \lambda_2, \ldots, \lambda_j$ be a partition of $n + \frac{j^2-j}{2}$ into j unequal parts none exceeding $i+j-1$, written in ascending order, on subtracting from it $0, 1, 2 \ldots (j-1)$ it becomes a partition of n into equal or unequal (say relatively independent) parts none exceeding i.

Hence the complete system of partitions of n into j unlike parts none exceeding i has a one-to-one correspondence with the complete system of the partitions of $n - \frac{j^2-j}{2}$ into j parts none exceeding $i-j+1$. Consequently the coefficient of a^j in the expansion of $(1-ax)\ldots(1-ax^i)$ may be found from that of a^j in the expansion of its reciprocal by changing i into $i-j+1$ and introducing the factor $x^{\frac{j^2-j}{2}}$.

(11) The expansion of the reciprocal may of course be found algebraically from the multiplication of the expansion which has been given of $\frac{1}{(1-a)(1-ax)\ldots(1-ax^i)}$ by $(1-a)$, or immediately by the correspondence between partitions into an exact number j of parts limited not to exceed i, and partitions into j or fewer parts so limited.

By subtracting a unit from each term of $k_1, k_2, \ldots, k_j$ a partition of n where no k exceeds i results a partition $q_1, q_2, \ldots q_j$, a partition of $n-j$ where no q exceeds $i-1$. Hence the coefficient of a^j in $\frac{1}{1-ax.1-ax^2\ldots1-ax^i}$ may be found from that in $\frac{1}{1-a.1-ax\ldots1-ax^i}$ by introducing the factor x^j and changing i into $i-1$, so that choosing for the former, the alternative form $\frac{1-x^{j+1}.1-x^{j+2}\ldots1-x^{j+i}}{1-x.1-x^2\ldots1-x^i}$ the latter becomes $\frac{1-x^{j+1}.1-x^{j+2}\ldots1-x^{j+i-1}}{1-x.1-x^2\ldots1-x^{i-1}}x^j$, and consequently the coefficient of a^j in $1-ax.1-ax^2\ldots1-ax^i$ will be

$$\frac{1-x^{j+1}.1-x^{j+2}\ldots.1-x^i}{1-x.1-x^2\ldots.1-x^{i-j}}x^{\frac{j^2+j}{2}}.$$

(12) Before quitting this part of the subject it is desirable to make mention of Dr. F. Franklin's remarkable method of proving Euler's celebrated expansion of $(1-x)(1-x^2)(1-x^3)\ldots$ *ad inf.* by the method of correspondence. This has been given by Dr. Franklin himself in the *Comptes Rendus* of the Institut (1880), and by myself in some detail in the last February Number of the *J. H. U.*

Circular. The method is in its essence absolutely independent of graphical considerations, but as it becomes somewhat easier to apprehend by means of graphical description and nomenclature, I shall avail myself here of graphical terminology to express it.

If a regular graph represent a partition with unequal elements, the lines of magnitude must continually increase or decrease. Let the annexed figures be such graphs written in ascending order from above downwards.

```
* *
* * *
* * * * *
* * * * * *
* * * * * * *
```
(A)

(B)
```
* *
* * *
* * * * * *
* * * * * * *
```

```
* * *
* * * *
* * * * * *
* * * * * * *
```
(C)

In A and B the graphs may be transformed without altering their content or regularity by removing the nodes at the summit and substituting for them a new slope line at the base. In C the slope line at the base may be removed and made to form a new summit; the graphs so transformed will be as follows:

```
* * *
* * * * *
* * * * * * *
* * * * * * * *
```
(A')

(B')
```
* * *
* * * * * * *
* * * * * * * *
```

```
* *
* * *
* * * *
* * * * *
* * * * * *
```
(C')

A' and B' may be said to be derived from A, B by a process of contraction, and C' from C by one of protraction.

Protraction could not now be applied to A' and B', nor contraction to C' without destroying the regularity of the graph; but the inverse processes may of course be applied, viz. of protraction to A' and B' and contraction to C', so as to bring back the original graph A, B, C.

In general (but as will be seen not universally), it is obvious that when the number of nodes in the summit is inferior or equal to the number in the base-slope, contraction may be applied, and when superior to that number, protraction: each process alike will alter the number of parts from even to odd or from odd to even, so that barring the exceptional cases which remain to be considered where neither protraction nor contraction is feasible, there will be a one-to-one correspondence between the partitions of n into an odd number and the parti-

tions of n into an even number of unrepeated parts; the exceptional cases are those shown below where the summit meets the base-slope line, and contains either the same number or one more than the number of nodes in that line; in which case neither protraction nor contraction will be possible, as seen in the annexed figures which are written in regular order of succession, but may be indefinitely continued:

```
*    * *    * *        * * *        * * *          * * * *
            * * *      * * * *      * * * *        * * * * *
                                    * * * * *      * * * * * *

            * * * *                 * * * * *
            * * * * *               * * * * * *
            * * * * * *             * * * * * * *
            * * * * * * *           * * * * * * * *
```

for the contraction process which *ought* (*ex gr.*), according to the general rule, to be applicable to the last of the above graphs, cannot be applied to it, because on removing the nodes in the slope line and laying them on the summit, in the very act of so doing the summit undergoes the loss of a node and is thereby incapacitated to be surmounted by the nodes in the slope, which will have not now a less, but the same number of nodes as itself; and in like manner, in the last graph but one, the nodes in the summit cannot be removed and a slope line be added on containing the same number of nodes without the transformed graph ceasing to be regular, in fact it would take the form

```
            *
*     * * *
* * * * * *
* * * * * * *
```

and so the last graph transformed according to rule would become:

```
* * * *
* * * *
* * * * *
* * * * * *
* * * * * * *
```

which, although regular, would cease to represent a partition into unlike numbers.

The excepted cases then or unconjugate partitions are those where the number of parts being j, the successive parts form one or the other of the two arithmetical series

$$j,\ j+1,\ j+2,\ \ldots\ 2j-1 \ \text{ or } \ j+1,\ j+2,\ \ldots\ 2j,$$

in which cases the contents are $\frac{3j^2-j}{2}$ and $\frac{3j+j}{2}$ respectively, and consequently since in the product of $1-x.1-x^2.1-x^3\ldots$ the coefficient of x^n is the number of ways of composing n with an even less the number of ways of com-

posing it with an odd number of parts, the product will be completely represented by $\sum\limits_{j=+\infty}^{j=-\infty}(-)^j x^{\frac{3j^2+j}{2}}$.*

(13) It has been well remarked by Prof. Cayley that barring the unconjugate partitions, the rest really constitute 4 classes, which using c and x to signify contractile and extensile and e and o to signify of-an-even or of-an-odd order, may be denoted by

$$\begin{array}{cc} c.e & c.o \\ x.e & x.o \end{array}$$

Hence as each $c.e$ is conjugate to an $x.o$ and *vice versâ*, and each $c.o$ to an $x.e$ and *vice versâ*, the theorem established really splits up into two, one affirming that the number of contractile partitions of an odd order is the same as the number of extensile ones of an even order, the other that the number of contractiles of an even is equal to the number of extensiles of an odd order. It might possibly be worth while to investigate the difference between the number of partitions which each set of one couple and the number of partitions which each set of the sub-contrary couple contain: the sets which belong to the same couple and contain the same number of partitions being those *both* of whose characters are dissimilar.

(14) There are one or two other simple cases of correspondence which are interesting, inasmuch as the construction employed to effect the correspondence involves the operations of division and multiplication, which have not occurred previously.

$$\text{If } fx = (1-x)(1-x^3)(1-x^3)(1-x^7)(1-x^9)\ldots$$
$$\text{and } \phi x = (1+x)(1+x^2)(1+x^3)(1+x^4)(1+x^5)\ldots$$
$$fx.\phi x = 1,$$

from which we obtain $\phi x = 1/fx$ and $fx = 1/\phi x$.

The latter of these equations has been noticed by Euler as involving the elegant theorem that a number may be partitioned in as many ways into only-once-occurring odd-or-even integers as into any-number-of-times-occurring only-odd integers.

The second, which I think he does not dwell upon, expresses that the difference between the number of partitions with an even number of parts and that

* Another proof of this theorem, deduced as an immediate algebraical consequence of a more general one, obtained by graphical dissection, will be given in Act 2; and in the Exodion I furnish a purely arithmetical proof by the method of correspondence of Jacobi's series for $(1 \pm x^{n-m})(1 \pm x^{n+m})(1-x^{2n})(1 \pm x^{3n-m})(1 \pm x^{3n+m})(1-x^{4n})\ldots$ (which includes Euler's theorem as a particular case.). I prove this theorem in a more extended sense than was probably intended by its immortal author, inasmuch as I regard m and n as absolutely general symbols.

of partitions with an odd number of parts of the same number n is the same as the number of partitions of n into exclusively odd numbers [such difference being in favor of the partitions of even or of odd order, according as the partible number is even or odd].

This latter theorem brings out a point of analogy between repetitional and non-repetitional partition systems which appears to me worthy of notice.

Any one of the former contains a class of what may be termed singular partitions, in the sense that they are their own associates, or more briefly, *self-conjugate* in respect to the Ferrers transformation. Any one system of the latter may also be *said* to contain a set of singular partitions (0 or 1 in number) in the sense of being *unconjugate* in respect to the Franklin process of transformation. Since then in this case the difference between the number of partitions of an odd and those of an even order of the same number is equal to the number (1 or 0) of singular partitions of that number, so we might anticipate as not improbable that the like difference for the repetitional partitions of a number should be equal to the number of singular partitions of that number—and such is actually the case; for it will be shown in a future section that the number of self-conjugate partitions of a number is the same as the number of ways in which it can be composed with odd integers.

(15) The correspondence indicated by the equation $\phi x = 1/fx$ can be established as follows:

Let $2^\lambda.l$, $2^\mu.m$, $2^\nu.n$, ... be any partition of unrepeated general numbers, where l, m, n ... are any odd integers not exceeding unity; and let $k^{[q]}$ in general denote q parts k, then without changing its content the above partition can be converted into $l^{[2^\lambda]}$, $m^{[2^\mu]}$, $n^{[2^\nu]}$, ... which consists exclusively of odd numbers.

It will of course be understood that the original partition may contain any the same odd number as l multiplied by different powers 2^λ, $2^{\lambda'}$, $2^{\lambda''}$ ... of 2, with the sole restriction that the λ, λ', λ'' ... must be all unequal.

Conversely, any such partitions as $l^{[\sigma]}$, $m^{[\tau]}$, $n^{[\upsilon]}$ may be converted back into one and only one partition of the former kind. For there will be one and but one way of resolving σ into the sum of powers of 2 [the zero power not excluded], and supposing σ to be equal to $2^\lambda + 2^{\lambda'} + 2^{\lambda''} + \ldots$ $l^{[\sigma]}$ may be replaced by $2^\lambda l$, $2^{\lambda'} l$, $2^{\lambda''} l$, and the same process of conversion may be simultaneously applied to each of the other products $m^{[\tau]}$, $n^{[\upsilon]}$, ...

Hence each partition of either one kind is conjugate to one of the other, and the number of partitions in the two systems will be the same, as was to be shown.

(16) But we have here another example of the fact that the theory of correspondence reaches far deeper than that of mere numerical congruity with which it is associated as the substance with the shadow. For a correspondence exists of a much more refined nature than that above demonstrated between the two systems, and which, moreover (it is important to notice) does not bring the same individuals into correlation as does the former method.

The partition system made up of unrepeated general numbers may be divided into groups of the first, second, . . . i^{th}. . . class respectively, those of the i^{th} class containing i distinct sequences of consecutive numbers having no term in common, with the understanding that no two sequences must form part of a single sequence (so that the difference of the largest term of one sequence and the smallest one of the next largest must differ by more than a single unit), and that a single number unpreceded and unfollowed by a consecutive number is to count as a sequence.

The partition system, made up of repeatable odd numbers may, in like manner, be resolved into groups of the 1st, 2nd, . . . i^{th}, . . . class respectively, those of the i^{th} class containing i distinct numbers; and the new theorem of correspondence is that there is a correlation between the numbers of the i^{th} class of one system and the i^{th} class of the other; so that the number of partitions in a class of the same name must be the same to whichever system it belongs, and thus Euler's theorem becomes a corollary to this deeper-reaching one, obtained from it by *adding together* the numbers of partitions in all the several classes in the one system and in the other.

(17) As regards the first class, the theorem amounts to the statement that the number of single sequences of consecutive numbers into which n may be resolved is equal to the number of odd factors which n contains; so that if $N = 2^c.l^\lambda.m^\mu.n^\nu. . .$ where $l, m, n, \ldots$ are odd numbers N can be represented by the sum of $(\lambda+1)(\mu+1)(\nu+1)\ldots$ such sequences; thus *ex gr.* if $N = 15 = 3.5$ we have $1+2+3+4+5 = 4+5+6 = 7+8 = 15$, so $30 = 4+5+6+7+8 = 6+7+8+9 = 9+10+11$. If $N = 27 = 3^3$; $27 = 2+3+4+5+6+7 = 8+9+10 = 13+14$. If $N = 45$, $45 = 1+2+3+\ldots+9 = 5+6+7+8+9+10 = 7+8+9+10+11 = 14+15+16 = 22+23$. So too if N is a prime number it can only be resolved into the two sequences $\frac{N-1}{2} + \frac{N+1}{2}$ and N. More generally N can be resolved into as

many different sets of i distinct sequences as there are solutions in positive integers of the equation $2(x_1y_1+x_2y_2+\ldots+x_iy_i)+x_1+x_2+\ldots+x_i=N$, of the truth of which remarkable theorem, in its general form, I have for the present only obtained empirical evidence, but may possibly be able to discover the proof in time to annex it in the form of a note at the end, so as not to keep the press waiting.*

(18) The proof for the case of the first class and the mode of establishing the correspondence between the partitions of this class of the two kinds is not far to seek. I use as previously $a^{(b)}$ to signify a repeated b times.

Consider then any sequence of consecutive numbers for the cases where the number of terms is odd and where it is even, separately calling s the sum of the first and last term, and i the number of terms, where i is odd, so that s is even, the sequence may be replaced by $i^{(\frac{s}{2})}$, and where i is even (so that s is odd) by $s^{(\frac{i}{2})}$. Hence each partition of the first class of the first kind may be transformed into one of the first class of the second kind.

It is necessary to show the converse of this, which may be done as follows: Let λ^μ be any partition of the second kind so that λ is necessarily odd. I say that this must be transformable into one or the other (but not into both) of two sequences, viz. one of λ terms of which the sum of the first and last is 2μ, the other of which the sum of the first and last terms is λ and the number of terms 2μ. The former supposition is admissible if 2μ is equal to or greater than $\lambda+1$, inadmissible if 2μ is less than $\lambda+1$. The second supposition is admissible if λ is equal to or greater than $2\mu+1$, inadmissible if λ is less than $2\mu+1$.

The two conditions of admissibility coexisting would imply that 2μ is equal to or greater than $2\mu+2$; the two conditions of inadmissibility the one that 2μ is equal to or less than $\lambda-1$, the other that λ equal to or less than $2\mu-1$, *i. e.* $\lambda-1$ equal to or less than $2\mu-2$, which are inconsistent. Hence one of the two transformations is always possible and the other impossible to be effected; which proves the correlation that was to be established. A single example will serve to show that this correspondence is entirely different from that offered by the first and (so to say) grosser method; suppose $N=15$, then $1.2.3.4.5$ will be a partition of the first kind and will be counted by the new rule into $5.5.5$, whereas, by the former rule, it would be inverted into $1.1.1.3.1.1.1.1.5$, *i. e.* into $1^7.3.5$ belonging to the third class instead of to the first.

*A complete proof of the general theorem will be given in the 3d Act.

(19) I will now pass on to the conjugate theorem corresponding to $fx = 1/\phi x$.

It may be well here to recall that this identity essentially depends upon the identity $1 - x = 1/(1+x)(1+x^2)(1+x^4)\ldots$ which, interpreted,* signifies that any number greater than unity may be made up in as many ways with an odd as with an even number of points restricted to the geometrical progression $1, 2, 4, 8\ldots$ This may be called, for brevity, a geometric partition. The correspondence to which this points is itself worthy of notice—one mode of establishing it would be to proceed to decompose N into such parts in regular dictionary order—it would easily be seen that each pair of partitions thus deduced would be of contrary *parities*, but it would not be easy, or at all events evident how to determine at once the conjugate to a given partition by reference to this principle; but if we observe that it is possible to pass from the geometric partitions of n immediately to those of $n+1$ by the addition of a unit to each of the former, and consequently to those of $n+2$ from the partitions of $E\frac{n}{2}, E\frac{n-2}{2}, E\frac{n-4}{2}, \ldots 2, 1$ by an obvious process of doubling and adding complementary ones, another rule or law of correspondence, which proves itself as soon as stated (and is not identical in effect with that supplied by the dictionary-order methed), looms into the field of vision, than which nothing can be simpler. Hence we may derive a transcendental equation in differences for u_n, the number of geometric partitions (with radix 2) to n, viz. to find the conjugate of any geometric partition, look at its greatest part—if it is repeated add two of them together: if it is unrepeated split it into two equal parts; these processes are obviously reversible, just as in Dr. Franklin's method of correspondence for the pentagonal-series-theorem, and is equally open to the remark made thereon by Prof. Cayley; that is to say, there will be four classes, extensile even, extensile odd, contractile even and untractile odd, and the number of partitions in any class will be the same as in the class in which both its characters are reversed. The application of this transformation to the construction indicated by the equation $fx = 1/\phi x$ will be obvious. Let any partition containing only unrepeated numbers consist of odd numbers $p, q, r, \ldots t$, each multiplied by one or more powers of 2; form batches of these terms which have the same greatest odd divisor $(p, q, r, \ldots t)$ and arrange those batches in a line according to the order of magnitude of $p, q, r, \ldots t$. Then we may

* Just so the equation $1/1 - x = (1+x)(1+x^2)(1+x^4)\ldots$ teaches that there is one and only one way of effecting the unrepetitional geometric partition of any number—a theorem which has been applied in the preceding theory.

agree to proceed either from left to right or from right to left in reading off the batches; and that convention being established once for all, as soon as a batch is reached which does not consist of a single odd term, if it contain one term larger than all the rest that term is to be split into two equal parts, but if it contain two terms not less than any others in the batch, those two are to be amalgamated into one. In this way the order of a partition consisting of terms not all of them distinct odd numbers, will have its *parity* (quality of being odd or even) reversed, and it is obvious that if A has been under the operation of the rule converted into B, B by the operation of the same rule will be converted back into A. Hence it follows that (making abstraction of the partitions consisting exclusively of unrepeated odd numbers) all the rest will be separable into as many contractile of an odd as into extensile of an even order, and into as many extensile of an odd as into contractile of an even order, so that the difference between the entire number of the partitions of N into an odd and those of an even order of repeatable numbers (odd or even) will be the number of partitions of N into unrepeated odd numbers, and that those of an odd or of an even order will be in the majority according as N itself is odd or even.*

It will be convenient to interpolate here Dr. F. Franklin's constructive proof of the theorems referred to in page 254 of what precedes, as there will be frequent occasion to refer to them in what follows. The theory is thus made completely self-contained. I give the proofs in the author's own words, which I think cannot be bettered.

(20) *Constructive Proof of the Formula for Partitions into Repeatable Parts, limited in Number and Magnitude.* The partitions herein spoken of are always partitions into a fixed number, j, of parts, *written in descending order.*

* Dr. F. Franklin has remarked that "the theorem admits of the following extensions," which the method employed in the text naturally suggests, and "which are very easily obtained either by the constructive proof or by generating functions":

1. The number of ways in which w can be made up of any number of odd and k distinct even parts is equal to the number of ways in which it can be made up of any number of unrepeated and k distinct repeated parts.

2. The number of ways in which w can be made up of parts not divisible by m is equal to the number of ways in which it can be made up of parts not occurring as many as m times.

3. The number of ways in which w can be made up of an infinite number of parts not divisible by m, together with k parts divisible by m, is equal to the number of ways in which it can be made up of an indefinite number of parts occurring less than m times, together with k parts occurring m or more times. (3) of course comprehends (1) and (2) as special cases.

Dr. Franklin adds, "another extension is naturally contained in the mode of proof, which it is perhaps not worth while to state." See *Johns Hopkins Circular* for March, 1883.

Take any partition of w in which the first excess is greater than i; subtracting $i+1$ from the first part we get a partition of $w-(i+1)$; and conversely if to the first part in a partition of $w-(i+1)$ we add $i+1$ we get a partition of w in which the first excess is greater than i.* Hence the number of partitions of w in which the first excess is greater than i is equal to the whole number of partitions of $w-(i+1)$; so that if the generating function for the partitions of w is $f(x)$, that for those partitions in which the first excess is *not greater* than i is $(1-x^{i+1})f(x)$. Confining ourselves now to this class of partitions, consider any one of them in which the second excess is greater than i; subtracting $i+1$ from the first part and 1 from the next, and putting the reduced first part into the second place we have a partition of $w-(i+2)$ in which the first excess is not greater than i; and conversely if in any partition of $w-(i+2)$ in which the first excess is not greater than i, we add $i+1$ to the second part and 1 to the first part and transfer the augmented second part to the first place, we get a partition of w in which the first excess is not greater than i and the second excess *is* greater than i. Hence the generating function for those partitions in which the second excess is *not* greater than i is $(1-x^{i+1})(1-x^{i+2})f(x)$. Considering now exclusively the partitions last mentioned, any one of them in which the third excess is greater than i may be converted into a partition of $w-(i+3)$ in which the second excess is not greater than i, by subtracting $i+1$ from the first part, 1 from the second part, and 1 from the third part, and removing the reduced first part to the third place, and, as before, by the reverse operation, the latter class of partitions are converted into the former. Hence the generating function for the partitions in which the third excess is not greater than i is $(1-x^{i+1})(1-x^{i+2})(1-x^{i+3})f(x)$. So in like manner, the generating functions for the partitions in which the k_2^{th} excess is not greater than i is

$$(1-x^{i+1})(1-x^{i+2})(1-x^{i+3})\ldots(i-x^{i+k})f(x);$$

and for the partitions in which the j_2^{th} or absolute excess is not greater than i, that is in which the greatest part does not exceed i, the generating function is

$$(1-x^{i+1})(1-x^{i+2})(1-x^{i+3})\ldots(1-x^{i+j})f(x).$$

(21) *Constructive Proof of the Formula for Partitions into Unrepeated Parts, limited in Number and Magnitude.* All the partitions to be considered consist of a fixed number, j, of *unrepeated* parts, written in *descending order*.

* The first excess signifies the excess of the largest part over the next largest; the second excess the excess of the largest over the next part but one, and so on.

Take any partition of w in which the first excess is greater than $i+1$; subtracting $i+1$ from the first part we get a partition of $w-(i+1)$; conversely, if to the first part in any portion of $w-(i+1)$ we add $i+1$, we get a partition of w in which the first excess is greater than $i+1$; hence the number of partitions of w in which the first excess is greater than $i+1$ is equal to the whole number of partitions of $w-(i+1)$; so that, if the generating function for all the partitions is $\phi(x)$, the generating function for partitions whose first excess is *not* greater than $i+1$ is $(1-x^{i+1})\phi(x)$. Considering now only partitions subject to this condition, if in any such partition of w the second excess is greater than $i+2$, we obtain by subtracting $i+2$ from the first part and removing the part so diminished to the second place a partition of $w-(1+2)$ subject to the condition; and conversely from any partition of $w-(1+2)$ in which the first excess is not greater than $i+1$, we obtain, by adding $i+2$ to the second part and removing the augmented part to the first place, a partition of w, in which the first excess is not greater than $i+1$ and the second excess *is* greater than $i+2$; hence the generating function for the partitions in which the second excess is *not* greater than $i+2$ (which restriction includes the condition that the first excess is not greater than $i+1$) is $(1-x^{i+1})(1-x^{i+2})\phi(x)$. Confining ourselves now to this class of partitions, and taking any partition of w in which the third excess is greater than $i+3$, we obtain, by subtracting $i+3$ from the first part and removing the diminished part to the third place, a partition of $w-(i+3)$ belonging to the class now under consideration; and reversely. Hence the number of partitions in which the third excess is not greater than $i+3$ is given by the generating function $(1-x^{i+1})(1-x^{i+2})(1-x^{i+3})\phi(x)$. Proceeding in this manner, we have finally that the generating function giving the number of partitions into j unrepeated parts, in which the absolute excess, *i. e.* the magnitude of the greatest part, is not greater than $i+j$, is $(1-x^{i+1})(1-x^{i+2})(1-x^{i+3})\ldots(1-x^{i+j})\phi(x)$.

For example, if $w=18$, $j=3$, $i=4$, the partitions

15, 2, 1 14, 3, 1 13, 4, 1 13, 3, 2 12, 5, 1 12, 4, 2 11, 5, 2 11, 4, 3

in which the first excess is greater than 5, becomes by subtraction of 5 from their first part,

10, 2, 1 9, 3, 1 8, 4, 1 8, 3, 2 7, 5, 1 7, 4, 2 6, 5, 2 6, 4, 3

which are *all* the partitions of 13; the partitions

11, 6, 1 10, 7, 1 10, 6, 2 10, 5, 3 9, 8, 1 9, 7, 2

in which the first excess is not greater than 5, but the second excess is greater

than 6 become, by the subtraction of 6 from the first part and its removal to the second place,

6, 5, 1   7, 4, 1   6, 4, 2   5, 4, 3   8, 3, 1   7, 3, 2

which are all the partitions of 12 whose first excess is not greater than 5; the partitions   9, 6, 3   9, 5, 4   8, 7, 3   8, 6, 4

in which the second excess is not greater than 6, but the third excess (the greatest part) is greater than 7, become, by the subtraction of 7 from the first part and its removal to the last place,

6, 3, 2   5, 4, 2   7, 3, 1   6, 4, 1

which are all partitions of 11 whose second excess is not greater than 6. The only remaining partition of 18 is 7, 6, 5.

Interact.

Notes on certain Generating Functions and their Properties.

(22) (A) It may be as well to reproduce here (so as to keep the whole subject together) the entire proof of the well-known expansions of $1 + ax.1 + ax^2.1 + ax^3 \ldots (1 + ax^i)$ and the reciprocal of $1 - a.1 - ax.1 - ax^2.1 - ax^3 \ldots 1 - ax^i$, which appeared in *part* in the Johns Hopkins Circular for February last. This is, I think, distinguishable from the ordinary proofs as being, so to say, *classical* in form (using the word in an algebraical sense), inasmuch as it establishes the identity of two rational integral functions, one explicitly, the other implicitly given, by comparison of their zeros.

1°. Let the coefficient of a^j in the expansion of $(1+ax)(1+ax^2)\ldots(1+ax^i)$ say $F(x, a)$ be called J_x and $\dfrac{1-x^i.1-x^{i-1}\ldots 1-x^{i-j+1}}{1-x.1-x^2\ldots 1-x^j}$ be called X_j.

J_x being the sum of the j, any combinations of $x, x^2, \ldots x^i$ will necessarily contain $x^{1+2+\cdots+j}$, *i. e.* $x^{\frac{j^2+j}{2}}$ and will be of the degree $i + (i-1) + \ldots + (i-j+1)$ in x, and therefore of the same degree as $X_j x^{\frac{j^2+j}{2}}$.

All the linear factors of X_j are obviously of the form $x - \rho$, where $x - \rho$ is a primitive factor of some binomial expression $x^r - 1$: the number of times that any $x - \rho$ occurs in X_j will obviously be equal to $E\dfrac{i}{r} - E\dfrac{j}{r} - E\dfrac{i-j}{r}$ which is either 1 or 0. Now consider $F(\rho, a)$ the value of $F(x, a)$ when x becomes ρ. Let $i = kr + \delta$ where $\delta < r$, then $F(\rho, a) = (1 \pm a)^k$ multiplied by δ linear functions of a, and consequently if $j = k'r + \delta'$, where $\delta' < r$, J_x vanishes when $\delta' > \delta$, in which case $E\dfrac{i}{r} - E\dfrac{j}{r} - E\dfrac{i-j}{r} = 1$.

Hence any linear factor $x-\rho$ of X_j possesses the two-fold property of being unrepeated and of being contained in J_x. Hence J_x must contain $X_j x^{\frac{j^2+j}{2}}$ and being of the same degree as it is in x must bear to it a constant ratio, which, by making $x=1$, is seen to be the coefficient of a_j in $(1-a)^i$, $\left[i.\ e.\ \text{ratio of } \frac{i(i-1)(i-2)\ldots(i-j+1)}{1.2.3\ldots j}\right]$ to the product of the fractions in their vanishing state $\frac{1-x^i}{1-x}, \frac{1-x^{i-1}}{1-x^2}, \ldots, \frac{1-x^{i-j+1}}{1-x^5}$, *i. e.* is a ratio of equality, so that $J_x = X_j x^{\frac{i^2+i}{2}}$. Q. E. D.

(23) Again let X_j and J_x now stand for $\frac{1-x^{i+1}.1-x^{i+2}\ldots 1-x^{i+j}}{1-x.1-x^2\ldots 1-x^j}$ and the coefficient of a^j in the reciprocal of $1-a.1-ax\ldots 1-ax^i$ (say $F_{(x,a)}$) which latter is the sum of homogeneous products of the j^{th} order of $1, x, x^2, \ldots x^i$ and is therefore of the degree ij which is also the degree (as is obvious) of X_j in x. For like reason as in what precedes $x-\rho$ any linear factor of x^r-1 is contained 1 or 0 times in X_j according as $E\frac{i+j}{r} - E\frac{i}{r} - E\frac{j}{r} = 1$ or 0.

Let the minimum negative residue of $i-1$ to modulus r be $-\delta$, $F(\rho, a)$ may be expressed as the product of δ linear functions of a, divided by a power of $1-a^r$, and the only power of a (say a^θ) which appears in its development will accordingly be those for which the residue of θ in respect to r is 0, 1, 2, … δ, and consequently if a^θ appears in the development $E\frac{i+\theta}{r} - E\frac{i}{r} - E\frac{\theta}{r} = 0$, or conversely if $x-\rho$ is a factor of X_j so that $E\frac{i+\theta}{r} - E\frac{i}{r} - E\frac{\theta}{r} = 1$, J_x vanishes. Hence J_x contains each linear factor of X_j, and these being simple, contains X_j itself, and on account of their degrees in x being the same must bear to it a ratio independent of x, which, by making $x=1$ [so that the things to be compared are the coefficient of a_j in $\frac{1}{(1-a)^{i+1}}$ and the product of the vanishing fractions $\frac{1-x^{i+1}}{1-x}, \frac{1-x^{i+2}}{1-x^2}, \ldots, \frac{1-x^{i+j}}{1-x^j}$] is readily seen to be a ratio of equality, so that $J_x = X_j$. Q. E. D.

(24) (B) *On the General Term in the Generating Function to Partitions into parts limited in number and magnitude,* by DR. F. FRANKLIN.

To prove that the coefficient of a^j in the development of

$$\frac{1}{(1-a)(1-ax)(1-ax^2)\ldots(1-ax^i)} \text{ is } \frac{(1-x^{j+1})(1-x^{j+2})\ldots(1-x^{j+i})}{(1-x)(1-x^2)\ldots(1-x^i)},$$

I showed that the number of partitions of w into i or fewer parts, subject to the condition that the first excess (the excess of the first part over the second) is not greater than j, is the coefficient of x^w in $\frac{1-x^{j+1}}{(1-x)(1-x^2)\dots(1-x^i)}$, and in general that the number of partitions in which the r^{th} excess (the excess of the first part over the $(r-1)^{\text{th}}$) is not greater than j, is the coefficient in

$$\frac{(1-x^{j+1})(1-x^{j+2})\dots(1-x^{j+r})}{(1-x)(1-x^2)\dots(1-x^i)}.$$

If we look at the question reversely, namely, if the coefficient of a^j in $\frac{1}{(1-a)(1-ax)(1-ax^2)\dots(1-ax^i)}$ being known to be $\frac{(1-x^{j+1})(1-x^{j+2})\dots(1-x^{j+i})}{(1-x)(1-x^2)\dots(1-x^i)}$, we ask what is the significance of the fractions

$$\frac{1-x^{j+1}}{(1-x)(1-x^2)\dots(1-x^i)}, \dots \frac{(1-x^{j+1})(1-x^{j+2})\dots(1-x^{j+r})}{(1-x)(1-x^2)\dots(1-x^i)},$$

the answer is immediately given by the generating function itself. For

$$\frac{1-x^{j+1}}{(1-x)(1-x^2)\dots(1-x^i)} = \frac{1}{(1-x^2)(1-x^3)\dots(1-x^i)} \cdot \frac{1-x^{j+1}}{1-x}$$

$$= \frac{1}{(1-x^2)(1-x^3)\dots(1-x^i)}\left(\text{co. of } a^j \text{ in } \frac{1}{(1-a)(1-ax)}\right)$$

$$= \text{co. of } a^j \text{ in } \frac{1}{(1-a)(1-ax)(1-x^2)(1-x^3)\dots(1-x^i)}.$$

But the coefficient of $a^j x^w$ in the last written fraction is obviously the number of ways in which w can be composed of the numbers $1, 2, 3, \dots i$, using not more than j 1's. And the number of 1's in a given partition is equal to the excess of the first part over the second part in its conjugate. In like manner,

$$\frac{(1-x^{j+1})(1-x^{j+2})\dots(1-x^{j+r})}{(1-x)(1-x^2)\dots(1-x^i)} = \text{co. of } a^j \text{ in}$$

$$\frac{1}{(1-a)(1-ax)\dots(1-ax^r)(1-x^{r+1})\dots(1-x^i)};$$

and the coefficient of $a^j x^w$ in the fraction on the right is the number of ways in which w can be composed of the parts $1, 2, 3, \dots i$, not more than j of the parts being as small as r. But the number of 1's in a given partition is equal to the excess of the first part over the second in its conjugate; the number of 2's to the excess of the second part over the third, and so on. Hence the number of 1's plus the number of 2's ... plus the number of r's in a given partition is equal to the excess of the first part over the r^{th} part in its conjugate; and we

have thus proved that the coefficient of x^w in the development of

$$\frac{(1-x^{j+1})(1-x^{j+2})\ldots(1-x^{j+r})}{(1-x)(1-x^2)\ldots(1-x^i)}$$

may be indifferently regarded as the number of partitions of w into parts none greater than i and not more that j of them as small as r or as the number of partitions of w into j or fewer parts, the excess of the first part over the r^{th} part being as small as j. These results may obviously be extended by introducing the a in non-consecutive factors of the product $(1-x)(1-x^2)\ldots(1-x^i)$.

(25) (C) *On the theorem of one-to-one and class-to-class correspondence between partitions of n into uneven and its partitions into unequal parts, by* DR. F. FRANKLIN.

The number of partitions of w into k distinct odd numbers, each repeated an indefinite number of times, is evidently the coefficient of $a^k x^w$ in the development of

$$\left(1+a\frac{x}{1-x}\right)\left(1+a\frac{x^3}{1-x^3}\right)\left(1+a\frac{x^5}{1-x^5}\right)\ldots$$

It is not easy to form the generating function for the number of partitions containing k sequences, but it is plain that the number of partitions of w containing *one* sequence is the coefficient of x^w in $S_1+S_2+S_3+\ldots$, where

$$S_1=x+x^2+x^3+x^4+x^5+\ldots=\frac{x}{1-x}$$

$$S_2=x^3+x^5+x^7+x^9+x^{11}+\ldots=\frac{x^3}{1-x^2}$$

$$S_3=x^6+x^9+x^{12}+x^{15}+x^{18}+\ldots=\frac{x^6}{1-x^3}$$

$$S_4=x^{10}+x^{14}+x^{18}+x^{22}+x^{26}+\ldots=\frac{x^{10}}{1-x^4}$$

$$S_5=x^{15}+x^{20}+x^{25}+x^{30}+x^{35}+\ldots=\frac{x^{15}}{1-x^5}:$$

and in general

$$S_r=x^{1+2+3+\ldots+r}+x^{2+3+4+\ldots+(r+1)}+\ldots=\frac{x^{\frac{1}{2}n(n+1)}}{1-x^n}.$$

So much of Prof. Sylvester's theorem as relates to a single sequence follows from inspection of the above scheme. For $S_1=\frac{x}{1-x}$; adding to S_3 the first term of S_2, we get $\frac{x^3}{1-x^3}$; adding to S_5 the first term of S_4 and the second term of S_2, we get $\frac{x^5}{1-x^5}$; adding to S_{2m+1} the first term of S_{2m}, the second term of $S_{2(m-1)}$, the third term of $S_{2(m-2)}$, ... and the m^{th} term of S_1, we get $\frac{x^{2m+1}}{1-x^{2m+1}}$;

thus the proposition is proved. The fact is made more evident to the eye if we write the scheme as follows:

$$
\begin{array}{ll}
S_1 = x + x^2 + x^3 + x^4 + x^5 + \ldots & S_2 = x^3 + x^5 + x^7 + x^9 + x^{11} + \ldots \\
S_3 = x^6 + x^9 + x^{12} + x^{15} + x^{18} + \ldots & S_4 = \qquad x^{10} + x^{14} + x^{18} + x^{22} + \ldots \\
S_5 = x^{15} + x^{20} + x^{25} + x^{30} + x^{35} + \ldots & S_6 = \qquad\qquad x^{21} + x^{27} + x^{33} + \ldots \\
S_7 = x^{28} + x^{35} + x^{42} + x^{49} + x^{56} + \ldots & S_8 = \qquad\qquad\qquad x^{36} + x^{44} + \ldots \\
S_9 = x^{45} + x^{54} + x^{63} + x^{72} + x^{81} + \ldots & S_{10} = \qquad\qquad\qquad\qquad x^{55} + \ldots
\end{array}
$$

Here $\frac{x^9}{1-x^9}$, for instance, is obtained by adding the 4th column on the right to the 5th now on the left.

It may be noted that we have thus found that

$$\frac{x}{1-x} + \frac{x^3}{1-x^3} + \frac{x^5}{1-x^5} + \ldots + \frac{x^{2m+1}}{1-x^{2m+1}} + \ldots$$

$$= \frac{x}{1-x} + \frac{x^3}{1-x^2} + \frac{x^6}{1-x^3} + \ldots + \frac{x^{\frac{1}{2}n(n+1)}}{1-x^n} + \ldots$$

(26) [Compare Jacobi's theorem contained in the last-but-one two lines of the last but one page of the *Fundamenta Nova*, which may be easily reduced to the form $\frac{x}{1+x} - \frac{x^3}{1+x^3} + \frac{x^5}{1+x^5} \ldots = \frac{x}{1+x} - \frac{x^3}{1+x^2} + \frac{x^5}{1+x^3} - \ldots$ J. J. S.]

Act II. On the Graphical Conversion of Continued Products into Series.

Naturelly, by composiciouns
Of anglis, and slie reflexiouns.
The Squieres Tale.

(27) The method about to be explained of representing the elements of partitions by means of a succession of angles fitting into one another arose out of an investigation (instituted for the purpose of facilitating the arrangement of tables of symmetric functions)* as to the number of partitions of n which are their own conjugates. The ordinary graphs to such partitions must obviously be symmetrical in respect to the two nodal boundaries, as seen below.

```
* * * * * * *
* * * * * * *
* * * *
* * *
* *
* *
* *
```

* By Mr. Durfee, of California (Fellow of the Johns Hopkins University), to whom I suggested the desirability of investigating more completely than had been done the method of arrangement of such tables founded upon the notion of self-conjugate partitions, which M. Faà de Bruno had the honor of initiating. The very valuable results of Mr. Durfee's inquiries, embodying, systematising and completing the theory of arrangement originated by Prof. Cayley, and further illustrated by the labors of Professors Betti and De Bruno, will probably appear in the next number of the Journal.

Let the above figure be any such graph; it may be dissected into a square (which may contain one or any greater square number) of say i^2 nodes, and two perfectly similar appended graphs, each having the content $\frac{n-i^2}{2}$, and subject to the sole condition that the number of its lines (or columns), *i. e.* that the number (or magnitude) of the parts in the partition which it represents shall be i or less; such number is the coefficient of $x^{\frac{n-i^2}{2}}$ in $\frac{1}{1-x.1-x^2\ldots 1-x^i}$ which is the same as that of x^{n-i^2} in $\frac{1}{1-x^2.1-x^4\ldots 1-x^{2i}}$ or of x^n in $\frac{x^{i^2}}{1-x^2.1-x^4\ldots 1-x^{2i}}$.

Hence giving i all possible values we see that the coefficient of x^n in the infinite series $1+\frac{x}{1-x^2}+\frac{x^4}{1-x^2.1-x^4}+\frac{x^9}{1-x^2.1-x^4.1-x^6}+\ldots$ is the number of self-conjugate partitions of n, or which is the same thing of symmetrical groups whose content is n.

(28) But any such graph in which there is a square of i^2 nodes with its two appendices may be dissected in another manner into i angles or bends, each containing any continually decreasing odd number of nodes, and *vice versâ*, any set of equilateral angles of nodes continually decreasing in number (which condition is necessary in order that the lower lines and posterior columns may not protrude beyond the upper lines and anterior columns) when fitted into one another in the order of their magnitudes will form a regular graph. Thus the actual figure (where there is a square of 9 nodes) formed by the intersections of the lines and columns may be dissected into 3 angles containing respectively 15, 7, 3 nodes; and so in general the number of ways in which n can be made up of odd and unrepeated parts will be the same as the number of ways in which $\frac{n-j^2}{2}$ can be partitioned into not more than j parts; hence we see that the coefficients of $x^n a^j$ in $(1+ax)(1-ax^3)(1-ax^5)\ldots$ and in $\frac{x^{j^2}}{1-x^2.1-x^4\ldots 1-x^{2j}}$ are the same, so that the continued product above written is equal to

$$1+\frac{x}{1-x}a+\ldots+\frac{x^{j^2}}{1-x^2.1-x^6\ldots 1-x^{2j}}a^j+\ldots$$

as is well known.

(29) In like manner if the expansion in a series of ascending powers of a of the finite continued product $(1+ax)(1+ax^3)\ldots(1+ax^{2i-1})$ be required, the coefficient of x^n in the coefficient of a^j will be the number of ways in which n can be made up with j of the unrepeated numbers $1, 3, \ldots 2i-1$, and as $2i-1$ is the number of nodes in an equilateral angle whose nodes contain i nodes, it follows that this coefficient will be the number of ways in which $\frac{n-j^2}{2}$ can be composed

with parts none *exceeding* $i-j$ in magnitude, and will therefore be the same as the coefficient of $x^{\frac{n-j^2}{2}}$ in $\dfrac{1-x^{i-j+1}.1-x^{i-j+2}\ldots 1-x^{i}}{1-x.1-x^2\ldots 1-x^j}$, and consequently the finite continued product above written is equal to

$$1+\ldots+\frac{1-x^{2i-2j+2}.1-x^{2i-2j+4}\ldots 1-x^{2i}}{1-x^2.1-x^4\ldots 1-x^{2j}}x^{j^2}a^j+\ldots$$

(30) If it be required to ascertain how many self-conjugate partitions of n there are containing exactly i parts, this may be found by giving j all possible values and making p_j equal to the number of ways in which $\dfrac{n-j^2}{2}$ can be composed with j or fewer parts the greatest of which is $i-j$, *i. e.* $n-j^2+2j-2i/2$ with $j-i$ or fewer parts none greater than $i-j$, so that p_j will be the coefficient of $x^{n-j^2+2j-2i/2}$ in $\dfrac{1-x^{i-j+1}.1-x^{i-j+2}\ldots 1-x^{i-1}}{1-x.1-x^2\ldots 1-x^{j-1}}$ or of

x^n in $\dfrac{1-x^{2i-2j+2}.1-x^{2i-2j+4}\ldots 1-x^{2i-2}}{1-x^2.1-x^4\ldots 1-x^{2j-2}}x^{j^2-2j+2i}$; the sum of the values of p_j for all values of j will be the number required: this number, therefore, writing ω for $2i-1$, will be the coefficient of x^n in

$$1+\frac{1-x^{\omega+1}}{1-x^2}x^{\omega}+\frac{1-x^{\omega+1}.1-x^{\omega+3}}{1-x.1-x^4}x^{\omega+1}+\frac{1-x^{\omega+1}.1-x^{\omega+3}.1-x^{\omega+5}}{1-x^2.1-x^4.1-x^6}x^{\omega+4}$$
$$+\frac{1-x^{\omega+1}.1-x^{\omega+3}.1-x^{\omega+5}.1-x^{\omega+7}}{1-x^2.1-x^4.1-x^6.1-x^8}x^{\omega+9}$$

the coefficient of the outstanding factor in the q^{th} term after the first in this series being $x^{\omega+(q-1)^2}$ we may suppose q the least integer number not less than $1+\sqrt{n-\omega}$, and then the subsequent term to the $(q+1)^{\text{th}}$ being inoperative may be neglected.

(31) In order to see how any self-conjugate graph may be recovered, so to say, from the corresponding partition consisting of unrepeated odd numbers, consider the diagrammatic case of the partition 17, 9, 5, 1 represented by the angles of the graph below written

```
* * * * * * * * *
* * * * * *
* * * * *
* * * *
* * *
* *
*
*
*
```

the number of angles is the number of the given parts, *i. e.* is 4, and the first

four lines of the graph will be obtained by adding 0, 1, 2, 3 to the major half (meaning the integer next above the half) of 17, 9, 5, 1, *i. e.* will be 9, 6, 5, 4, the total number of lines will be the major half of the highest term (17) and the remaining lines will have the same contents, viz. 3, 2, 1, 1, 1 as the columns of the graph found by subtracting 4 (the number of the parts) from the numbers last found, *i. e.* will be the lines of the graph which is conjugate to 5, 2, 1. And so in general the self-conjugate graph corresponding to any partition of unrepeated odd numbers $q_1, q_2, \ldots q_j$ will be found by the following rule:

Let P be the system of partitions $k_1, k_2, \ldots k_j$, in which any term k_θ is the major half of q_θ augmented by $\theta - 1$, and P' another system of $k', k'_2, \ldots k'_j$, obtained by subtracting j from each term in P, then P and the conjugate to P' will be the self-conjugate partition corresponding to the given q partition. Thus as an example, 19, 11, 7, 5 being given, P, P' will be 19, 7, 6, 6; 6, 3, 2, 2 respectively, and the self-conjugate system required will be 10, 7, 6, 6, 4, 4, 2, 2, 1. Of course P' might also be obtained by taking the minor halves of the given parts in inverse (ascending) order and subtracting from them the numbers 0, 1, 2, ... respectively.

To pass from a given self-conjugate to the corresponding unrepeated odd numbers-partition is a much simpler process, the rule being to take the numbers in descending order and from their doubles subtract the successive odd numbers in the natural scale until the point is reached at which the difference is about to become negative; thus the partition 6 6 5 4 3 2 is self-conjugate, and the correspondent to it is 11 9 5 1.

(32) The expansion of the reciprocal to $(1 - ax)(1 - ax^3) \ldots (1 - ax^{2i-1})$ may be read off with the same facility as the direct product. In this case we are concerned with partitions of odd numbers capable of being repeated in the same partition; now, therefore, if we use the same method of equilateral angles as before, and fit them into one another in regular order of magnitude, it will no longer be the case that their sum will form a regular graph, for if there be θ parts alike, each line and column which ranges with either side of any (but the first one) of these will jut out one step beyond the anterior line and column (respectively), so that the line joining the extremities of the lines or columns will be parallel to the axis of symmetry. The figure then corresponding to i odd parts can no longer be dissected into a *square* of nodes and two equal regular graphs, but it may be dissected into a *line* of nodes lying in the axis of symmetry, and

two regular graphs one of which has for its boundaries one of the original boundaries and a line of nodes parallel to the axis of symmetry, and the other one the other original boundary and a line of nodes parallel to the same axis, as seen in the annexed figure, where the axial points are distinguished by being made larger than the rest.

```
*  *  *  *  *  *
*  *  *  *  *  *  *
*  *  *  *  *  *  *  *
*  *  *  *  *  *  *
*  *  *  *  *  *  *
*  *  *  *  *  *  *
   *  *  *     *
      *
```

The graph read off in angles represents the partition 11 11 11 7 3 3. On removing the 6 diagonal nodes it breaks up into two regular graphs, of which one is 5 5 5 3 1 1, and the other the conjugate thereto; hence the coefficient of x^n in the coefficient of a^j in the expansion of the reciprocal of $1-ax.1-ax^3\ldots.1-ax^{2i-1}$ in ascending powers of a is the number of ways in which $\frac{n-j}{2}$ can be resolved into j parts limited not to exceed $i-1$, which is the coefficient of $x^{\frac{n-j}{2}}$ in $\frac{1-x^i.1-x^{i+1}\ldots1-x^{i+j-1}}{1-x.1-x^2\ldots1-x^j}$ or of x^n in $\frac{1-x^{2i}.1-x^{2i+2}\ldots1-x^{2i+2j-2}}{1-x^2.1-x^4\ldots1-x^{2j}}x^j$

(33) Although I shall not require any intermediate expansion whatever in order to obtain the transcendant $\Theta_1 x$ product in the form of a series, I will give another of those which are sometimes employed together in combination (see Cayley, Elliptic Functions, pp. 296–7) to obtain this result: thus to prove that the continued product of the reciprocal of $(1-ax)(1-ax^2)(1-ax^3)\ldots$ is identical with

$$1+\frac{x}{1-x}\cdot\frac{a}{1-xa}+\frac{x^4}{1-x.1-x^2}\cdot\frac{a^2}{1-xa.1-x^2a}$$
$$+\frac{x^9}{1-x.1-x^2.1-x^3}\cdot\frac{a^3}{1-xa.1-x^2a.1-x^3a}+\text{etc.}$$

if n is partitioned into j parts, the regular graph which represents the result of any such partition must consist either of $1, 2, 3, \ldots j-1$ or of not less than j columns, and its graph may accordingly in these several cases be dissected into a square of $1, 4, 9, \ldots j^2$ nodes; suppose that such square consists of θ parts, then there will be $n-\theta^2$ nodes remaining over subject to distribution into two groups limited by the condition as to one of the groups that it may contain an unlimited number of parts none exceeding θ in magnitude, and as to

the other that it must contain $j-\theta$ parts none exceeding θ in magnitude, as seen in the following diagrams:

```
*  *  *  *  *  *  *      *  *  *  *  *  *  *  *  *
*  *  *  *  *  *  *      *  *  *  *  *
*  *  *  *  *  *         *  *  *
×  ×                     ×  ×  ×
×  ×                     ×  ×  ×
×                        ×  ×
×                        ×
```

```
*  *  *  *  *  *
*  *  *  *  *  *
*  *  *  *  *
×  ×  ×
×  ×  ×
×
×
```

in all of which the partible number is 26, and j and θ are 7 and 3 respectively. Now the number of such distributions is the coefficient of

$$x^{n-\theta^2}a^{j-\theta}\frac{1}{1-x.1-x^2\ldots1-x^\theta}\cdot\frac{1}{1-ax.1-ax^2\ldots1-ax^\theta}$$

i. e. of $x^n a^j$ in
$$\frac{x^{\theta^2}}{1-x.1-x^2\ldots1-x^\theta}\cdot\frac{a^\theta}{1-ax.1-ax^2\ldots1-ax^\theta},$$

and consequently giving θ all values from 1 to ∞, the proposed equation is verified.

(34) It may be desired to apply the same method to obtain a similar development for the reciprocal of the limited product $(1-ax)(1-ax^2)\ldots(1-ax^i)$; the construction will be the same as in the last case; the distribution into two groups can be made as before; the second group will remain subject to the same condition as in the preceding case (seeing that the number of parts being less than $j-\theta$ will necessarily be less than $i-\theta$, for j cannot exceed i), but the first group will be subject to the condition of being partitioned not now into an unlimited but into $i-\theta$ (or fewer) parts none exceeding θ in magnitude, and the number of such distributions into the two groups will accordingly become the coefficient of $x^{n-\theta^2}a^{j-\theta}$ in

$$\frac{1-x^{i-\theta+1}.1-x^{i-\theta+2}\ldots1-x^i}{1-x.1-x^2\ldots1-x^\theta}\cdot\frac{1}{1-ax.1-ax^2\ldots1-ax^\theta}$$

in the last written fraction multiplied by $x^{\theta^2}.a^\theta$, so that the required expansion will be

$$1+\frac{1-x^i}{1-x}\cdot\frac{xa}{1-ax}+\frac{1-x^i.1-x^{i-1}}{1-x.1-x^2}\cdot\frac{x^4a^2}{1-ax.1-ax^2}$$
$$+\frac{1-x^i.1-x^{i-1}.1-x^{i-2}}{1-x.1-x^2.1-x^3}\cdot\frac{x^9a^3}{1-ax.1-ax^2.1-ax^3}+\ldots$$

(35) It is interesting to investigate what will be the form of the mixed development resulting from an application of the same method to the *direct* product $1 + ax . 1 + ax^2 \ldots 1 + ax^i$.
For greater clearness I shall first suppose i infinitely great. Consider the diagram:

```
*  *  *  *  *  *  *  *  *
*  *  *  *  *  *
*  *  *  *  *
*  *  *
*  *
```

In the above graph j and θ used in the same sense as *ante* are 5 and 3 respectively, so that there is a square of 9 points; an appendage to the right of and another appendage below the square, which I shall call the lateral and subjacent appendages respectively. The content of the graph being $25 - 9$, there are 16 points to be distributed between these two appendages. What now are the conditions of the distribution of the $n - \theta^2$ points between them?

I say that there will be two sorts of such distribution—one in which the lateral appendage will consist of θ unrepeated parts none of them zero, as in the graph above, and the subjacent appendage of $j - \theta$ unrepeated parts, limited not to exceed θ in magnitude, and another sort as in the graph below written.

```
*  *  *  *  *  *  *  *  *  *  *  *
*  *  *  *  *  *  *
*  *  *
*  *
*
```

in which the j^{th} line of the lateral appendage is missing, and consequently the subjacent graph will consist of $j - \theta$ unrepeated parts limited not to exceed $\theta - 1$ in magnitude, for there could not be a part so great as θ without the last line of the square having the same content as the first line of the subjacent appendage.

It should be observed that only the *last* admissible line of the lateral appendage can be wanting, for if more than this were wanting, two lines of the square would belong to the graph, and consequently there would be two equal parts θ.

Hence there are two kinds of association of the appendages, one leading to a distribution of $n - \theta^2$ between one group of θ unrepeated but unlimited parts, and another of $j - \theta$ unrepeated parts limited not to exceed θ; the other to a distribution of $n - \theta^2$ between one group of $\theta - 1$ unrepeated but unlimited parts, and another of $j - \theta$ unrepeated parts limited not to exceed $\theta - 1$.

The number of distributions of the first kind is the coefficient of

$$x^{n-\theta^2} . a^{j-\theta} \text{ in } \frac{x^{\frac{\theta^2+\theta}{2}}}{1-x . 1-x^2 \ldots 1-x^\theta} . (1 + ax)(1 + ax^2) \ldots (1 + ax^\theta)$$

the other of $x^{n-\theta^2}.a^{j-\theta}$ in $\dfrac{x^{\frac{\theta^2-\theta}{2}}}{1-x.1-x^2\ldots(1-x^{\theta-1})}.(1+ax)(1+ax^2)\ldots(1+ax^{\theta-1})$; hence the sum of the distributions of the two kinds is the coefficient of the same argument in

$$\frac{x^{\frac{\theta^2-\theta}{2}}}{1-x.1-x^2\ldots1-x^\theta}\{x^\theta(1+ax^\theta)+(1-x^\theta)\}\{1+ax.1+ax^2\ldots1+ax^{\theta-1})\},$$

i. e. of $\quad x^n a^j$ in $x^{\frac{3\theta^2+\theta}{2}}\left(\dfrac{1+ax.1+ax^2\ldots1+ax^{\theta-1}}{1-x.1-x^2\ldots1-x^{\theta-1}}\cdot\dfrac{1+ax^{2\theta}}{1-x^\theta}\right)$

and consequently we obtain the equation

$$1+ax.1+ax^2.1+ax^3\ldots=1+\frac{1+ax^2}{1-x}xa+\frac{1+ax.1+ax}{1-x.1-x^2}x^5a^2+\ldots$$
$$+\frac{1+ax.1+ax^2\ldots1+ax^{j-1}.1+ax^{2j}}{1-x.1-x^2\ldots1-x^{j-1}.1-x^j}x^{\frac{3j^2-j}{2}}a^j+\ldots$$

and thus by a very unexpected route we arrive at a proof of Euler's celebrated pentagonal-number theorem; for in making $a=-1$ the above equation becomes

$$1-x.1-x^2.1-x^3\ldots=1-(1+x)x+(1+x^2).x^5\ldots+(-)^j(1+x^j)x^{\frac{3j^2-j}{2}}+\ldots$$

Such is one of the fruits among a multitude arising out of Mr. Durfee's ever-memorable example of the dissection of a graph (in the case of a symmetrical one) into a square, and two regular graph appendages.

Even the trifling algebraical operation above employed to arrive at the result might have been spared by expressing the continued product as the sum of the two series (which flow immediately from the graphical dissection process), left uncombined, viz.

$$1+\frac{1+ax}{1-x}x^2a+\frac{1+ax.1+ax^2}{1-x.1-x^2}x^7a^2+\frac{1+ax.1+ax^2.1+ax^3}{1-x.1-x^2.1-x^3}x^{15}a^3+\ldots$$

together with $\quad +xa+\dfrac{1+ax}{1-x}x^5a^2+\dfrac{1+ax.1+ax^2}{1-x.1-x^2}x^{12}a^3+\ldots$

which for $a=-1$ unite into the single series $1-x-x^2+x^5+x^7-x^{12}-x^{15}$ etc.

(36) I will now proceed to find the expression in a mixed series of the limited product $1+ax.1+ax^2\ldots(1+ax^i)$.

In each of the two systems of distribution (as shown already in the theory of the reciprocal of such product) the second group will remain unaffected by the new limitation, but the first group will now consist of partitions (limited in number as before), but in magnitude instead of being unlimited, limited not to exceed $(i-\theta)$, so that we will have to take the coefficient of $x^{n-\theta^2}.a^{j-\theta}$ in the sum of

$$x^{\frac{\theta^2+\theta}{2}}\frac{1-x^{i-\theta}.1-x^{i-\theta-1}\ldots1-x^{i-2\theta+1}}{1-x.1-x^2\ldots1-x^\theta}.(1+ax)(1+ax^2)\ldots(1+ax^\theta)\text{ and}$$

$$x^{\frac{\theta^2-\theta}{2}}\frac{1-x^{i-\theta}.1-x^{i-\theta-1}\ldots 1-x^{i-2\theta+2}}{1-x.1-x^2\ldots 1-x^{\theta-1}}.(1+ax)(1+ax^2)\ldots(1+ax^{\theta-1}).$$

This will be the same as the coefficient of $x^n a^j$ in

$$x^{\frac{3\theta^2-\theta}{2}}a^j(1+ax)(1+ax^2)\ldots(1+ax^{\theta-1})\frac{1-x^{i-\theta}.1-x^{i-\theta-1}\ldots 1-x^{i-2\theta+2}}{1-x.1-x^2\ldots 1-x^{\theta-1}.1-x^{\theta}}$$
$$\{1-x^{\theta}+(1-x^{i-2\theta+1})(x^{\theta}+ax^{1\theta})\}$$

where the quantity within the final bracket is equal to $1-x^{i+1}a-x^{i-\theta+1}+x^{2\theta}a$. Hence the required series is

$$\left\{1+\frac{1-x^i}{1-x}xa+\frac{1-x^{i-1}.1-x^{i-2}}{1-x.1-x^2}(1+ax)x^5a^2\right.$$
$$\left.+\frac{1-x^{i-2}.1-x^{i-3}.1-x^{i-4}}{1-x.1-x^2.1-x^3}.1+ax.1+ax^2.x^{12}a^3+\ldots\right\}$$
$$+\left\{\frac{1-x^{i-1}}{1-x}x^3a^2+\frac{1-x^{i-2}.1-x^{2-3}}{1-x.1-x^2}(1+ax)x^9a^3\right.$$
$$\left.+\frac{1-x^{i-3}.1-x^{i-4}.1-x^{i-5}}{1-x.1-x^2.1-x^3}.1+ax.1+ax^2.1+ax^3.x^{18}a^4+\ldots\right\}$$

the indices in the outstanding powers of x being the pentagonal numbers in the first, and the triangular numbers trebled, in the second of the above series.

In obtaining in the preceding articles mixed series for continued products, it will be noticed that the graphical method has been employed, not to exhibit correspondence, but as an instrument of transformation. The graphs are virtually segregated into classes, and the number of them contained in each class separately determined. (The magnitude of the square in the Durfee-dissection serves as the basis of the classification.)

(37) Now let us consider the famous double product of

$(1+ax)(1+ax^3)(1+ax^5)\ldots$ by $(1+a^{-1}x)(1+a^{-1}x^3)(1+a^{-1}x^5)\ldots$

Here it will be expedient to introduce a new term and to explain the meaning of a bi-partition and a system of parallel bi-partitions of a number. The former indicates that the elements are to be distributed into two groups, say into a left and right-hand group: the latter that the number of the elements (on one, say) on the left-hand side of each bi-partition of the system is to be equal or exceed by a constant difference the number (on the other, say) on the right-hand side of the same bi-partition. If we use dots, regularly spaced, to represent the elements (themselves numbers and not units), we get a figure or pair of figures such as the following:

$$\begin{array}{cccccc}
* & * & * & * & * & * \\
* & * & * & * & * & * \\
* & * & * & * & * & \\
* & * & * & * & &
\end{array}
\qquad
\begin{array}{cccc}
* & * & * & * \\
* & * & * & * \\
* & * & * & * \\
* & * & &
\end{array}$$

for which the corresponding lines of the contour are respectively parallel—hence the name. When the number of elements on the two sides are identical, I call the system an equi-bi-partition-system—in the general case, a parallel bi-partition-system to a constant difference j, where j is the excess of the number of elements in the left-hand over that in the right-hand part of any of the bi-partitions.

(38) Consider now the given double product—it is obvious that it may be expanded in terms of paired powers $a^j + a^{-j}$ of a, and the coefficient of x^n in the term not involving a will evidently be the number of equi-bi-partitions of n that can be formed with unrepeated odd numbers; and so the coefficient of x^n associated with a^j or a^{-j} will be the number of parallel bi-partitions of n to the constant difference j that can be so formed.

1°. For the equi-bi-partitions; suppose $l_1, l_2 \ldots l_i, \lambda_1, \lambda_2 \ldots \lambda_i$ is an equi-bi-partition, all the elements being odd and unrepeated; take successive angles whose (say horizontal and vertical) sides are the major halves of $l_1, \lambda_1; l_2, \lambda_2 \ldots; l_i, \lambda_i$; these angles will fit on to one another so as to form a regular graph by reason of the relations

$$l_1 > l_2 + 1 \quad l_2 > l_3 + 1 \ldots l_{i-1} > l_i + 1$$
$$\lambda_1 > \lambda_2 + 1 \quad \lambda_2 > \lambda_3 + 1 \ldots \lambda_{i-1} > \lambda_i + 1.$$

Conversely any regular graph may be resolved into angles whose horizontal sides shall be the major halves of one set of odd numbers, and their vertical sides of the major halves of another set of as many odd numbers, and these two sets of odd numbers will each form a decreasing series; hence there is a one-to-one conjugate correspondence between any bi-partition of n written in regular order, and the totality of regular graphs whose content is $\frac{n}{2}$, so that the number of the equi-bi-partitions of n will be the coefficient of $x^{\frac{n}{2}}$ in $\frac{1}{1-x.1-x^2.1-x^3\ldots}$, *i. e.* of x^n in $\frac{1}{1-x^2.1-x^4.1-x^6\ldots}$ which fraction is therefore equal to the totality of the terms not involving a.

(39) Next for the coefficient of a^j.

Let $l_1, l_2, \ldots l_j, l_{j+1}, l_{j+2}, \ldots l_{j+\theta}; \lambda_1, \lambda_2, \ldots \lambda_\theta$ be an equi-parallel bi-partition to the difference j (with the elements on each side written in descending order) with the equi-bi-partition $l_{j+1}, l_{j+2}, \ldots l_{j+\theta}; \lambda_1, \lambda_2, \ldots \lambda_\theta$, form a graph, as in the preceding case; say for distinctness, with major halves of the l series horizontal and of the λ series vertical; over the highest horizontal

line the successive quantities* $\frac{l_j-1}{2}, \frac{l_{j-1}-3}{2}, \frac{l_{j-2}-5}{2}, \ldots \frac{l_1(2-j-1)}{2}$ may be laid so as to form a regular graph of which the content will be $\frac{n-j^2}{2}$.

Conversely every regular graph whose content is $\frac{n-j^2}{2}$ will correspond to a parallel bi-partition of unrepeated odd numbers to a difference j; to obtain the bi-partition the first j lines of the graph must be abstracted,† and the graph thus diminished resolved into angles; the doubles of the contents of each vertical side of these angles diminished by unity will constitute the right-hand side of the bi-partition, and the doubles of the contents of each horizontal side preceded by the doubles of the lines of the abstracted portion of the graph increased by $1, 3, 5, \ldots 2j-1$ respectively, will form the left-hand portion. Hence the number of such bi-partitions will be the number of ways of resolving $\frac{n-j^2}{2}$ into unrestricted parts, *i. e.* will be the coefficient of x^n in $\frac{1}{1-x^2.1-x^4.1-x^6\ldots} x^{j^2}$, and this being true for all values of n and j, we see that the double product in question will be identical with the infinite series

$$\frac{1}{1-x^2.1-x^4.1-x^6\ldots}\{1+x(a+a^{-1})+x^4(a^2+a^{-2})+x^9(a^3+a^{-3})+\ldots\}$$

(40) To expand the limited double product

$(1+ax)(1+ax^3)\ldots(1+ax^{2i-1})$ into $(1+a^{-1}x)(1+a^{-1}x)\ldots(1+a^{-1}x^{2i-1})$ the procedure and reasoning will be precisely the same as in the extreme case of i infinite, the only difference being that the elements of the bi-partition instead of being unlimited odd numbers will be limited not to exceed $2i-1$. In the case of $j=0$ the equi-bi-partition will furnish a series of nodal angles in which neither side can exceed the major half of $2i-1$, *i. e.* i, and the coefficient of x^n in the term not containing any power of a will consequently be the number of ways in which n can be divided into parts limited as well in number as in magnitude not to exceed i, and will therefore be the same as the coefficient of x^n in the development of $\frac{1-x^{i+1}.1-x^{i+2}\ldots 1-x^{2i}}{1-x.1-x^2\ldots 1-x^i}$, or which is the same thing, of x^n in the development of $\frac{1-x^{2i+2}.1-x^{2i+4}\ldots 1-x^{4i}}{1-x^2.1-x^4\ldots 1-x^{2i}}$, and when the bi-partition system has a constant difference j, the corresponding graph will be of the same form, except that it will be overlaid with j lines, obtained as in the preceding case by subtracting $1, 3, \ldots 2j-1$ from the first j left-hand elements, and taking the halves of the remainders; the graphs thus

* Any number of these quantities may happen to become zero.

† If the actual number of horizontal lines in the graph is less than j, it must be made to count as j, by understanding lines of zero content to be supplied underneath the graph.

formed will be subject to the condition of having a content $\frac{n-j^2}{2}$, and parts limited not to exceed $i-j$ in magnitude nor $i+j$ in number [$i-j$ in magnitude because the topmost line cannot exceed $\frac{(2i-1)-(2j-1)}{2}$ in content; $i+j$ in number because without reckoning the j superimposed lines the subjacent portion of the graph cannot contain more than i lines]. The converse that out of every regular graph fulfilling these conditions may be spelled out a parallel bi-partition with a difference j, and containing only unrepeated odd numbers limited not to exceed $2i-1$ in magnitude may be shown as in the preceding case. Hence the coefficient of x^n in the coefficient of a^j+a^{-j} in the expansion, is the number of ways of resolving $\frac{n-j^2}{2}$ into parts none exceeding $i-j$ in magnitude nor $i+j$ in number, *i. e.* is the coefficient of x^n in

$$\frac{1-x^{2i+2j+2}.1-x^{2i+2j+4}\ldots 1-x^{4i}}{1-x^2.1-x^4\ldots 1-x^{2i-2j}}x^{j^2}.$$

Hence by the process of reasoning which has been so often applied, we see that the finite double product

$$1+ax.1+ax^3\ldots 1+ax^{2i-1} \text{ into } 1+a^{-1}x.1+a^{-1}x^3\ldots 1+a^{-1}x^{2i-1}$$

$$=\frac{1-x^{2i+2}.1-x^{2i+4}\ldots 1-x^{4i}}{1-x^2.1-x^4\ldots 1-x^{2i}}\left\{1+\frac{1-x^{2i}}{1-x^{2i+2}}x+\frac{1-x^{2i}.1-x^{2i-2}}{1-x^{2i+2}.1-x^{2i+4}}x^4 + \frac{1-x^{2i}.1-x^{2i-2}.1-x^{2i-4}}{1-x^{2i+2}.1-x^{2i+4}.1-x^{2i+6}}x^9+\ldots\right\}$$

Compare Hermite, Note sur les fonctions elliptiques, p. 35, where Cauchy's method is given of arriving at this and the preceding identity.

ACT III. ON THE ONE-TO-ONE AND CLASS-TO-CLASS CORRESPONDENCE BETWEEN PARTITIONS INTO UNEVEN AND PARTITIONS INTO UNEQUAL PARTS.

> . . . mazes intricate,
> Eccentric, intervolved, yet regular
> Then most, when most irregular they seem.
>
> *Paradise Lost*, V. 622.

(41) It has been already shown that any partition of n into unequal parts may be converted into a partition consisting of odd numbers equal or unequal by, first, expressing any even part by its longest odd divisor, say its nucleus and a power of 2, and, second, adding together the powers of 2 belonging to the same nucleus, there will result a sum of odd nuclei, each occurring one or more times; a like process is obviously applicable to convert a partition in which any number occurs 1, 2, . . . or $(r-1)$ times into one in which only numbers not divisible by r occur with unrestricted liberty of recurrence. The nuclei will here be

numbers not divisible by r multiplied by powers of r, and by adding together the powers of r belonging to the same nucleus there results a series of nuclei, each occurring one or more times. Conversely when the nuclei and the number of occurrences of each are given, there being only one way in which any such number can be expressed in the scale whose radix is r, it follows that there is but one partition of the previous kind in which one of the latter kind can originate, and there is thus a one-to-one correspondence, and consequently equality of content between the two systems of partitions.

(42) To return to the case of $r=2$, with which alone we shall be here occupied, we see that the number of parts in the unequal partition which corresponds after this fashion with a partition made up of given odd numbers depends on the sum of the places occupied when the number of occurrences of each of the odd numbers is expressed in the notation of dual arithmetic. Such correspondence then is eminently arithmetical and transcendental in its nature, depending as it does on the forms of the numbers of repetitions of each different integer with reference to the number 2.

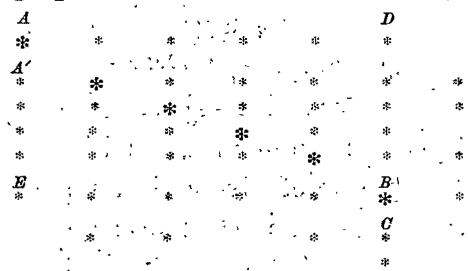
Very different is the kind of correspondence which we are now about to consider between the self-same two systems, as well in its nature, which is essentially graphical, as in its operation, which is to bring into correspondence the two systems, not as wholes but as separated each of them into distinct classes; and it is a striking fact that the pairs arithmetically and graphically associated will be entirely different, thus evidencing that correspondence is rather a creation of the mind than a property inherent in the things associated.*

(43) I shall call the totality of the partitions of n consisting of odd numbers the U, and that consisting of unequal numbers the V system.

1°. I say that any U may be converted into a V by the following rule: Let each part of the given U be converted into an equilateral bend, and these bends fitted into another as was done in the problem of converting the reciprocal of $(1-ax)(1-ax^3)(1-ax^5)\ldots$ into an infinite series considered in the preceding section. We thus form what may be called a bent graph. Then, as there shown, such graph may be dissected into a diagonal line of points and two precisely similar regular graphs. The graph compounded of the diagonal and one of these, it is obvious, will also be a regular, and I shall call it the major component of the bent graph; the remaining portion may be called the minor component. Each of these graphs will be bounded by lines inclined to each other

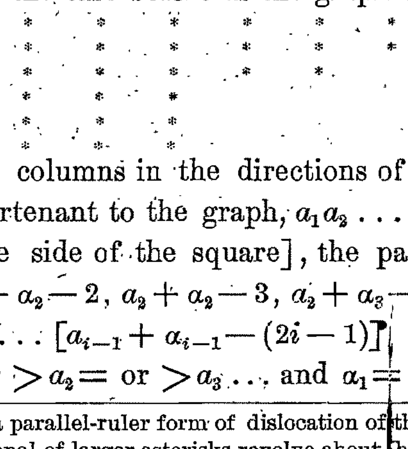
* Just so it is possible for two triangles to stand in a treble perspective relation to each other, as I have had previous occasion to notice in this Journal.

at an angle one-half of that contained between the original bounding lines, and each may be regarded as made up of bends fitting into one another. The contents of these bends taken in alternate succession, commencing with the major graph, will form a series of continually decreasing numbers, that is to say, a V partition. As an example, let 11 11 9 5 5 5 be the given U partition, this gives rise to the graph



Reading off the bends on the major and minor graphs alternately, commencing with BAD, $CA'E$ respectively, there results the regularized partition into unequal numbers 11 10 9 8 6 2.

(44) The application of the rule is facilitated to the eye by at once constructing a graph, the number of points in whose horizontal lines are the major halves of the given parts, and construing this to signify two graphs, one the graph actually written down, the other the same graph with its first column omitted; for instance in the case before us the graph will be*



If we call the lines and columns in the directions of the lines and columns of the Durfee-square appurtenant to the graph, $a_1 a_2 \ldots a_i$, $\alpha_1 \alpha_2 \ldots \alpha_i$ [i (here 3) being the extent of the side of the square], the partition given by the rule will be $a_1 + \alpha_1 - 1$, $a_1 + \alpha_2 - 2$, $a_2 + \alpha_2 - 3$, $a_2 + \alpha_3 - 4$, $a_3 + \alpha_3 - 5$, ... $[a_{i-1} + \alpha_{i-1} - (2i-1)]$, $[a_{i-1} + \alpha_i - 2i]$, $[a_i - i]$, and inasmuch as $a_1 =$ or $> a_2 =$ or $> a_3 \ldots$ and $\alpha_1 =$ or $> \alpha_2 =$ or $> \alpha_3 \ldots$ the

* This may be regarded as a parallel-ruler form of dislocation of the figure produced by making the portion to the right of the diagonal of larger asterisks revolve about that diagonal until it coincides with the portion to the left of the diagonal; the graph thus formed (merely as a matter of convenience to the eye) may be then made to revolve about an axis perpendicular to the plane, so as to bring the diagonal out of its oblique into the more usual horizontal position. All this trouble of description might have been saved by beginning not with a bent graph but with a graph formed with *straight* lines of points written *symmetrically* under each other, which is made possible by the fact of there being an *odd* number of points in each line. The graph so formed then resolves itself naturally into a major and minor regular graph.

above series is necessarily made up of continually decreasing numbers, at all events until the the last term is reached. But this term will form no exception, for the fact of i being the content of the side of the square belonging to the transverse graph $\alpha_1, \alpha_2 \ldots, \alpha_i, \alpha_{i+1} \ldots$ implies that $\alpha_i =$ or $> i$, hence $(a_{i-1} + \alpha_i - 2i) - (a_i - i) = (a_{i-1} - a_i) + \alpha_i - i > 0$.

In the above example the side of the square *nucleus* in the original total graph was supposed to be the same for the major and minor graphs of which it is composed. If we suppose that graph to contain only i nodes in the i^{th} line, then the side of the square to the minor graph which it contains will be $i-1$, and the number of parts given by the angular readings of the two graphs combined will be $2i-1$ instead of $2i$, as *ex gr.* if the 3d line in the graph above written be 3 instead of 5, the resulting partition will be 11 10 9 8 2, but we may, if we please, regard this as 11 10 9 8 2 0 and the last term will then still be $a_i - i$, and the general expression will remain unchanged from what it was before.

Next I proceed to the converse of what has been established, viz. that every U may be transformed by the rule into a V, and shall show that any V may be derived from some one (and only one) U.

Whether the number of effective parts in the given V be odd or even, we may always suppose it to be even by supplying a zero part if necessary, and may call the parts $l_1, \lambda_1, l_2, \lambda_2 \ldots, l_i, \lambda_i$. Suppose that it is capable of being derived from a certain U: form with the parts of U a graph expressed in the usual way by equilateral bends or elbows, then the side of the square appurtenant to the regular graph formed by the major half of this, say G, must have for content the given number i.

Let $a_1, a_2 \ldots a_i, \alpha_1, \alpha_2 \ldots \alpha_i$, be the contents of the first i rows and first i columns respectively of G, then the equations to be satisfied are

$$a_1 + \alpha_1 - 1 = l_1, \quad a_2 + \alpha_2 - 3 = l_2, \quad a_3 + \alpha_3 - 5 = l_3 \ldots, \quad a_i + \alpha_i - (2i-1) = l_i$$

$$a_1 + \alpha_2 - 2 = \lambda_1; \quad a_2 + \alpha_3 - 4 = \lambda_2, \quad a_3 + \alpha_4 - 6 = \lambda_3 \ldots, \quad a_i \qquad - i \qquad = \lambda_i.$$

Hence $a_1 - a_2 = \lambda_i - l_2 - 1 \quad a_2 - a_3 = \lambda_2 - l_3 - 1 \ldots$

$$a_{i-1} - a_i = \lambda_{i-1} - l_{i-1} \quad a_i = \lambda_i + i$$

$$\alpha_1 - \alpha_2 = l_1 - \lambda_1 - 1 \quad \alpha_2 - \alpha_3 = l_2 - \lambda_2 - 1 \ldots$$

$$\alpha_{i-1} - \alpha_i = l_{i-1} - \lambda_{i-1} - 1 \quad \alpha_i = l_i - \lambda_i + i - 1$$

and for all values of θ, $l_\theta > \lambda_\theta > l_{\theta+1}$.

Hence $a_1, a_2 \ldots a_i$ are all positive, and $\alpha_1, \alpha_2 \ldots \alpha_i$ are all at least equal to i. There will therefore be one and only one graph G satisfying the required

conditions, namely a graph the contents of whose lines are

$$\alpha_1,\ \alpha_2,\ \ldots\ \alpha_i,\quad A_1,\ A_2,\ \ldots\ A_{\alpha_1}-i$$

[where $A_1, A_2, \ldots A_{\alpha_i}-i$ is the conjugate partition to $\alpha_1-i,\ \alpha_2-i,\ \ldots\ \alpha_i-i$]; the partition U will be found by subtracting unity from the doubles of each of those parts. Thus then it has been shown that every U will give rise to some one V, and every V be derived from a determinate U; hence there must exist a one-to-one correspondence between the U and V system. In a certain sense it is a work of supererogation to show that there is a U corresponding to each V; it would have been sufficient to infer from the linear form of the equations that there could not be more than one U transformable into a V; for each U being associated with a distinct V it would follow that there could be no V's not associated with a U, since otherwise there would be more V's than U's, which we know *aliunde* is impossible.

As an example of what precedes let the partible number be 12. The U system computed exhaustively will be

$11.1\ \ 9.3\ \ 9.1^3\ \ 7.5\ \ 7.3.1^2\ \ 7.1^5\ \ 5^2.1^2\ \ 5.3.1^4\ \ 5.3^2.1\ \ 5.1^7\ \ 3^4\ \ 3^3.1^3\ \ 3^2.1^6\ \ 3.1^9\ \ 1^{12}$

Underneath of these partitions I will write the major component graph, and underneath this again the corresponding V; we shall thus have the table

11.1	9.3	9.1^3	7.5	$7.3.1^2$	7.1^5
* * * * * * *	* * * * * * *	* * * * * * * *	* * * * * * *	* * * * * * * *	* * * * * * * * *
7.5	6.5.1	8.4	5.4.2.1	7.4.1	9.3

$5^2.1^2$	$5.3.1^4$	$5.3^2.1$	5.1^7	3^4	$3^3.1^3$	$3^2.1^6$	3.1^9	1^{12}
* * * * * * * *	* * * * * * * * *	* * * * * * * *	* * * $(*)^7$	* * * * * * * *	* * * * * * $(*)^3$	* * * * $(*)^6$	* * $(*)^9$	$(*)^{12}$
6.3.2.1	8.3.1	6.4.2	10.2	5.4.3	7.3.2	9.2.1	11.1	12

Thus we obtain for the V system:

7.5 6.5.1 8.4 5.4.2.1 7.4.1 9.3 6.3.2.1 8.3.1 6.4.2 10.2 5.4.3 7.3.2 9.2.1 11.1 12

which are all the ways in which 12 can be broken up into unequal parts.*

The U's corresponding to those given by the arithmetical method of effecting correspondence would be:

$7.5\ \ 1.3^2.5\ \ 1^{12}\ \ 1^7.5\ \ 1^5.7\ \ 3.9\ \ 1^3.3^3\ \ 1^9.3\ \ 1^6.3^2\ \ 1^2.5^2\ \ 3.1^4.5\ \ 1^2.3.7\ \ 1^3.3^3\ \ 11.1\ \ 3^4$

* In Note D, *Interact.* Part 2, I show how this transformation can be accomplished by the continually doubling of a string of itself.

instead of

$11.1 \quad 9.3 \quad 9.1^3 \quad 7.5 \quad 7.3.1^2 \quad 7.1^5 \quad 5^2.1^2 \quad 5.3.1^4 \quad 5.3^2.1 \quad 5.1^7 \quad 3^4 \quad 3^3.1^3 \quad 3^2.1^6 \quad 3.1^9 \quad 1^{12}$

so that there is absolutely not a single pair the same in the two methods of conjugation.

(45) The object, however, of instituting the graphical correspondence is not to exhibit this variation, however interesting to contemplate, but to find a correspondence between the two systems which shall resolve itself into correspondences between the classes into which each may be subdivided.

Thus we may call U_i that class of U's in which there are i distinct odd numbers, and V_i that class of V's in which there are i sequences with a gap between each two successive ones: the theorem now to be established is that the V corresponding to any U_i is a V_i, so that class corresponds with class, and as a corollary, that the number of ways in which n can be made up by a series of ascending numbers constituting i distinct sequences is the same as the number of ways in which it can be composed with any i distinct odd numbers each occurring any number of times. This part of the investigation which I will presently enter upon is purely graphical. A few remarks and illustrations may usefully precede.

In the example above worked out it will be observed that there are three classes of U's, viz.

$1^{12} \quad 3^4: \quad 11.1 \quad 9.3 \quad 9.1^3 \quad 7.5 \quad 7.1^5 \quad 5^2.1^2 \quad 3^3.1^3 \quad 3^6.1^6 \quad 3.1^9: \quad 7.3.1^2 \quad 5.3.1^4 \quad 5.3^2.1$

and three class of V's agreeing with those above in the number of partitions in each, viz.

$12 \quad 3.4.5: \quad 11.1 \quad 9.3 \quad 10.2 \quad 8.4 \quad 7.5 \quad 9.2.1 \quad 7.3.2 \quad 6.5.1 \quad 5.4.2.1: \quad 8.3.1 \quad 7.4.1 \quad 6.4.2$

So again for $n = 16$ there will be found to be eleven partitions into odd parts of the third class, which, with their quasi-graphs and corresponding partitions into unequal parts are exhibited below:

```
 11.3.1²        9.5.1²       9.3².1        9.3.1⁴      7.5.1⁴
* * * * * *    * * * * *    * * * * *    * * * * *    * * * *
 * *           * * *        * *           * *          * * *
(*)²          (*)²          * *          (*)⁴         (*)⁴
                            *
  9.6.1         8.5.2.1      8.6.2        10.5.1       9.6.2.1
```

```
 7.3.1⁶     7.3².1³     5².3.1³     5.3³.1²     5.3².1⁵     5.3.1⁸
* * * *    * * * *     * * *       * * *       * * *       * * *
* *        * *         * * *       * *         * *         * *
(*)⁶       * *         * *         * *         * *        (*)⁸
          (*)³        (=)³         * *        (*)⁵
                                  (*)²
 11.4.1     9.5.2       8.4.3.1     8.5.3       10.4.2      12.3.1
```

The transformed partitions above written are all of them of the third class (*i. e.* consist of three distinct sequences) and comprise all that there exist of that class. 16 will correspond to 1^{16} and 1.3.5.7 to itself. All the other partitions of each of the two systems will be of the second class, and will necessarily have a one-to-one graphical correspondence inasmuch as the entire systems have been proved to have such correspondence.

It is worthy of preliminary remark that the association of the first classes of U's and V's given in the previous section will be identical with the association furnished by the graphical method—but whereas in converting V into U by the antecedent process, the two cases of the sequence being of an odd or even order had to be separately considered, the graphical method is uniform in its operation.

Thus 9 8 7 6 a sequence of an even order will be given graphically by

* * * * * * * *
* * * * * * * *

corresponding to 15^2, and 9 8 7 6 5 a sequence of an odd order will be given graphically by

* * *
* * *
* * *
* * *
* * *
* * *
* * *

corresponding to 5^7, whereas it will be observed that $15^2 = (9+6)^{\frac{4}{2}}$ and $5^7 = 5^{\frac{9+5}{2}}$.

It may be noticed that when the major component is an oblate rectangle it gives rise to a sequence of an even order, and when a quadrate or prolate rectangle to one of an odd order.

I subjoin an example of the algorithm by means of which a given V can be transformed into its corresponding U, taking as a first example $V=10\ 9\ 8\ 5\ 4\ 1$.

The process of finding U is exhibited below:

3	3	5	5	(9)
2	2	3	3	(8)
	4	4	2	(7)
	1	3	3	(6)
	10	8	4	(1)
	9	5	1	(2)
	1	1	1	(3)
	4	4	4	(4)
	7	7	7	(5)

$3^3.5^2.7^3$ will be the U required.

As a second example let $V = 12\ 10\ 9\ 8\ 5\ 4\ 1$; the algorithm will be as shown below:

$$\begin{array}{cccc|c}
 & & & 1 & (9) \\
 & & & 1 & (8) \\
1 & 0 & 0 & 0 & (7) \\
2 & 1 & 1 & 1 & (6) \\
\hline
12 & 9 & 5 & 1 & (1) \\
10 & 8 & 4 & 0 & (2) \\
\hline
1 & 3 & 3 & 0 & (3) \\
8 & 8 & 6 & 4 & (4) \\
15 & 15 & 11 & 7 & (5)
\end{array}$$

1 7 11 15 15 will be the U required. Lines (1) and (2) are the parts of the given V written alternately in the upper and lower line; lines (3) and (6) are obtained by oblique and direct subtraction performed between (1) and (2); line (4) is obtained from (3) by adding the number of terms in (1) to the last term in (3) which gives the last term in (4) and then adding in successively the other terms in (3) each diminished by one unit; (7) is derived from (6) by diminishing each term in the latter by a unit and taking the continued sum of the terms thus diminished; (8) is found by the usual rule of "calling"* from its conjugate (7); and finally (5) and (9) are obtained by subtracting a unit from the doubles of the several terms in (4) and (8).

It thus becomes apparent that the passage back from a V to a U is a much more complicated operation than that of making the passage from a U to a V, so much more so that it would seemingly have been labor in vain to have attacked the problem of transformation by beginning from the V end.

(46) I now proceed to the main business, which is to show that any U containing i distinct odd numbers will, by the method described, be graphically converted into a V containing i distinct sequences.

Let G be any regular graph: H what G becomes when the first column of G is removed; $a, b, c, d \ldots$ the contents of the angles of G, H taken in succession.

Also let i be the number of lines of unequal content in G, j the number of distinct sequences in $a, b, c, d, e, \ldots$

The two first lines of G, say L, L', and also the two first columns, say K, K', may be equal or unequal.†

---

*I borrow this term from the vernacular of the American Stock Exchange.

† For brevity I use line and column to signify the extent of (*i. e.* the number of nodes in) either.

If $L=L'$ and $K=K'$, $a-1=b$, $b-1=c$
If $L=L'$ and $K>K'$, $a-1=b$, $b-1>c$
If $L>L'$ and $K=K'$, $a-1>b$, $b-1=c$
If $L>L'$ and $K>K'$, $a-1>b$, $b-1>c$

Let G', H' represent what G, H become on removing the first bend, *i. e.* the first line and first column, and let i', j' be the values of i, j for G', H', so that j' is the number of sequences in $c, d, \epsilon, \ldots$

It is obvious from what precedes that in the four cases considered $j'=j$, $j'=j-1$, $j'=j-1$, $j'=j-2$ respectively. But in these four cases $i'=i$, $i'=i-1$, $i'=i-1$, $i'=i-2$ respectively.

Hence on each supposition $i-j=i'-j'$, and continuing the process by removing each bend in succession, $i-j$ must for any number of bends have the same values as it has for *one* bend; but in that case if h and k are the contents of the line and column of the bend, the reading of the corresponding G, G' will be $h+k-1$, $h-1$, so that for that case j will be 1 or 2 according as h and k are not or are both greater than 1, *i. e.* according as i is 1 or 2.*

Hence $i-j$ is always equal to zero, consequently a U of the i^{th} class will be transformed by the graphical process into a V of the i^{th} class, as was to be proved.

(47) I have previously noticed that the simplest case of $i=j=1$ leads to the formula

$$\frac{q}{1-q}+\frac{q^3}{1-q^3}+\frac{q^5}{1-q^5}+\frac{q^7}{1-q^7}\ldots=\frac{q}{1-q}+\frac{q^5}{1-q^2}+\frac{q^6}{1-q^3}+\frac{q^{10}}{1-q^4}\ldots$$

which is a sort of pendant to Jacobi's formula

$$\frac{q}{1+q}-\frac{q^3}{1+q^3}+\frac{q^5}{1+q^5}-\frac{q^7}{1+q^7}\ldots=\frac{q}{1+q}-\frac{q^3}{1+q^2}+\frac{q^6}{1+q^3}-\frac{q^{10}}{1+q^4}\ldots\dagger$$

These formulæ may be derived from one another or both obtained simultaneously as follows: From addition of the left-hand sides of the two equations there results the double of

$$\frac{q}{1-q^2}+\frac{q^6}{1-q^6}+\frac{q^5}{1-q^{10}}+\frac{q^{14}}{1-q^{14}}\ldots \text{ or of} \sum_{i=1}^{i=\infty}\left(\frac{q^{4i-8}}{1-q^{8i-6}}+\frac{q^{8i-2}}{1-q^{8i-2}}\right)$$

and from addition of the right-hand sides of the same there results the double of

$$\frac{q}{1-q^2}+\frac{q^5}{1-q^4}+\frac{q^6}{1-q^6}+\frac{q^{14}}{1-q^8}\ldots \text{ or of} \sum_{i=1}^{i=\infty}\left(\frac{q^{i(2i-1)}}{1-q^{4i-2}}+\frac{q^{i(2i+3)}}{1-q^{4i}}\right)$$

* The final graph after denudation pushed as far as it will go must be either a single bend, a column, a line or a single node. In the first case $i=2$, $j=2$, in each of the remaining three cases $i=1$, $j=1$.

† My formula is what Jacobi's becomes when every middle *minus* sign in it is changed into *plus* and every inferior *plus* sign into *minus*.

Consequently in order by the operation of addition of the two equations to deduce one from the other we must be able to show that these expressions are identical: observing then that $4i-3$ and $8i-2$ are odd and even respectively for all values of i, but $i(2i-1)$ and $i(2i+3)$ odd or even, according as for i, $2i-1$ or $2i$ be written, it has to be shown that

$$(A)\quad \sum_1^\infty \frac{q^{4i-3}}{1-q^{8i-6}} = \sum_1^\infty \left(\frac{q^{2i-1.4i-3}}{1-q^{8i-6}} + \frac{q^{2i-1.4i+1}}{1-q^{8i-4}}\right)$$

$$\text{and } (B)\quad \sum_1^\infty \frac{q^{8i-2}}{1-q^{8i-2}} = \sum_1^\infty \left(\frac{q^{i(8i-2)}}{1-q^{8i-2}} - \frac{q^{i(8i+6)}}{1-q^{8i}}\right)$$

(A) is equivalent to $\displaystyle\sum_1^\infty q_{4i-3}\frac{1-q^{i-1.8i-6}}{1-q^{8i-6}} = \sum_1^\infty \frac{q^{2i-1.4i+1}}{1-q^{8i-4}}$

or $\displaystyle\sum_1^\infty q_{4i+1}\frac{1-q^{i(8i+2)}}{1-q^{8i+2}} = \sum_1^\infty \frac{q^{2i-1.4i+1}}{1-q^{8i-4}}.$

Hence if i signify any number from 1 to ∞ and k signify any number from 0 to $i-1$, it has to be shown that $(4i+1)(2k+1)$ contains the same integers and each taken the same number of times as $(2m-1)(4m+1+4n)$, where m is any number from 1 to ∞ and n is any number from 0 to ∞. But the $(4i+1)$ $.(2k+1)$ is the same as $(2k+1)(4\overline{k+l+1}+1)$ where k and l each extend from 0 to ∞, and the $(2m-1)(4m+4n+1)$ is the same as $(2m+1)$ $.(\overline{4m+n+1}+1)$ where m and n each extend from 0 to ∞, and the two latter expressions on writing $k=m$, $l=n$ become identical.

Again (B) is equivalent to $\displaystyle\sum_1^\infty q^{8i-2}\frac{1-q^{i-1.8i-2}}{1-q^{8i-2}} = \sum_1^\infty \frac{q^{i(8i+6)}}{1-q^{8i}}$. Hence we have to show that $(8i-2)(1+j)$ when $i=2, 3, \ldots \infty$ and $j=0, 1, 2, \ldots;$ $(i-2)$, or say $(8i+6)(1+j)$, where $i=1, 2, \ldots \infty$ and $j=0, 1, 2, \ldots (i-1)$ is identical with $l(8l+6+8m)$, where $l=1, 2, \ldots \infty$ and $m=0, 1, 2, \ldots \infty$, the former of these is identical with $(1+j)(8\overline{j+k+1}+6)$, where $j=0, 1, \ldots \infty$; $k=0, 1, \ldots \infty$, and the latter is identical with $(1+l)(8\overline{l+m+1}+6)$, where $l=0, 1, \ldots \infty$; $n=0, 1, \ldots \infty$, consequently the two expressions are coextensive, which proves (B), and (A) has been already proved. Hence we see that either of the two original equations can be deduced from the other from the fact that their sum leads to an identity.

In like manner subtraction performed between the two allied equations leads to the fissiparous equation

$$\sum_0^\infty \left\{ \frac{x^{8i+2}}{1-x^{8i+2}} + \frac{x^{4i+3}}{1-x^{8i+6}} \right\} = \sum_0^\infty \left\{ \frac{x^{(i+2)(2i+1)}}{1-x^{4i+2}} + \frac{x^{i+1.2i+3}}{1-x^{4i+4}} \right\}$$

which gives birth to the pair

$$\sum_0^\infty \frac{x^{4i+3}}{1-x^{8i+6}} = \sum_0^\infty \left\{ \frac{x^{2i+3.4i+3}}{1-x^{8i+6}} + \frac{x^{2i+1.4i+3}}{1-x^{3i+4}} \right\} \ldots (C)$$

and

$$\sum_0^\infty \frac{x^{8i+2}}{1-x^{8i+2}} = \sum_0^\infty \left\{ \frac{x^{2i+2.4i+1}}{1-x^{8i+2}} + \frac{x^{2i+2.4i+5}}{1-x^{4i+8}} \right\} \ldots (D)$$

(C) is equivalent to $\displaystyle\sum_0^\infty \frac{x^{4i+3}(1-x^{i+1.8i+6})}{1-x^{8i+6}} = \sum_0^\infty \frac{x^{2i+1.4i+3}}{1-x^{8i+4}}$

which is an identity by virtue of the equivalence of
$(4i+3)\{1+2(j<\overline{i+1})\}$ *i. e.* $(4j+4k+3)(1+2j)$ to $(2\lambda+1)(4\lambda+3+4\mu)$ where j, k, λ, μ each extend from zero to infinity, and

(D) is equivalent to $\displaystyle\sum_0^\infty \frac{x^{8i+2}(1-x^{i(8i+2)})}{1-x^{3i+2}} = \sum_0^\infty \frac{x^{2i+2.4i+5}}{1-x^{3i+8}}$

which is an identity by virtue of the equivalence of
$(8i+2)\{1+(j<i)\}$ *i. e.* $(8\overline{j+k+1}+2)(1+j)$ to $(2\lambda+2)(4\lambda+5+4\mu)$
each symbol j, k, μ having as before the same range, viz. from zero to infinity. Thus then the difference of the two allied equations (as previously their sum) is reduced to an identity which establishes the validity of each of them.

INTERACT, PART 2.

With notes of many a wandering bout,
Of linkèd sweetness long drawn out.
L'Allegro.

(48) D. *Transformation of Partitions by the Cord Rule.*—The figures below are designed to show how it is possible by means of the continuous doubling of a string upon itself to pass from an arrangement of groups of repetitions of r distinct odd integers to the corresponding one with like sum, made up of r distinct sequences. Each of the two figures duplicated by relation about its upper horizontal boundary of nodes through two right angles will represent an

arrangement of repeated odd numbers, the parts being represented by the contents of the *vertical* lines in the figures so duplicated.

Fig. 1.

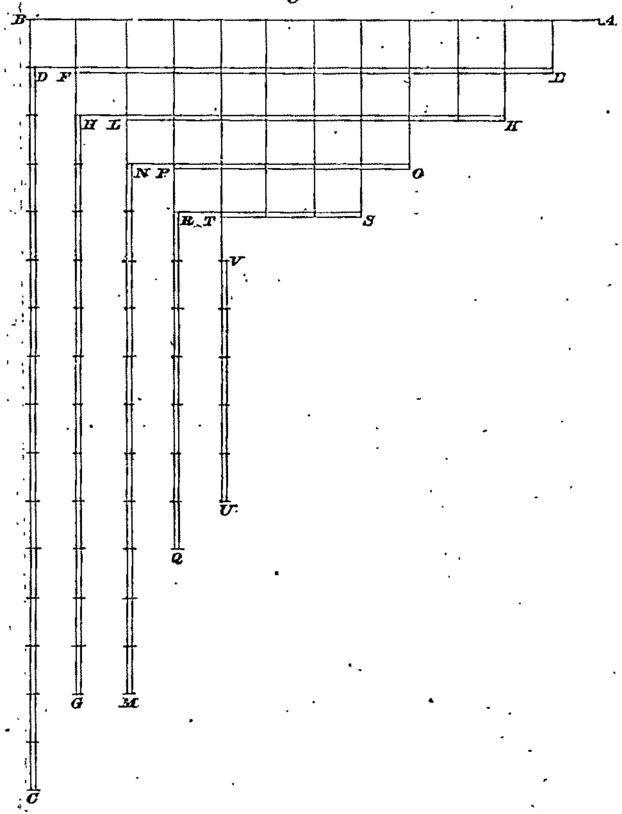
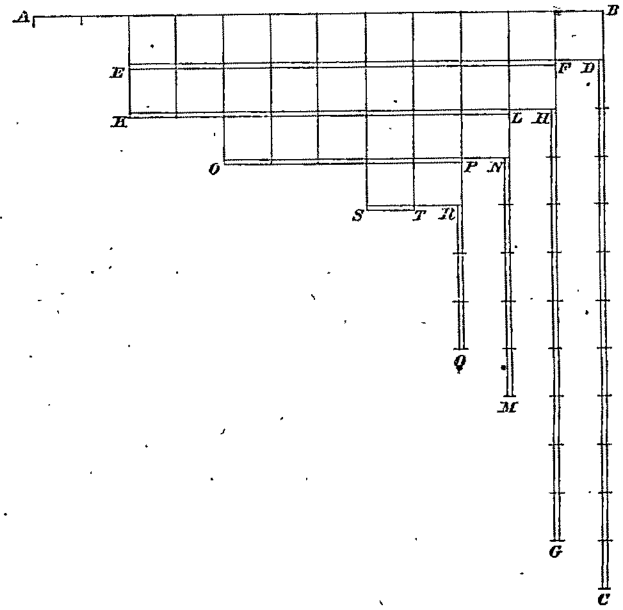


Fig. 2.



The first duplicated figure represents the arrangement 33, 29^2, 23, 21, 9^3, 7, 5^2, 3, 1 whose sum is 183; its correspondent will be the contents of the lengths of * ABC, CDE, EFG, GHK, KLM, MNO, OPQ, QRS, STU, UV, viz. the arrangement 29, 27, 24 (22, $21_,$), 18, 14, 12, 10, 6 which is the same number 183 partitioned into (ten parts but) nine sequences: the second duplicated figure represents the arrangement 25, 23, 17, 15, 9^2, 7^3, 5^2, 1^2 whose sum is 130; its correspondent is represented by the lengths of

ABC, CDE, DEF, FGH, HKL, LMN, NOP, PQR, RST, TU,

which is the same number 130 partitioned into the (nine parts but) eight sequences 25, 22 (20, $19_,$), 15, 12, 10, 6, 1.

E. *On Graphical Dissection.*

(49) It may be not unworthy of notice that there is a sort of potential anticipation of Mr. Durfee's dissection of a symmetrical graph, in a method which, whether

* A line containing i units of length represents $(i+1)$ nodes.

it is generally known or not I cannot say, but is substantially identical with Dirichlet's for finding approximately $\sum_1^n \left[\frac{n}{i}\right]$ and other such like series (a bracketed quantity being used to signify that quantity's integer part). Constructing the hyperbola $xy = n$, drawing its ordinates to the abscissas $1, 2, 3, \ldots n$, and in each of them planting nodes to mark the distances $1, 2, 3, \ldots$ from its foot, there results a *symmetrical graph* included between one branch of the curve, its two assymptotes, and lines parallel to and cutting each of them at the distance n from the original. Its content will be the sum in question. The Durfee-square to its will be limited by the square whose side is $[\sqrt{n}]$, and this added to the original area gives twice over the area in which the number of nodes is $\sum_1^{\sqrt{n}} \left[\frac{n}{i}\right]$, and consequently neglecting magnitudes of the order $\sqrt{n}$, $\sum_1^n \left[\frac{n}{i}\right]$ $= 2n \sum_1^{\sqrt{n}} \frac{1}{i} - i^2 = n(\log n + 2C - 1)$ and as a corollary $\sum_1^{\sqrt{n}} \left\{\frac{n}{i} - \left[\frac{n}{i}\right]\right\}$ $= n(C - 2C - 1) = (1 - C)n$, where C is Euler's number .57721, so that $1 - C$ for large values of n will be the average value of the fractional part of n divided by an inferior number. Furthermore a similar graph, but with $xy = 2n$ diminished by the portion contained between a branch of the new curve, one of its assymptotes and two parallel ordinates cutting that assymptote at distances n and $2n$ from the origin (which portion obviously contains $(2n - n)$ *i. e.* n nodes) will represent $\sum_1^n \left[\frac{2n}{i}\right]$; and consequently the sum $\sum_1^n \left\{\left[\frac{2n}{i}\right] - 2\sum\left[\frac{n}{i}\right]\right\}$, *i. e.* (see *Berl. Abhand.* 1849, p. 75) the number of times that $\frac{n}{i} - \left[\frac{n}{i}\right]$ equals or exceeds $\frac{1}{2}$, as i progresses from 1 to n (within the same limits of precision as previously) $= 2n(\log 2n + 2C - 1) - n$ less $2n(\log n + 2C - 1)$, *i. e.* $=$ $(\log 4 - 1)n$, so that the probability of the fractional part of n divided by an inferior number not falling under $\frac{1}{2}$ is $\log 4 - 1$.*

* What precedes I recall as having been orally communicated to me many years ago by the late ever to be regretted Prof. Henry Smith, so untimely snatched away when in the very zenith of his powers, and so to say, in the hour of victory, at the moment when his intellectual eminence was just beginning to be appreciated at its true value, by the outside world. I was under the impression until lately that he was quoting literally from Dirichlet when so communicating with me, but as the geometrical presentation given in the text is not to be found in the memoir cited from the *Berlin Transactions*, I infer that it originated with himself. In comparing Mertens' memoir, *Crelle*, 1874, with Dirichlet's (1849), upon which it is a decided step in advance, one cannot fail to be struck with surprise that the point to

(50) F. *Mr. Ely's method of finding the assymptotic value of the number of a very large given numerator which are nearer to the integer below than to the integer above.**

"Let a number n be divided by all the numbers from 1 to n; then a value is required for the number of residues which are equal to or greater than $\frac{1}{2}$. An example will make evident a method by which we may obtain limits to the value sought. If n be 100 the residues $=> \frac{1}{2}$ are

(1)	$\frac{49}{51}$	$\frac{48}{52}$	$\frac{47}{53}$	$\frac{46}{54}$	$\frac{45}{55}$	$\frac{44}{56}$	$\frac{43}{57}$	$\frac{42}{58}$	$\frac{41}{59}$	$\frac{40}{60}$	$\frac{39}{61}$	$\frac{38}{62}$	$\frac{37}{63}$	$\frac{36}{64}$	$\frac{35}{65}$	$\frac{34}{66}$
(2)	$\frac{32}{34}$	$\frac{30}{35}$	$\frac{28}{36}$	$\frac{26}{37}$	$\frac{24}{38}$	$\frac{22}{39}$	$\frac{20}{40}$									
(3)	$\frac{22}{26}$	$\frac{19}{27}$	$\frac{16}{28}$													
(4)	$\frac{16}{21}$	$\frac{12}{22}$														
(5)	$\frac{15}{17}$	$\frac{10}{18}$														
(6)	$\frac{10}{15}$															
(a)	$\frac{4}{6}$	$\frac{4}{8}$	$\frac{9}{13}$													

In which it will be observed that the residues $=> \frac{1}{2}$ occur in batches. Let X be the whole number, and x_i the number in batch i. In batch i the numerators decrease by i and the denominators increase by 1. (Those marked (a) of which the denominators are less than $\sqrt{200}$ are left out of account for the present.)

which the closer drawing of the limits to the values of certain transcendental arithmetical functions achieved by the former is owing, should have escaped the notice of so profound and keen an intellect as Dirichlet's, and those who came after him in the following quarter of a century. The point I refer to is the almost self-evident fact that if in the cases under consideration $\Sigma\phi(Fi.x) = \psi x$ then $\phi x = \Sigma\mu(i)\psi(Fi.x)$ where $\mu(i)$ means 0, if i contains any repeated prime factors, but otherwise 1 or $\bar{1}$ according as the number of prime factors in i is even or odd. Dirichlet works with a function given implicitly by an equation, Mertens with the same function expressed in a series, wherein exclusively lies the secret of his success.

*It is proper to state that what follows in the text was handed into me by Mr. Ely on the morning after I had proposed to my class to think of some "common sense method" to explain the somewhat startling fact brought to light by Dirichlet, of more than $\frac{3}{5}$ of the residues of n in regard to $i = 1, 2, 3, \ldots n$ being less than $\frac{1}{2}$. Mr. Ely's method shows at once, in a very common sense manner, why the proportion must be considerably greater than the half, inasmuch as whilst the terms in the first few harmonic ranges are approximately $\frac{n}{1.2}, \frac{n}{2.3}, \frac{n}{3.4}$, etc. in number, the number of them which employed as denominators to n give fractional parts greater than $\frac{1}{2}$, instead of being the halves of these are only $\frac{n}{2.3}, \frac{n}{3.5}, \frac{n}{4.7}$, etc. The mean value in both methods to quantities of the order of $\sqrt{n}$ *inclusive*, turns out to be the same, whichever method is employed, but the margin of unascertained error by the use of Mr. Ely's method (as compared with Dirichlet's) is reduced in the proportion of $1 : 1 + \sqrt{2}$, *i. e.* nearly $2 : 5$.

It is evident for the general case we have approximately

$$\frac{\left[\frac{n}{i+1}\right]-ix_i}{\left[\frac{n}{i+1}\right]+x_i}=\frac{1}{2}$$

or accurately $\qquad x_i=\left[\frac{n}{(i+1)(2i+1)}\right]$ or $\left[\frac{n}{(1+1)(2i+1)}\right]+1$."*

Mr. Ely is then able to show that by limiting the calculation of x_i to the values of i which do not exceed $[\sqrt{n/2}]$, so that roughly speaking the character of $\sqrt{2n}$ of the remainders is left undetermined (and no account taken of them in finding the value of X), and giving to x_i its approximate value $\frac{n}{(i+1)(2i+1)}$ and then extending the series $\frac{n}{2.3}+\frac{n}{3.5}+\frac{n}{4.7}$ beyond the $[\sqrt{n/2}]^{\text{th}}$ term where it ought to stop, to infinity, the errors arising from each of these three sources† and therefore their combined effect will be of the order $\sqrt{n}$, so that the assymptotic value of X will be $\left(\frac{1}{2.3}+\frac{1}{3.5}+\frac{1}{4.7}+\ldots\right)n$, which is $(2\log 2-1)\,n$, with an uncertainty of the order $\sqrt{n}$, as was to be shown.

(51) It may be seen that Mr. Ely's method consists in distributing the n numbers from n to 1 into termed *harmonic ranges* and determining what portion of the several ranges employed as denominators to n give fractional parts, greater or less than $\frac{1}{2}$. It may assist in forming a more vivid idea of this kind of distribution, if the reader takes a definite case, say of $n=121$, the first (10) harmonic ranges will then comprise all the numbers from 121 to 12 inclusive, and the remaining 111 harmonic ranges will comprise the remaining 11 numbers from 11 to 1; that is to say 11 of them will contain a single number, and the remaining 100 ranges be vacant of content.

So again if $n=20$ the first four ranges will contain all the numbers from 20 to 5 inclusive; the 5th, 6th, 9th and 20th range will consist of the sole

*I find by an exact calculation that if R is the remainder of n in regard to $(i+1)(2i+1)$ and $R=\lambda(i+1)+\mu$, where $\lambda<2i+1$ and $\mu<i+1$, then for $\lambda=2\theta-1$ or 2θ, $x_i=\left[\frac{n}{(i+1)(2i+1)}\right]+1$; if $\mu=i-1$ or $i-2\ldots$ or $i-\theta$, and $x_i=\left[\frac{n}{(i+1)(2i+1)}\right]$ for all other values of μ. Hence it follows that out of $(2i^2+3i+1)$ successive values of n, (i^2+i) and (i^2+2i+1) will be the respective numbers of the cases for which the one or the other of these two values of x_i is employed, so that for larger value of i the chances for the two values are nearly the same, but with a slight preponderance in favor of the smaller value. See p. 303.

†The error from the first cause makes the determination of X too small by an unknown amount, that from the third cause too large by a known amount, and that from the second too large or too small (as it may happen) by an unknown amount.

numbers 4, 3, 2, 1, and the remaining 12 ranges will be vacant. I shall proceed to compare the accuracy of Mr. Ely's method with that of Dirichlet's—for this purpose it will be enough to determine the assymptotic value of the uncertainty and to take no account of quantities of a lower order than $\sqrt{n}$.

Let us then suppose that $\sqrt{kn}$ ranges are preserved, and consequently $\sqrt{\frac{n}{k}}$ fractions left out (k being an arbitrary constant which will eventually be determined so as to make the uncertainty a minimum).

The first cause of error necessitates a correction of which the limits are 0 and $\sqrt{\frac{n}{k}}$; the second cause a correction of which the limits are $\sqrt{kn}$ and $-\sqrt{kn}$; and the third, viz. the overreckoning of $\frac{n}{(j+1)(2j+1)}+\frac{n}{(j+2)(2j+3)}+\ldots$ where $j=\sqrt{kn}$, a correction of which the value is $-\frac{n}{2j}$ or $-\frac{1}{2}\sqrt{\frac{n}{k}}$.

Hence making $(\log 4-1)n=U$, the superior limit of X is $U+\frac{1}{2}\sqrt{\frac{n}{k}}+\sqrt{kn}$, and the inferior limit $U-\frac{1}{2}\sqrt{\frac{n}{k}}\sqrt{kn}$. Consequently $X=U+\rho n^{\frac{1}{2}}$ where $\rho<\sqrt{k}+\frac{1}{2}\sqrt{\frac{1}{k}}$ of which the minimum value is found by making $k=\frac{1}{2}$ so that $\rho<\sqrt{2}$ and the *uncertainty* is $\sqrt{2}.n^{\frac{1}{2}}$. Adopting Mertens' assymptotic value of the *uncertainty* of $\sum_{1}^{n}\left[\frac{n}{i}\right]$, viz. $\sqrt{n}$, and using Dirichlet's formula $\sum_{n}^{1}\left[\frac{2n}{i}\right]-2\sum_{n}^{1}\left[\frac{n}{i}\right]$, X has the same mean value as above, but the uncertainty becomes $(\sqrt{2}+2)n^{\frac{1}{2}}$ which is nearly two and a half times as great as that given by the direct method employed by Mr. Ely.

I use the word *uncertainty*, it will be noticed, in a different sense from *error*; the latter is objective, referring to fact, the former subjective, referring to knowledge. Both methods in the case here presented give the same mean value, and therefore the *error* is the same, but the uncertainty is widely different according to the method made use of. Of two formulæ referring to the same fact one might very well give the smaller error and the other the smaller uncertainty.

I have shown above that for considerable values of i, the average value of x_i is $\frac{n}{(i+1)(2i+1)}+\frac{1}{2}$; if then it may be assumed (and there seems no reason

for suspecting the contrary) that for $i=1, 2, \ldots, \sqrt{2n}$, the mean value of $\frac{n}{i}-\left[\frac{n}{i}\right]$ is $\frac{1}{2}$ U will not only be the mean value of the known limits of X but also the mean value of X itself. The value found for k shows that the most advantageous mode of employing Mr. Ely's method is to make the series $\frac{12}{2.3}+\frac{12}{3.5}+\ldots\frac{n}{(i+1)(2i+1)}+\ldots$ stop at one of the terms which is approximately equal to unity.

(52) It is not without interest to consider the exact law for the extent of a harmonic range of a given denomination, say i: this it is easily seen will be always equal to $\left[\frac{n}{i^2+i}\right]$ or $\left[\frac{n}{i^2+i}\right]+1$.

I shall regard i as given and determine the values of n which correspond to the one or the other of the two formulæ: this will depend not on the absolute value of n but on its remainder in respect to the modulus i^2+i. To fix the ideas, let $i=4$ so that $i^2+i=20$, and let n take in successively all values from to 59 inclusive.

Then corresponding to n equal to

40	44	48	52	56
41	45	49	53	57
42	47	50	54	58
43	46	51	55	59

the fourth range will be

10,9	11,10,9	12,11,10	13,12,11	14,13,12
10,9	11,10	12,11,10	13,12,11	14,13,12
10,9	11,10	12,11	13,12,11	14,13,12
10,9	11,10	12,11	13,12	14,13,12

i. e. in half the terms of the period $\left[\frac{n}{i^2+i}\right]$ and in the other half $\left[\frac{n}{i^2+i}\right]+1$ gives the extent of the range.

So in general, if $n=k(i^2+i)+\lambda i+\mu$, where $\lambda=0, 1, 2, \ldots i$, and $\mu\equiv 0, 1, 2, \ldots (i-1)$, when the remainder of n to modulus (i^2+i) is of the form $\lambda(i^2+i)+(0, 1, 2, \ldots (\lambda-1))$ *i. e.* in $\frac{i^2+i}{2}$ cases the extent of the i^{th} harmonic range to n is $\left[\frac{n}{i^2+i}\right]+1$, and when of the form $\lambda(i^2+i)+(\lambda, \lambda+1, \ldots (i-1))$, *i. e.* in the remaining $\frac{i^2+i}{2}$ cases it is $\left[\frac{n}{i^2+i}\right]$.

As the sum of the harmonic ranges to n is n itself, and $\frac{n}{1.2}+\frac{n}{2.3}+\ldots+\frac{n}{n(n+1)}=n-\frac{n}{n+1}$, it follows that if we separate all the numbers from

1 to n into two classes, say i's and j's, i being any number for which n is of the form $k(i^2+i)+\lambda i+0, 1, 2, \ldots (\lambda-1)$, and j any other number within the prescribed limits, then $\sum_1^n \frac{n}{t} - \sum_1^n \left[\frac{n}{t}\right] =$ number of i's $-\frac{n}{n+1}$, and consequently the number of the i terms has $(1-C)n$ for its assymptotic value.

(53) In like manner the law previously stated in a footnote p. 300 for giving the extent of that portion of the i^{th} range for which $\frac{n}{t}$ contains a fractional part not less than $\frac{1}{2}$ may be verified. Thus let $i=3$ then $(i+1)(2i+1)=28$, let $n=56, 57, \ldots 83$. Then for the values of n

28	32	36	40	44	48	52
29	33	37	41	45	49	53
30	34	38	42	46	50	54
31	35	39	43	47	51	55

the portion of the third range having the required character will contain the numbers

8	9	10	11	12	13	14
8	9	10	11	12	14,13	15,14
8	9	10	12,11	12,11	14,13	15,14
8	10,9	11,10	12,11	13,12	14,13	15,14

so that there are $2(1+2+3)$, *i. e.* 3.4 forms of n out of 7.4 for which the formula $\left[\frac{n}{4.7}\right]+1$ has to be employed, and so in general if R is the residue of n in respect to $(i+1)(2i+1)$, there are i^2+i cases where the formula $\left[\frac{n}{(i+1)(2i+1)}\right]+1$ and $(i+1)^2$ where the formula $\left[\frac{n}{(i+1)(2i+1)}\right]$ has to be employed.

G. *On Farey Series.*

(54) This note is a natural sequel to and has grown out of the two which precede; it has also a collateral affinity with the subject matter of the Acts, inasmuch as a graph affords the most simple mode of viewing and stating the fundamental property of an ordinary Farey series, and any series *ejusdem generis.* For instance, let A, B, C be a reticulation in the form of an equilateral triangle where B is a right angle and n the number of nodes in the base or height of the triangle; if the hypothenuse be made to revolve in the plane of the triangle about (either end say about) A, the triangle formed by joining A with any two consecutive nodes of greatest proximity to the centre of rotation traversed by the rotating line will be equal in area to the minimum triangle which has any three nodes for its apices, *i. e.* its double will be equal to unity. This law of

uniform description of areas (say of *equal areas in equal jerks*) is identical with the characteristic law of an ordinary Farey series which deals with terms whose number is the sum-totient ϖn: but it will also hold good if the triangle be scalene instead of equilateral, which corresponds to Glaisher's extension of a Farey series, to the case where the numerator and denominator of each term has its own separate limit (*Phil. Mag.* 1879), or again, when the rotation takes place about the right angle B as centre, which gives rise to a Farey series of a totally different species, defined by the inequality $ax+by<n$: or again when the hypothenuse is replaced by the quadrant of a circle or ellipse, and in an infinite variety of other cases as *ex gr.* when the graph is contained between a branch of an equilateral hyperbola and the assymptotes, which case corresponds to the subject matter of the theory of Dirichlet (*Berl. Abhand.* 1844) concerning the sum of the number of ways in which all integers up to n can be resolved into the product of two relative primes which is the same thing as the half of the numbers of divisors (containing no repeated prime factors) which enter into the several integers up to n, or as the entire number of solutions in relative primes of the inequality $xy =$ or $< n$. The law of equal description of areas $(pq'-p'q = \pm 1)$, Mr. Glaisher has shown very acutely is an immediate inference (by an obvious induction) from the well-known fact that between a fraction and its two nearest convergents (viz. the one ordinarily so called and that which is obtained by substituting $\delta - 1$ and 1 for the last partial quotient), no other fraction can be interposed whose denominator is not greater than that of the one first named.

From the areal-law obviously follows the equation $\frac{p''}{q''} = \frac{xp'-p}{xq'-q}$ (where $\frac{p}{q}, \frac{p'}{q'}, \frac{p''}{q''}$ are any three consecutive terms of the series), so that in order to construct explicitly such a series from the two first terms, all we have to do is to give to x at each step the highest value it can assume, consistent with the imposed limit or limits. Thus *ex gr.* I have found by this method when the limiting inequality is $x + y =$ or < 15, the series

$$\frac{0}{1}\ \frac{1}{15}\ \frac{1}{14}\ \frac{1}{13}\ \frac{1}{12}\ \frac{1}{11}\ \frac{1}{10}\ \frac{1}{9}\ \frac{1}{8}\ \frac{1}{7}\ \frac{1}{6}\ \frac{2}{11}\ \frac{1}{5}\ \frac{2}{9}\ \frac{1}{4}\ \frac{3}{11}\ \frac{2}{7}\ \frac{3}{10}\ \frac{1}{3}\ \frac{4}{11}\ \frac{3}{8}\ \frac{2}{5}\ \frac{3}{7}\ \frac{4}{9}\ \frac{1}{2}\ \frac{1}{1}\ ^{*}$$

* It is advisable for the purpose of securing generality in reasoning upon Farey series not to omit the initial and final terms $\frac{0}{1}$, $\frac{1}{1}$ which seem generally to have been lost sight of by previous writers on the subject. Even then the series is only half complete, for after $\frac{1}{1}$ should follow the reciprocals of the preceding terms until $\frac{1}{0}$ is reached. Thus a complete *ordinary* Farey series beginning with $\frac{0}{1}$ and ending with $\frac{1}{0}$ consists of two symmetrical branches with $\frac{1}{1}$ as their point of junction, each made up of two symmetrical sub-branches meeting respectively in the terms $\frac{1}{2}$ and $\frac{2}{1}$, and such that the sum of a corresponding pair of fractions on the one side of $\frac{1}{1}$ and of their reciprocals on the other side is equal to unity: whereas in the two complete branches the *product* of each corresponding pair is unity.

and the complements in respect to unity of the several terms which precede $\frac{1}{2}$ taken in reverse order, and again for $xy =$ or < 15 the series (which might be called the Dirichlet-Farey series).

$$\frac{0}{1}\ \frac{1}{15}\ \frac{1}{14}\ \frac{1}{13}\ \frac{1}{12}\ \frac{1}{11}\ \frac{1}{10}\ \frac{1}{9}\ \frac{1}{8}\ \frac{1}{7}\ \frac{1}{6}\ \frac{1}{5}\ \frac{1}{4}\ \frac{2}{7}\ \frac{1}{3}\ \frac{2}{5}\ \frac{1}{2}\ \frac{3}{5}\ \frac{2}{3}\ \frac{3}{4}\ \frac{1}{1}.$$

In general if we agree to understand respectively by the *decernent* and the *secernent* to x, the number of divisors without restriction, and the number of divisors restricted to contain no square number, that go into x, and denote the sum-secernent and sum-decernent of n by Sn and Dn respectively, Dirichlet's mode of looking at the question leads immediately to the equation $\sum_1^n S\frac{n}{i^2} = Dn$. Mertens' equation $\left[Sn = \sum_1^n \mu i D\frac{n}{i^2}\right]$ obtained by a longer and somewhat more difficult process is in point of fact merely that equation *reverted*. On pointing out to Mr. F. Franklin this elegant passage in Dirichlet's memoir, he remarked to me to the effect that it was an example, which might admit of wide generalization, of a concept resembling that inherent in the subject matter of the ordinary Farey series; which excellent and keen-witted observation led me to look into the subject from the point of view herein explained. The present theory diverges from the ordinary one in quite another and more natural direction (I imagine) than that pursued by M. Darboux, whose article on the subject of quasi-Farey series (Bulletin de la Société Mathématique de France, tome 6) I have not been able to obtain sight of, and can only conjecture its purport through the reference made to it in a subsequent article which I *have* been able to procure in the same journal by M. Edouard Lucas.

(55) I prove the persistency of the fundamental property of ordinary Farey series for such series generalized in the manner supposed above, as follows.

Let us use $O.\ F.\ S_i$ to denote an ordinary Farey series for which the limit is i, and $G.\ F.\ S$ a Farey series in which calling the numerator and denominator of any term x, y, $\phi(x, y) <= i$, $\phi(x, y)$ meaning a rational function which increases when either x or y increases. If an $O.\ F.\ S_i$ any two consecutive terms be $\frac{a}{b}, \frac{c}{d}$, and in $O.\ F.\ S_{i+1}$ $\frac{p}{q}$ intervenes between $\frac{a}{b}, \frac{c}{d}$ we know p being greater than b and d, the two nearest convergents to $\frac{p}{q}$ must be contained in $O.\ F.\ S_i$, and consequently must be $\frac{a}{b}, \frac{c}{d}$ themselves, so that $p = a + c$,

$q = b + d$, and as a corollary if $\frac{a}{b}$, $\frac{c}{d}$ be consecutive terms in any *O. F. S*, and $\frac{p}{q}$ be any one of the terms which subsequently intervene between $\frac{a}{b}$, $\frac{c}{d}$, we must have $p =$ or $> a + c$, $q =$ or $> b + d$. In order to fix the ideas let us suppose $\phi(x, y)$ to represent $x + y$, so that $x + y <= n$.

For the values 2, 3, 4, 5, 6, 7, 8, 9 . . . of n, the *G. F. S* will be

$$\frac{0}{1}\ \frac{1}{1};\ \frac{0}{1}\left(\frac{1}{2}\right)\frac{1}{1};\ \frac{0}{1}\left(\frac{1}{3}\right)\frac{1}{2}\ \frac{1}{1};\ \frac{0}{1}\left(\frac{1}{4}\right)\frac{1}{3}\ \frac{1}{2}\left(\frac{2}{3}\right)\frac{1}{1};\ \frac{0}{1}\left(\frac{1}{5}\right)\frac{1}{4}\ \frac{1}{3}\ \frac{1}{2}\ \frac{2}{3}\ \frac{1}{1};$$
$$\frac{0}{1}\left(\frac{1}{6}\right)\frac{1}{5}\ \frac{1}{4}\ \frac{1}{3}\left(\frac{2}{5}\right)\frac{1}{2}\ \frac{2}{3}\left(\frac{3}{4}\right)\frac{1}{0},\ \frac{0}{1}\left(\frac{1}{1}\right)\frac{1}{6}\ \frac{1}{5}\ \frac{1}{4}\ \frac{1}{3}\ \frac{2}{5}\ \frac{1}{2}\left(\frac{3}{8}\right)\frac{2}{3};$$
$$\frac{0}{1}\left(\frac{1}{8}\right)\frac{1}{7}\ \frac{1}{6}\ \frac{1}{5}\ \frac{1}{4}\left(\frac{2}{7}\right)\frac{1}{3}\ \frac{2}{5}\left(\frac{3}{7}\right)\frac{1}{2}\left(\frac{3}{5}\right)\frac{2}{3}\ \frac{3}{4}\left(\frac{4}{5}\right)\frac{1}{3};\ldots$$

where the terms in parenthesis are the new terms which intervene as n increases from any value to the next following integer, and where it will be noticed that if $\frac{p}{q}$ be any such parenthesised fraction lying between $\frac{a}{b}$ and $\frac{c}{d}$, $p = a + c$ and $q = b + d$, just as in the successive form of an *O. F. S.* The theorem to be proved may be made to depend on the following lemma.

If for any given value of n every two consecutive terms in a *G. F. S.* appear as consecutive terms in an *O. F. S.* for the same or any smaller value of n; this will continue to be true for all superior values of n.

The proof is immediate, for let $\frac{a}{b}$, $\frac{c}{d}$ be any two consecutive terms in the *G. F. S*$_j$ which are also consecutive terms in *O. F. S*$_i$ where $i =$ or $< j$. If a term $\frac{p}{q}$ intervene between $\frac{a}{b}$, $\frac{c}{d}$ in *G. F. S*$_{j+1}$, $p =$ or $> a + c$, $q =$ or $> b + d$, by virtue of the lemma. But if $p > a + c$ and $q > b + d$, $\phi(a + c, b + d) < \phi(p, q) < j + 1$, but $\frac{a+c}{b+d}$ is intermediate in value between $\frac{a}{c}$, $\frac{b}{d}$, hence $\frac{a+c}{b+d}$ must have appeared in a *G. F. S*$_{j}$, where $j' < j$, which is contrary to hypothesis.

Hence $\frac{a}{b}$, $\frac{p}{q}$, $\frac{c}{d}$ will have been consecutive terms in some *O. F. S.* and in like manner any two consecutive terms in *G. F. S.* either remain consecutive in *G. F. S*$_{j+1}$ or admit a new term between them which is consecutive to each of them in some *O. F. S.*, so that the supposed relation of it hold good for j is true for all superior values of j, but $\frac{0}{1}$, $\frac{1}{1}$ will in any of the supposed cases be a *G. F. S.*, consequently in all these cases no two terms are consecutive in

any *G. F. S.* which are not so in some *O. F. S.*, and therefore the law of equal description of areas will apply to them equally as to the *O. F. S.*, as was to be proved.

The theory may be extended to *G. F. S.*, defined by several concurrent limiting equations. Thus *ex gr.* Mr. Glaisher has proved this for the case of $x \leqq m$, $y \leqq n$: I have not had time as yet to consider what are the restrictions to which the limiting functions may be subject, but the theorem is obviously an extremely elastic one, and the above proof suffices for all the special cases which I have enumerated.*

(56) I am indebted to Mr. Ely for the following additional example of Farey series, in the enlarged sense, which may interest some of my readers.

Ex. (1). $x + y =$ or < 20

$$\frac{1}{19} \ldots \frac{1}{9}\ \frac{2}{17}\ \frac{1}{8}\ \frac{2}{15}\ \frac{1}{7}\ \frac{2}{13}\ \frac{1}{6}\ \frac{3}{17}\ \frac{2}{11}\ \frac{3}{16}$$
$$\frac{1}{5}\ \frac{3}{14}\ \frac{2}{9}\ \frac{3}{13}\ \frac{1}{4}\ \frac{4}{15}\ \frac{3}{11}\ \frac{2}{7}\ \frac{3}{10}\ \frac{4}{13}\ \frac{1}{3}\ \frac{5}{14}\ \frac{4}{11}\ \frac{3}{8}\ \frac{5}{13}\ \frac{2}{5}$$
$$\frac{5}{12}\ \frac{3}{7}\ \frac{4}{9}\ \frac{5}{11}\ \frac{6}{13}\ \frac{1}{2}\ \frac{7}{13}\ \frac{6}{11}\ \frac{5}{9}\ \frac{1}{7}\ \frac{7}{12}\ \frac{3}{5}\ \frac{5}{8}\ \frac{7}{11}\ \frac{2}{3}\ \frac{7}{10}\ \frac{5}{7}$$
$$\frac{8}{11}\ \frac{3}{4}\ \frac{7}{9}\ \frac{4}{5}\ \frac{9}{11}\ \frac{5}{6}\ \frac{6}{7}\ \frac{7}{8}\ \frac{8}{9}\ \frac{9}{10}.$$

Ex. (2). $x^2 + y =$ or < 20

$$\frac{1}{19} \ldots \frac{1}{9}\ \frac{1}{8}\ \frac{2}{15}\ \frac{1}{7}\ \frac{2}{13}\ \frac{1}{6}\ \frac{2}{11}\ \frac{1}{5}\ \frac{2}{9}\ \frac{1}{4}\ \frac{3}{11}\ \frac{2}{7}\ \frac{3}{10}\ \frac{1}{3}\ \frac{3}{8}\ \frac{2}{5}\ \frac{3}{7}\ \frac{1}{2}\ \frac{2}{3}\ \frac{3}{4}.$$

Ex. (3). $y - \sqrt{x} =$ or < 15

$$\frac{1}{16} \ldots \frac{1}{8}\ \frac{2}{15}\ \frac{1}{7}\ \frac{2}{13}\ \frac{1}{6}\ \frac{2}{11}\ \frac{3}{16}\ \frac{1}{5}\ \frac{3}{14}\ \frac{2}{9}\ \frac{3}{13}\ \frac{4}{17}\ \frac{1}{4}\ \frac{4}{15}\ \frac{3}{11}\ \frac{2}{7}$$
$$\frac{5}{17}\ \frac{3}{10}\ \frac{4}{13}\ \frac{5}{17}\ \frac{1}{3}\ \frac{6}{17}\ \frac{5}{14}\ \frac{4}{11}\ \frac{3}{8}\ \frac{5}{13}\ \frac{2}{5}\ \frac{7}{17}\ \frac{5}{12}\ \frac{3}{7}\ \frac{7}{17}\ \frac{4}{0}\ \frac{5}{11}\ \frac{6}{13}$$
$$\frac{7}{15}\ \frac{8}{17}\ \frac{1}{2}\ \frac{9}{17}\ \frac{8}{15}\ \frac{7}{13}\ \frac{6}{11}\ \frac{5}{9}\ \frac{9}{16}\ \frac{4}{7}\ \frac{7}{12}\ \frac{10}{17}\ \frac{3}{5}\ \frac{11}{18}\ \frac{8}{13}\ \frac{5}{8}\ \frac{7}{11}$$
$$\frac{9}{14}\ \frac{11}{17}\ \frac{2}{3}\ \frac{11}{16}\ \frac{9}{13}\ \frac{7}{10}\ \frac{12}{17}\ \frac{5}{7}\ \frac{13}{18}\ \frac{8}{11}\ \frac{11}{15}\ \frac{3}{4}\ \frac{13}{17}\ \frac{10}{13}\ \frac{7}{9}\ \frac{11}{14}$$
$$\frac{4}{5}\ \frac{13}{16}\ \frac{9}{11}\ \frac{14}{17}\ \frac{5}{6}\ \frac{16}{19}\ \frac{11}{13}\ \frac{6}{7}\ \frac{13}{15}\ \frac{7}{8}\ \frac{15}{17}\ \frac{8}{9}\ \frac{17}{19}\ \frac{9}{10}\ \frac{10}{11}\ \frac{11}{12}\ \frac{12}{13}\ \frac{13}{14}$$
$$\frac{14}{15}\ \frac{15}{16}\ \frac{16}{17}\ \frac{17}{18}\ \frac{18}{19}\ \frac{1}{1}.$$

* Since the above was in type I have discovered the true principle of Farey series, for which see Note H following the Exodion.

EXODION. *On the Correspondence between certain Arrangements of Complex Numbers.*

At which he wondred much and gan enquere
What stately building durst so high extend
Her lofty towres, unto the starry sphere.
Faerie Queene I, x, 56.

(57) Starting from the expansion in a series of $\Theta_1 x$, multiplying in the usual notation both sides of the equation by $(1-q^2)(1-q^4)(1-q^6)\ldots$ and intercalating the factors of this product between those of $(1-qz)(1-q^3z)\ldots(1-qz^{-1})(1-q^3z^{-1})\ldots$ taken in alternate order there results the equation $(1-qz^{-1})(1-qz)(1-q^2)(1-q^3z^{-1})(1-q^3z)(1-q^4)\ldots = \sum\limits_{i=-\infty}^{i=+\infty}(-)^i q^{i^2}z$, and writing q^n in place of q and making $z=\pm q^m$ Jacobi (*Crelle*, Vol. 32, p. 166) derives the identity

$$(1\pm q^{n-m})(1\pm q^{n+m})(1-q^{2n})(1\pm q^{3n-m})(1-q^{4n})\ldots = \sum_{-\alpha}^{+\alpha}(\pm)^i q^{ni^2+mi}.$$

From this equation, using the lower sign and making $n=\frac{3}{2}$, $m=\frac{1}{2}$, he observes, may be deduced Euler's expression in a series for $(1-q)(1-q^2)(1-q^3)\ldots$ and using the *upper* sign and making $n=\frac{1}{2}$, $m=\frac{1}{2}$, another known series "given by Gauss in the first volume of the *Göttingen Commentaries* for the years 1808–11."

It is not without interest, I think, to observe that by making $n=\frac{1}{2}$, $m=\frac{1}{2}+\varepsilon$ (where ε is an infinitesimal), and using the *lower* sign we may immediately deduce Jacobi's own celebrated postscript (so to say) to Euler's equation, viz.

$$(1-q)^3(1-q^2)^3(1-q^3)^3\ldots = \sum_{-\infty}^{+\infty}(-)^i q^{\frac{i^2+i}{2}+i\varepsilon} \div (1-q^{-\varepsilon})$$
$$= 1-3q+5q^3-7q^5\ldots$$

the general term being $\sum\limits_0^\infty (-)^i \left\{\left(q^{\frac{i^2+i}{2}+i\varepsilon} - q^{\frac{i^2+i}{2}-(i-1)\varepsilon}\right) \div \frac{1}{1-q^{-\varepsilon}}\right\}$

which is $$(-)^i(2i+1)q^{\frac{i^2+1}{2}}.$$

(58) It is obvious, that by the same right and within the same limits of legitimacy as the equation involving q, n, m (or if we please to say so in q, m) has been derived from the equation in (q, z) the equation in q, z may be recovered

from the equation in q and m, if this latter can be shown to be true, morphologically interpreted for general values of m. I shall show that regarding m and n as absolutely general symbols, such as $\sqrt{-1}$ or $\sqrt{2}$ or ρ or the quaternion units, or any other heterogeneous or homogeneous units we please, the equation in question which I shall write under the equivalent form

$$(1 \mp q^a)(1 \mp q^b)(1 - q^c)(1 \mp q^{a+c})(1 \mp q^{b+c})(1 \mp q^{2c}) \ldots = \sum_{i=-\infty}^{i=+\infty} (\mp)^i q^{\frac{i^2}{2}c + \frac{i}{2}(a-b)}$$

[where $c = a + b$ and a, b are absolutely general symbols or species of units entirely independent of one another] does hold good as a morphological identity.* Thus interpreted, it amounts to a theorem in complex quantities, dealing with arrangements of three sorts of elements which I shall call C's, B's, A's respectively, meaning by a C any non-negative integer (*i. e.* zero or any positive integer) multiple of c, by a B such multiple augmented by a single b, and by an A such multiple augmented by a single a.

The C's, the B's and the A's in any such arrangement will be regarded as three separate series, the terms in each of which flow from left to right in descending order, *i. e.* the multiples of c which represent totally or with the exception of a single b or a single a, the terms in each such series taken in severalty are to form a continually decreasing series.

The total number of elements and the number of C's will be called the major and minor parameters respectively—the relation to the modulus 2 (*i. e.* the parity or imparity) of either one of them its character: and for brevity, the terms major and minor character will be used to signify the character of the major or minor parameter. The totality of all arrangements whatever of A's, B's, C's in which *no element is repeated*, will constitute the sphere of the investigation, limited only by the absence of what I term the exceptional or isolated arrangements, consisting exclusively of a series of *consecutive* B's ending in b, or of consecutive A's ending in a. Within the prescribed sphere I shall prove that a process may be instituted for transforming any arrangement which shall satisfy the five following conditions:

1° That it shall be capable of acting on every licit and unexceptional arrangement.

* This theorem is less transcendental than Newton s binomial theorem when the same latitude is given to the meaning of the symbols in either case: for $(1+x)^m = 1 + mx + \frac{m^2-m}{2}x^2 + \ldots$ does not admit of *direct* interpretation when m is a general symbol. The passage from numerical proximate equality to absolute identity, prepared but not perfected nor capable of being explained by infinitesimal gradation, brings to mind the analogous transfiguration of sensibility into sensation, or of sensation into consciousness, or of consciousness into thought.

2^0 That it shall transform it into another such arrangement.

3^0 That operating once upon an arrangement and then again upon the operate, it brings back the original arrangement.

4^0 That it leaves the sum of the elements in the arrangement unaltered.

5^0 That it reverses each of its two characters.*

From (3) it will follow that all the arrangements within the prescribed sphere are associated in pairs, and from (1) that the sum of the elements in each such pair is the same. This being so, it is obvious from the fact of the parity of the total number of elements being opposite for any pair of associated arrangements, that in the development in a series of $(1-q^a)(1-q^b)(1-q^c)(1-q^{a+c})\ldots$, no term will appear in which the index of q is other than the sum of the terms in one of the exceptional (we may now call them unconjugated or unconjugable) arrangements, and from the fact of the parity of the number of the C's being opposite in any pair, the same will be true of the development in a series of $(1+q^a)(1+q^b)(1-q^c)(1+q^{a+c})\ldots$

As regards the coefficient in this latter series of any term whose index is the sum of the elements in an unconjugate arrangement it will manifestly be the number of ways in which the same complex number can be thrown under the form of a sum of the arithmetical series

$a,\ a+c, \ldots, a+\overline{i-1}c$ which is $\frac{i^2-1}{2}c+ia$, *i. e.* $\frac{i^2-1}{2}c+\frac{i}{2}(a-b)$, or of $b,\ b+c, \ldots, b+\overline{i-1}c$ which is $\frac{i^2-i}{2}c-\frac{i}{2}(a-b)$.

If $\frac{i^2-i}{2}c+\frac{i}{2}(a-b)=\frac{j^2-j}{2}c+\frac{j}{2}(a-b)$,

then $\frac{i^2}{2}a+\frac{i^2-2i}{2}b=\frac{j^2}{2}a+\frac{j^2-2j}{2}$ which necessitates $i=j$,

and if

$\frac{i^2-i}{2}c+\frac{i}{2}(a-b)=\frac{j^2-j}{2}c-\frac{j}{2}(a-b)$ then $\frac{i^2}{2}a+\frac{i^2-2i}{2}b=\frac{j^2-2j}{2}a+\frac{j^2}{2}b$,

so that $i^2-(i^2-2i)=(j^2-2j)-j^2$ or $i=-j$ and $\frac{i^2}{2}=\frac{i^2+2i}{2}$ or $i=j=0$.

* It will presently be seen that all the licit and unexceptional arrangements will be divided into 3 classes and a specific operator be found for each class capable of acting on each arrangement of that class and converting it into another of the same class, and which will satisfy also the 3d, 4th and 5th of the enumerated conditions. The total operator contemplated in the text may then be regarded as the sum of these specific ones, each of which, within its own sphere, will have to fulfil the five conditions of Catholicity, Homœogenesis, Mutuality, Inertia and Enantiotropy (the last a word used in the school of Heraclitus to signify "the conversion of the primeval being into its opposite"). See Kant's *Critique of Pure Reason* by Max Müller, Vol. I, p. 18.

Hence the general term is $(-)^i q^{\frac{i^2-i}{2}c \pm \frac{i}{2}(a-b)}$, where i is an integer stretching from zero to infinity, and in like manner, and for the same reason, the general term in the former series will be $q^{\frac{i^2-i}{2}c \pm \frac{i}{2}(a-b)}$ with the like interpretation: or which is the same thing, comprising both cases in one and interpreting i to be integer stretching from $-\infty$ to $+\infty$, the general term will be $(\mp)^i q^{\frac{i^2-i}{2}c + \frac{i}{2}(a-b)}$.

(59) The task before us then is to show the *possibility* of instituting, by *actually* instituting, a law of operation which shall satisfy the five preliminary conditions of catholicity, homœogenesis, reciprocity, reversal of characters and conservation of sum.

The following notation will be found greatly to conduce to clearness in effecting the needful separation into classes or species. A capital letter with a point above, as $\dot{X}$, will be used to signify the greatest value, and with a point below, as $\underset{\cdot}{X}$, the least value of any term in a series which that letter is used to denote. $X = 0$, $X > 0$, $X + Y = 0$, $X + Y > 0$ will signify respectively that there are no X's, that there are X's, that there are no X's and no Y's, that there are either X's or Y's or both in any arrangement under consideration. B's will be separated into $'B$ and B''s, or as we may write it $B = 'BB'$, where $'B$ is the general name for all the B's, which beginning with the highest term $\dot{B}$ form an arithmetical series of which c is the common difference. If there is a gap of more than one c between $\dot{B}$ and the next lowest B, $'B$ is of course the single term $\dot{B}$: B' is any B which is not a $'B$.

So again, A_1 is any A which belongs to a series of A's forming an arithmetical series whose constant difference is c and lowest term a, so that unless $\underset{\cdot}{A} = a$, $A_1 = 0$: any other A will be designated by $_1A$. The signs of accent and point may of course be separate or combined: thus *ex gr.* $\underset{\cdot}{C}$ will mean the smallest C in any given arrangement, $\dot{B}$ will mean the greatest B, $\underset{\cdot}{A}$ will mean the lowest A, $_1\underset{\cdot}{A}$ will mean the lowest of the $_1A$'s and $\dot{A}_1$ the highest of the A_1's. Every $'B$ is necessarily greater than any B', and every $_1A$ than any A_1. If $'\underset{\cdot}{B} - b = 0$, this will indicate that all the B''s will form a consecutive series of terms (*i. e.* having a constant difference c) and ending in b, so that here $B' = 0$, *i. e.* there are no B's except those that belong to the regular arithmetical progression ending in b. If $_1A = 0$, all the A's will form an arithmetical progression ending in a. Thus we see that the arrangements belonging to the 1st terms (those that I have called exceptional) will consist of two species denoted respectively by

${}_1A+B+C=0$ and $('\underset{\bullet}{B}-b)+A+C=0$. It may sometimes be found convenient to use a point to the left centre of a quantitative letter to signify that the quantity denoted is to be increased, and a point to the right centre to signify that the quantity denoted is to be diminished, by c. Thus $\dot{B}\cdot$ will mean $\dot{B}-c$, and $\cdot\underset{\bullet}{A}_1$ will mean $\underset{\bullet}{A}_1+c$, the first signifying the greatest B diminished by and the second the smallest A_1 increased by c. When any general letter, say X, is wanting as indicated by the equation $X=0$, $\underset{\bullet}{X}$ must be understood to mean zero. So for instance if $A=0$, and consequently ${}_1A=0$ and $A_1=0$, ${}_1\underset{\bullet}{A}=0$. Again, when there is a gap between the highest B and the one that follows it in any arrangement, the arithmetical progression of $'B$'s reduces as above remarked to a single term and there results $'\dot{B}='\underset{\bullet}{B}$. It may be noticed also that always $'\dot{B}=\dot{B}$, and $\underset{\bullet}{A}_1=\underset{\bullet}{A}$.

The arrangements which are comprised under the forms

$$(\alpha)\quad A,\ A-c,\ A-2c,\ \ldots,\ a$$
$$(\beta)\quad B,\ B-c,\ B-2c,\ \ldots,\ b$$

may be regarded as belonging to what I shall term the first genus.

The second genus, viz. that consisting of unexceptional combinations of unrepeated A's, B's, C's, may then be divided into the following three species, the conditions by which they are severally distinguished being attached to each in its proper place.

1st Species. Conditions (γ) $'\underset{\bullet}{B}-b>0$,

or (γ') $'\underset{\bullet}{B}-b=0$, $C>0$, $\underset{\bullet}{C}-c<='\dot{B}-b$.

2d Species. (δ) $'\underset{\bullet}{B}-b=0$, $A+C>0$, $C=0$ or $\underset{\bullet}{C}-c>'\dot{B}-b$.

or (δ') $B=0$, $C>0$, $A=0$, or ${}_1\underset{\bullet}{A}-a=>C$.

3d Species. (ε) $B=0$, $A>0$, ${}_1A+C>0$, $C=0$, or $\underset{\bullet}{C}>{}_1\underset{\bullet}{A}-a$.

Where it is to be understood that the conditions set out in the same line are simultaneous conditions. Thus *ex gr.* the conditions of an arrangement being of the second species are when all the conditions of the upper or else all the conditions of the lower of the two lines written under that species are fulfilled: the conditions of the upper line (be it noticed) are that $'\underset{\bullet}{B}$ is b, and that there are either some A's or some C's, and that if there are some C's, $\underset{\bullet}{C}-c>'\dot{B}-b$, and of the lower line, that there are no B's and some C's, and that if there are A's, $\underset{\bullet}{A}-a=>\underset{\bullet}{C}$, and so for the interpretation of the conditions of the existence of each of the other two species.

To these (7) systems of conditions α, β, γ, γ', δ, δ', ε, may be joined the trivial system (ω) $A=0$, $B=0$, $C=0$;* the (8) systems thus constituted will easily be seen to be mutually exclusive and between them to comprehend the entire sphere of possibility, leaving no space vacant to be occupied by any other hypothesis. I will now proceed to assign the operators ϕ, ψ, ϑ appropriate to the three species of the second genus.

Office of the Operator ϕ. $\phi = {}'\phi + \phi'$.

When in Genus 2, Species 1, $C=0$ or $\underset{\cdot}{C} - c > {}'\dot{B} - {}'\underset{\cdot}{B}$, ${}'\phi$ is to be performed, meaning that for each ${}'B$, ${}'B\cdot$ is to be substituted, and the inertia kept constant by forming a new $\underset{\cdot}{C}$ with the sum of the c's thus abstracted. In the contrary case ϕ' is to be performed, meaning that $\underset{\cdot}{C}$ is to be resolved into simple c's and as many of the ${}'B$'s, commencing with ${}'\dot{B}$ and taken in regular order to be converted into ${}^{\cdot\prime}B$ as are required to maintain the inertia constant, *i. e.* c is to be added to each B in succession, until all the c's which together make up $\underset{\cdot}{C}$ are absorbed.

Office of the Operator ψ. $\psi = {}'\psi + \psi'$.

When in Genus 2, Species 2, $C=0$ or $\underset{\cdot}{C} > {}'\dot{B} + \underset{\cdot}{A}$, ${}'\psi$ is to be performed, meaning that for ${}'\dot{B}$ and A their sum is to be substituted, producing a C [which, on the second hypothesis, will be a new $\underset{\cdot}{C}$]. In the contrary case ψ' is to be performed, meaning that for $\underset{\cdot}{C}$ is to be substituted ${}^{\cdot\prime}\dot{B}$ (which will form a new ${}'\dot{B}$) and $\underset{\cdot}{C} - {}^{\cdot\prime}\dot{B}$ which will form a new $\underset{\cdot}{A}_1$.

Office of the Operator ϑ. $\vartheta = {}'\vartheta + \vartheta'$.

When $C > 0$ and $\underset{\cdot}{C} + \dot{A}_1 < {}_1\underset{\cdot}{A}$, ${}'\vartheta$ is to be performed, meaning that for $\underset{\cdot}{C}$ and $\dot{A}_1$ their sum is to be substituted, producing a new ${}_1\underset{\cdot}{A}$. In the contrary case ϑ' is to be performed, meaning that for ${}_1\underset{\cdot}{A}$, $\cdot\dot{A}_1$ forming a new $\dot{A}_1$ and ${}_1\underset{\cdot}{A} - \dot{A}_1$ forming a new C are to be substituted.

(60) It will be seen that every species of the second genus consists of two contrary sub-species having opposite characters, and it will presently appear that any arrangement belonging to one of these sub-species under the effect of its appropriate operator passes over into the other, which operated upon in its turn by its appropriate operator becomes identical with the original one, so that any two contrary sub-species may be said to be of equal extent: in fact if the sum of

*It would be perfectly logical, and indeed is necessary to regard the trivial case as belonging to the cases of exception, and then we might say that there are two genera, each containing three species, those of the first genius solitary, and those of the second, each of them comprising two sub-species, namely the sub-species subject to the action of the left-accented and that subject to the operation of the right-accented operators. The *trivial* species of the first genus consists of a single individual.

the parts is supposed to be given there will be as many arrangements in any sub-species as in its opposite, for each one will be conjugated with some one of the others.

It may not be amiss to call attention here to the fact that the scheme of classification adopted is, in a certain sense, artificial. Thus, for instance, it proceeds upon an arbitrary choice between which shall be regarded as the A and which as the B series, so that by an interchange of these letters a totally different correspondence would be brought about between the arrangements of the second genus, those of the first genus remaining unaltered. Nor is there any reason for supposing that these are the only two correspondences capable of being instituted between the arrangements of the second genus—in particular there is great reason to suspect that a symmetrical mode of procedure might be adopted, remaining unaffected by the interchange between A and B. As a simple example of the effect of interchange, applying the method here given, suppose $A=0$, $B=0$, a case belonging to the second species and that sub-species thereof to which ψ' is applicable, and imagine further that the C series is monomial. Then C will be associated according to the scheme here given with b, $C-b$, but in the correlative scheme it would be associated with a, $C-a$.

(61) I need hardly say that so highly organized a scheme, although for the sake of brevity presented in a synthetical form, has not issued from the mind of its composer in a single gush, but is the result of an analytical process of continued residuation or successive heaping of exception upon exception in a manner dictated at each point in its development by the nature of the process and the resistance, so to say, of its subject-matter. The initial step (that applicable to species γ) is akin to the procedure applied by Mr. F. Franklin to the pentagonal-number theorem of Euler, of which I shall have more to say presently. It will facilitate the comprehension of the scheme to take as an example the particular case where a and b represent actual and real quantities, say, to fix the ideas, $b=1$, $a=2$. Nothing, it will be noticed, turns upon the fact of this specialization, which is adopted solely for the purpose of greater concision and to afford more ready insight into the *modus operandi.*

To illustrate the classes and laws of transformation consider (with $b=1$, $a=2$,* $c=a+b=3$) all the arrangements the sums of whose parts is 12, viz. 12, 11.1, 10.2, 9.2.1, 8.4, 8.3, 1.7.5, 7.4.1, 7.3.2, 6.5.1, 6.4.2, 5.4.3, 5.4.2.1.

*No use it will be seen is made of the *accidental* relation $a=b+b$.

One of these, 7.4.1, belongs to the exceptional genus. The rest will be conjugated and fall into species in the manner shown below, where the first species means where the conditions (γ) or (γ'), the second that where (δ) or (δ'), and the third where the conditions (ε) are satisfied. The C's, B's, A's are now numbers whose residues are 0, 1 or 2 in respect to the modulus 3. For greater clearness in each arrangement, numbers belonging to the same series are kept together, the law of descent only applying in this theory to elements belonging to the same series.

Species 1. 10.2 3.7.2; 4.8 3.1.8; 7.5; 3.4.5; 6.4.2 6.3.1.2; 5.7 3.2.7.

Species 2. 9.12 9.3; 6.15 4.1.5.2;

Species 3. Caret.

Or again let the collection of arrangements be one in which the sum is 18. The partitions of 18 are 18 17.1 16.2 15.3 15.2.1 14.4 14.3.1 13.5 13.4.1 13.3.2 12.6 12.5.1 12.4.2 12.3.2.1 11.7 11.6.1 11.5.2 11.4.3 11.4.2.1 10.8 10.7.1 10.6.2 10.5.3 10.5.2.1 10.4.3.1 9.8.1 9.7.2 9.6.3 9.6.2.1 9.5.4.1 9.5.3.1 9.4.3.2 8.7.3 8.7.2.1 8.6.4 8.6.3.1 8.5.4.1 8.5.3.2 8.4.3.2.1 7.6.5 7.6.4.1 7.6.3.2 7.5.4.2 7.5.3.2.1 7.4.3 6.5.4.3 6.5.4.2.1. In this case there are no exceptional arrangements.

1st Species. 16.2 3.13.2; 4.14 3.1.14; 13.5 3.10.5; 13.4.1 3.10.4.1; 7.11 3.4.11; 10.8 3.7.8; 12.4.2 12.3.1.2; 10.7.1 6.7.4.1; 6.10.2 6.3.7.2; 10.1.5.2 3.7.1.5.2; 9.4.5 9.3.1.5; 6.7.5 6.3.4.5; 7.1.8.2 3.4.1.8.2; 6.4.8 6.3.1.8; 7.4.5.2 6.4.1.5.2;

2d Species. 18 17.1; 15.3 15.1.2; 12.6 12.5.1; 6.1.11 4.1.11.2; 9.1.8 4.1.8.5; 9.7.2 9.3.4.2; 9.6.3 9.6.1.2; 11.5.2 3.8.5.2.

3d Species. Caret.

If the partible number is 11 of which the partitions are 11 10.1 9.2 8.3 8.2.1 7.4 7.3.1 6.5 6.4.1 6.3.2 5.4.2 5.3.2.1 there will be no exceptional arrangements and the pairs of unexceptional ones will be as below.

1st Species. 10.1 3.7.1; 7.4 6.4.1; 4.5.2 3.1.5.2.

2d Species. 3.8 1.8.2.

3d Species. 11 9.2; 6.5 6.3.2.

By interchanging a and b, *i. e.* making $a=1$, $b=2$, the correspondence changes into the following:

1st Species. 11, 3.8; 6.3.2, 6.5; 8.2.1, 3.5.2.1; 7.4, 6.4.1.

2d Species. Caret.

3d Species. 10.1, 6.4.1; 7.4, 3.7.1.

According to Mr. Franklin's process the correspondence takes a form quite distinct from either of the above, viz. 11, 10.1; 9.2, 8.2.1; 8.3, 7.3.1; 7.4, 6.4.1; 6.5, 5.4.2; 6.3.2, 5.3.2.1, all these arrangements constituting one single species.

A careful study of the preceding examples will sufficiently explain to the reader the ground of the divisions into species with their appropriate rules of transformation, and might almost supersede the necessity of a formal proof of the operator supplying the conditions of catholicity, homoeogenesis and mutuality; from their very definition they are seen to comply with the other two essential conditions of inertia and enantiotropy.

Signifying by Ω the total operator $\phi+\psi+\theta$, it has been already remarked that Ω will in the general case have two values which only come together when $a=b$, or which is the same thing each of them is 1; a special case of the special case when the complex reduce to simple numbers, viz. it is the case indicated in the well-known equation

$$(1-q)^2(1-q^3)^2(1-q^5)^2\ldots=\frac{1}{(1-q^2)(1-q^4)\ldots}\sum_{i=-\infty}^{i=\infty}q^{i^2}.$$

But besides the two correspondences given by the two values of Ω, if we take the actual (no longer a diagrammatic case) $b=2$, $a=1$, we revert to Euler's theorem concerning the partitions of all pentagonal and non-pentagonal numbers, and can obtain by Dr. Franklin's process, given in Art. (12) a totally different distribution into genera and species, viz. the first genus instead of containing arrangements of the species $1,\ 4,\ 7,\ \ldots\ 3i-2;\ 2,\ 5,\ 8,\ \ldots\ 3i-1$ will, as previously shown, consist of the very different arrangements (giving the same infinite series of numbers as those for other sums) $i,\ i+1,\ i+2,\ \ldots\ 2i-1;\ i+1,\ i+2,\ i+3\ \ldots;\ 2i$.

The character of each arrangement in the new solution depends in part on the relation to the modulus 2 of the whole number of parts and of the number of parts which are divisible by 3, so that we may divide the conjugate arrangements into four groups* designated respectively by *Oo*, *Oe*; *Eo*, *Ee*, using the capital letters to signify the oddness or evenness of the whole set of parts, and the small letters the same for the parts divisible by 3. There will thus be a cross classification of the arrangements of the second genus into groups over and above that into species, each species in fact consisting of four groups, which

* It will be seen later on that there is a division into sixteen groups analogous to the division into four groups first noticed by Prof. Cayley arising under the Franklin process.

may be denoted as above, and of which Oo and Ee are one associative couple, and Oe, Eo the other.*

(62) The following elegant investigation has been handed into me by Arthur S. Hathway, fellow and one of my hearers at the Johns Hopkins University, to which, although it does not exactly strike at the object of the constructive theory here expounded, I gladly give hospitality in these pages.

"The theorem to be proved is as follows:

$$1+\varepsilon x^a.1+\varepsilon x^{a+h}.1+\varepsilon x^{a+2h}\dots$$
$$\times\, 1+\varepsilon x^b.1+\varepsilon x^{b+h}.1+\varepsilon x^{b+2h}\dots$$
$$\times\, 1-x^h.1-x^{2h}.1-x^{3h}\dots=\sum_{\delta=-\infty}^{\delta=+\infty}\varepsilon^\delta.x^{\frac{a+b}{2}\delta^2+\frac{a-b}{2}\delta},$$

where $\varepsilon^2=1$ and $h=a+b$, a and b being any quantities whatever.

The general term contains, say i exponents of x selected from the first line, j from the second line and k from the third line, viz.

$$a+\alpha_0 h,\ \dots\ a+\alpha_{i-1}h,$$
$$b+\beta_0 h,\ \dots\ b+\beta_{j-1}h,$$
$$\gamma_1 h,\ \dots\ \gamma_k h,$$

where α_0, α_{i-1}, $\beta_0\dots\beta_{i-1}$, $\gamma_1\dots\gamma_k$ are respectively sets of i, j, k unequal integers arranged in ascending order, none representing a less integer than its subscript. This term is (remembering that $h=a+b$)

$$\varepsilon^{i+j}(-)^k x^{ma+nb},\ \text{where}$$

$$m=[(\alpha_0+1)+\dots(\alpha_{i-1}+1)]+[\beta_0\dots+\beta_{j-1}]+[\gamma_1+\dots\gamma_k]\dots \quad (1)$$
$$n=[\alpha_0+\dots\alpha_{i-1}]+[(\beta_0+1)\dots+(\beta_{j-1}+1)]+[\gamma_1+\dots\gamma_k]\dots \quad (2)$$

In addition to these we obtain by subtraction

$$m-n=i-j\equiv i+j \bmod 2\dots \quad (3)$$

Whence (since $\varepsilon^2=1$) $\varepsilon^{i+j}=\varepsilon^{m-n}$.

Thus all the above general terms having the same m and the same n divide themselves into positive and negative groups (corresponding to even and odd values of k), a term from one group cancelling a term from the other group. I

*The Oe and Eo conjugation has a very striking analogue in nature (as I am informed) in the existence of dissimilar hermaphrodite characters in two sorts of the wild English *primrose* and the American flower *Spring-beauty* or *Quaker-lady*—it being the law of nature that only those of different sorts can fertilize one another. Possibly the double symbolic character of Oo and Ee will justify or suggest the inquiry whether there may not be a latent duality in the unisexual specimens of such flowers as those just mentioned, where male and female are found codomiciled with the bisexual florets. There is also, it seems, a trace of analogy to the sparsely distributed unconjugate individuals of my first genus in Darwin's "complemental males."

"propose to prove that the number of terms in each of these groups are equal, except when a certain relation exists between m and n

$$\left(\text{viz. } m-\frac{(m-n)(m-n+1)}{2}=0 \text{ or } m=n=0\right),$$

corresponding to which there is but one general term having the same m and the same n which falls into the positive group ($k=0$). This establishes the theorem in question, as we see by putting $m-n=\delta$.

It is sufficient to consider (1) in connection with (3). In the first place the first two partitions in (1) may be converted by a (1:1) correspondence into an indefinite partition (bearing in mind (3)) with a decrease ($m-n>0$) in the sum or content of the integers by $\frac{(m-n)(m-n+1)}{2}$, as follows: extend α_0+1 in a horizontal line of dots and *under* the first dot extend β_0 in a vertical line of dots, thus forming an *elbow*; in a similar manner form elbows out of α_1+1, β_1 &c. until one of the partitions is exhausted [this will be according to (3), the first or the second, according as $m<$ or $>n$, leaving in the inexhausted partition $m-n$ integers]; place these elbows successively one without the other and place on top [$m-n>0$] horizontal lines of dots corresponding to the successive unmatched integers decreased respectively by $0, 1, \ldots (n-m-1)$ or $1, 2, \ldots (m-n)$, according as $m<$ or $>n$ [in either case the total decrease being $\frac{(m-n)(m-n+1)}{2}$]. In other words, the above tri-partition of m has a (1:1) correspondence with a bi-partition of

$$m-\frac{(m-n)(m-n+1)}{2} \text{ [or } m \text{ if } m=n],$$

consisting of an indefinite partition on one side and a partition of unrepeated integers on the other $(\gamma_1, \ldots \gamma_k)$. Such a bi-partition (on removing the line of demarkation) is an indefinite partition; and, conversely, every indefinite partition involving θ different integers gives rise as follows: to $(1+1)^\theta$ such bi-partitions, the number of those involving even and odd values of k being respectively the positive and negative parts of the expansion of $(1-1)^\theta$ [which are equal]: viz. first, the indefinite partition itself ($k=0$); second, the θ bi-partitions obtained by placing each of the θ integers successively on the k side ($k=1$); third, the $\frac{\theta(\theta-1)}{2}$ bi-partitions obtained by placing the $\frac{\theta(\theta-1)}{2}$ pairs of the θ integers successively on the k side ($k=2$), and so on. The only exception to this equality of the number of partitions for even and odd values of k is when the

"partible number $\left[m - \frac{(m-n)(m-n+1)}{2} \text{ or } m\right]$ is zero, for which case there is but one bi-partition $[0] + [0]$ $(k=0)$. Q. E. D. The tri-partition of m corresponding to the celibate case reduces to the natural sequence above subtracted whose content is $\frac{(m-n)(m-n+1)}{2}$ [or 0], which is the second or the first partition (according as $m <$ or $> n$), the others being wanting."

(63) The same infinitesimal method which applied to the expansion of $\Theta_1 x$ gives both as was shown to the expression for the cubes of the successive rational binomial functions may be applied to the development of $(1+ax)(1+ax^2)$ $(1+ax^3)\ldots$ given in Art. (35), but will not lead to any new result. Making $a = -x^{-1-\varepsilon}$, where ε is infinitesimal we obtain from the general theorem

$$(1-x^\varepsilon)(1-x)(1-x^2)(1-x^3)\ldots$$

$$= 1 - \frac{1-x^\varepsilon}{1-x}x + \frac{1-x^\varepsilon . 1-x}{1-x\ 1-x^2}x^5 - \frac{1-x^\varepsilon . 1-x . 1-x^2}{1-x . 1-x^2 . 1-x^3}x^{12}\ldots$$

$$-x^\varepsilon + \frac{1-x^\varepsilon}{1-x}x^3 - \frac{1-x^\varepsilon . 1-x}{1-x . 1-x^2}x^9\ldots$$

$$\text{or } (1-x)(1-x^2)(1-x^3) = 1 - \frac{x+x^3}{1-x} + \frac{x^5+x^9}{1-x^2}\ldots$$

$$= 1 - x(1-x) + x^5(1-x^2)\ldots$$

the same equation as results from writing $a = -1$.

To arrive at any new result it would be necessary to have recourse to processes of differentiation; the above calculation serves, however, as a verification if any were needed of the accuracy of theorem to which it refers.

(64) Since sending what precedes to press I have thought it would be desirable in the interest of sound logic to set out the marks or conditions of the several species of the arrangements of unrepeated A, B, C's, somewhat more fully and explicitly than before. And first, I may observe that since it has been convenient to understand that when there are no X terms $\underset{\cdot}{X}$ shall signify zero, the quantitative equation $\underset{\cdot}{X} = 0$ dispenses with the necessity of using the symbolical one $X = 0$, and in like manner $\underset{\cdot}{X} > 0$ supersedes the symbolical inequality $X > 0$, and, of course, the same remark extends to the equality or inequality $X + Y$ $=$ or > 0.

We have then for what I shall term the first, second and third species of genus 1, the conditions $\underset{\cdot}{C} + \underset{\cdot}{B} + \underset{\cdot}{A} = 0$, $\underset{\cdot}{C} + {}'\underset{\cdot}{B} + \underset{\cdot}{A} = b$, $\underset{\cdot}{C} + \underset{\cdot}{B} + \underset{\cdot}{A}_1 = 0$ respectively—the first, the trivial case of vacuous content; the second, of only a complete natural B, progression *i. e.* one ending with b, (the minimum value of

B) and the third, the same for A similarly ending with the minimum a. In what follows the conditions in each separate *line* are to be understood to be not disjunctive but simultaneous or accumulative; they of course refer to the species of the second genus.

Marks of species (1) (α) $\underset{\cdot}{B} - b > 0$ or
(β) $\underset{\cdot}{B} - b = 0,\ {}'\underset{\cdot}{B} - {}'\underset{\cdot}{B} = > \underset{\cdot}{C} - c,\ \underset{\cdot}{C} > 0.$

" " " (2) (α) $\underset{\cdot}{B} - b = 0,\ \underset{\cdot}{C} - c > {}'\dot{B} - {}'\underset{\cdot}{B}$ or
(β) $\underset{\cdot}{B} - b = 0,\ \underset{\cdot}{C} = 0\ [\underset{\cdot}{A} > 0]$ or
(γ) $\underset{\cdot}{B} = 0,\ \underset{\cdot}{A} - a = > \underset{\cdot}{C},\ \underset{\cdot}{C} > 0$, or
(δ) $\underset{\cdot}{B} = 0,\ \underset{\cdot}{A} = 0\ [\underset{\cdot}{C} > 0].$

" " " (3) (α) $\underset{\cdot}{B} = 0,\ \underset{\cdot}{C} > \underset{\cdot}{A} - a,\ \underset{\cdot}{A} > 0$ or
(β) $\underset{\cdot}{B} = 0,\ \underset{\cdot}{C} = 0,\ [{}_1\underset{\cdot}{A} - a > 0].$

The three inequalities included in brackets are only required in order to exclude arrangements belonging to the first genus. Leaving these out of account for the moment, merely for the sake of greater concision of statement, it is easy to see by mere inspection of the above table that the three species are mutually exclusive and share between them the total sphere of possibility, for (1) α exhausts the hypothesis of there being other B's besides those forming a complete natural progression, (1) β and (2) α of the B's forming such progression when there are existent C's, and (2) β when there are not. Also $((2)\gamma,\ 2(\delta)),\ 3(\alpha)$ exhaust between them the hypothesis of there being no B's when there are some existent C's, and (3) β of neither B's nor C's appearing in an arrangement.

Thus all unexceptional arrangements must bear the marks occurring in one or the other of the first four lines of the table, and all those where no B's occur either of the last line when there are neither B's nor C's, and of the three preceding ones when there are no B's but some C's, and the total sum of these hypotheses plus the hypothesis of the first genus together make up necessity, as was to be shown.

The convention $\underset{\cdot}{X} = 0$ when an arrangement contains no X with the consequent reduction of the conditions to a purely quantitative form has lent itself very advantageously to the above bird's eye view of the completeness of the scheme (as covering the whole ground of possibility); it also will be found to simplify the expression of the proof. I did not employ it until the necessity for so doing forced itself upon my notice, for a very obvious reason, viz. that X is a B (or an A) which is defined to be congruous to b (or a) [mod c], which zero is

not: there is thus an apparent paralogism in admitting that any X of these two *where there is a* B (or when there is an A) is congruent to b (or to a), but that *when there is no* B (or no A) then the conventional least B (or A) is zero. It will be seen, however, *ex post facto* that no inconvenience in working the scheme results from this extended definition which constitutes an important gain to the perfect evolution of the method. It is usually in the form of some apparent contradiction or paradox that a scientific advance makes its first appearance.

(65) Aided by this clearer and fuller expression of the definitions of the genera and species, I will now set out a logical proof that the respective operators fulfill the three additional necessary conditions. I may observe preliminarily that the Greek letterings α, β; α, β, γ, δ; α, β, do not express sub-species, for one distinguishing mark of species (or sub-species) may be taken to be that conjugation cannot take place except between individuals of the same species or sub-species, but it will be presently seen that individuals belonging to the differently lettered divisions of the above species are susceptible of mutual conjugation—and are therefore in conformity with biological precedent to be regarded as mere varieties. Besides these varieties of each of the species there is another entirely different principle of cross classification applicable to each of them, viz. in general an arrangement must belong to one of sixteen groups designated by combining together one out of each of the four pairs of opposite symbols X, C; x, c; O, E; o, e, where the large O, E refer to the oddness or evenness of the major, and the small o, e to the same for the minor parameter; and in like manner the large X and large C to the result of the operation appropriate to any arrangement, being to extend or contract the major, and x, c to extend or contract the minor parameter. There are thus eight pairs of groups, and conjugation can only take place between individuals belonging to the same pair.

The pairs are as follows:

$$\binom{XxOo}{Cc\,Ee},\ \binom{XxOe}{Cc\,Eo},\ \binom{XxEo}{Cc\,Oe},\ \binom{XxEe}{CcOo};$$

$$\text{and}\ \binom{XcOo}{CxEe},\ \binom{XcOe}{CxEe},\ \binom{XcEo}{CxOe},\ \binom{XcEe}{CxOo}.$$

Species (1) and species (3) it will be seen may each be separately divided into four sub-species denoted by the upper four, and species (2) into the four sub-species denoted by the lower four pairs of combined characters, so that there will be in all twelve (and not as might at first be supposed twenty-four) sub-species of conjugable arrangements. The different sub-species of the same

species do not admit of cross-conjugation; it is the property which they have in common of being subject to the same law of transformation when passage is made from an individual to its conjugate, which binds them together into a single species. In the arrangements peculiar to Euler's problem, we see that there was no division of the second genus at the outset, but that a separation would be made of it into two pairs of groups with conjugation possible only between individuals belonging to the same pair, and consequently there may be said in this case to be two species of the second genus, analogous, however, not to the species but the sub-species in the more general theory. The final separation of a pair of groups into its component elements has nothing to do with the concept of species, sub-species or variety, but may be regarded as similar to the separation of the sexes.

In what follows, a bracket enclosing a letter will be used to denote that it belongs to an arrangement after it has been operated upon by its appropriate operator, or what may be called its operate.

Species (1). When $\underset{\cdot}{B}-b>0$, if $\underset{\cdot}{C}-c>{}'\dot{B}-{}'\underset{\cdot}{B}$ or $\underset{\cdot}{C}=0$, ${}'\phi$ may be performed, giving $[\underset{\cdot}{C}]={}'\underset{\cdot}{B}-{}'B+C<\underset{\cdot}{C}$ so that the law of descending magnitude is maintained; we have then $[\dot{B}]-['\underset{\cdot}{B}]=$ or $>\dot{B}-{}'\underset{\cdot}{B}=>[\underset{\cdot}{C}]-c$; hence ϕ' has to be performed and will obviously restore the original arrangement. Again if in the original arrangement $\dot{B}-{}'\underset{\cdot}{B}=>\underset{\cdot}{C}-c$ and $\underset{\cdot}{C}>0$, ϕ' has to be applied; a resolution of $\underset{\cdot}{C}$ can take place into c's and the $\underset{\cdot}{C}/c$ first ${}^1B$'s, and will each be increased by c and $[\dot{B}]-{}'[\underset{\cdot}{B}]=\underset{\cdot}{C}-c$, so that either $[\underset{\cdot}{C}]=0$ or $[\underset{\cdot}{C}]-c<\underset{\cdot}{C}-c<[\dot{B}]-{}'[\underset{\cdot}{B}]$, and ${}'\phi$ being applicable to the new arrangement will convert it back to the original one.

First Species (β). When $\underset{\cdot}{B}-b=0$ and $\dot{B}-{}'\underset{\cdot}{B}=>\underset{\cdot}{C}-c$ and $\underset{\cdot}{C}>0$, ϕ' can be performed, and the new arrangement as before may be operated upon by ϕ' and so brought back to its original value. If $\underset{\cdot}{C}=0$ or $\underset{\cdot}{C}-c>\dot{B}-{}'\underset{\cdot}{B}$, ${}'\phi$ could not be performed, for then $\underset{\cdot}{B}=b$ and has no c to part with to help make up $[\underset{\cdot}{C}]$.

These two hypotheses belong to Species (2), which we will now proceed to consider throughout its full extent. When $\underset{\cdot}{B}-b=0$, then ${}'\underset{\cdot}{B}=b$, and I shall first suppose [(α) and (β)] that $\underset{\cdot}{C}+0$ or $\underset{\cdot}{C}-c>B-b$. When $\underset{\cdot}{C}=0$ or $\dot{B}+\underset{\cdot}{A}>\underset{\cdot}{C}$, then ${}'\psi$ will be applicable, making $[\underset{\cdot}{C}]=\dot{B}+\underset{\cdot}{A}$; if now $[\underset{\cdot}{B}]>0$ and $[\underset{\cdot}{A}]>0$, $[\underset{\cdot}{B}]+[\underset{\cdot}{A}]=>(\dot{B}-c)+(\underset{\cdot}{A}+c)=>\dot{B}+\underset{\cdot}{A}=>[\underset{\cdot}{C}]$, and $[\underset{\cdot}{C}]-c=\dot{B}+\underset{\cdot}{A}-c=[\underset{\cdot}{B}]+\underset{\cdot}{A}>[\underset{\cdot}{B}]-b$.

Hence we are still within Species 2 and have fallen upon the case to which the reversing operative ψ' has to be applied. If $[\underset{.}{B}]=0$, $[\underset{.}{A}]=0$ we must have $B[\underset{.}{C}]>0$, inasmuch as the original content (or inertia) is originally greater than zero and is kept constant, and this is a case which still belongs to Species 2 and falls under the operation of ψ'.

If $[\underset{.}{B}]=0$ so that $\dot{B}=\underset{.}{B}=b$ and $[\underset{.}{A}]>0$, then $[\underset{.}{A}]-a=>\underset{.}{A}+c-a =>\underset{.}{A}+\dot{B}=>\underset{.}{C}$, which also falls within the second species and is amenable to the reversing operator ψ'.

Finally, if $[\underset{.}{B}]>0$, *i. e.* $\underset{.}{B}-b=0$ and $[\underset{.}{A}]=0$, $[\underset{.}{C}]-c=\dot{B}+\underset{.}{A}-c =>[\dot{B}]-b$, *i. e.* $=>[\dot{B}]-{}'\underset{.}{B}$, and we are still within Species (2) and in the case amenable to the reversing operator ψ'.

If now on the other hand we begin with an arrangement of the second species in the case amenable to ψ' we must suppose either $\underset{.}{B}=0$ or $\underset{.}{A}=0$, or or else $\underset{.}{C}>0$ and $\underset{.}{C}<=\dot{B}+A$.

Take first this last supposition. The operation of ψ' gives $[\underset{.}{C}]=>\underset{.}{C}+c$, $[\dot{B}]=\dot{B}+c$ and $[\underset{.}{A}]=\dot{C}-c-\dot{B}>\dot{B}-b-\dot{B}>-b=>c-b=>a$

And $[\dot{B}]+[\underset{.}{A}]=\dot{B}+\underset{.}{C}-\dot{B}=\underset{.}{C}<[\underset{.}{C}]$.

$$[\underset{.}{C}]-c=>(\underset{.}{C}-c)+c=>B-b+c=>[B]-[\underset{.}{B}].$$

Hence the operate is licit, belongs to the second species and is amenable to the reversing operator $'\psi$.

If $\underset{.}{B}=0$ and $\underset{.}{A}=0$, $[\dot{B}]=[\underset{.}{B}]=b$ and $[\underset{.}{A}]=\underset{.}{C}-b$ and $[\underset{.}{C}]=0$ or $>\underset{.}{C}$.

If $[\underset{.}{C}]=0$ since $[\underset{.}{A}]>0$, the operate is included in variety (β) of the second species and amenable to the reversing operator $'\psi$, and if $[\underset{.}{C}]>\underset{.}{C}$ $[\underset{.}{C}-c]>\underset{.}{C}-c>0$, *i. e.* $>[\dot{B}]-\underset{.}{B}$ which belongs to variety (α) of the second species; and since $[\underset{.}{C}]>\underset{.}{C}>[\dot{B}]+[\underset{.}{A}]$ is amenable to the reversing operator $'\psi$.

If $\underset{.}{B}>0$ and $\underset{.}{A}=0$, then $\underset{.}{C}>0$ [otherwise it would be an arrangement in Genus 1, Species 2] $[\underset{.}{C}]=0$ or $>\underset{.}{C}$, $[\dot{B}]=\dot{B}+c$, $[\underset{.}{A}]=\underset{.}{C}-[\dot{B}]> (c+\dot{B}-b)-(c+\dot{B})=>a$, and either $[\underset{.}{C}]=0$ and $[\underset{.}{A}]>0$ or $[\underset{.}{C}]-c> (\underset{.}{C}-c)+c>\dot{B}+c-b>[\dot{B}]-{}'\underset{.}{B}$ and $[\underset{.}{A}]+[\dot{B}]=\underset{.}{C}>[\underset{.}{C}]$. Hence in either hypothesis the operate is still in Species (2) and amenable to the reversing operator $'\psi$.

Lastly, if $\underset{.}{B}=0$, $\underset{.}{A}-a=>\underset{.}{C}$ and $\underset{.}{C}>0$, the arrangement is amenable to the operator ψ', which will make $[\underset{.}{B}]=b$, $[\underset{.}{A}]=\underset{.}{C}-b<\underset{.}{C}+a<\underset{.}{A}$. We have then $[\underset{.}{B}]-b=0$ and $[\underset{.}{C}]=0$, and consequently also $\underset{.}{A}>0$ or $[\underset{.}{C}]$

$-c > \underset{.}{C} - c > 0$, *i. e.* $> [\dot{B}] - '[B]$, and the result is still contained within Species (2) and is amenable to the reverse operator $'\psi$.

(66) The following are examples of paired arrangements belonging to the first species, adapted to the case of $a = 2$, $b = 1$. The C and B terms are expressed; the A line is the same for each of any pair of this species, and may be filled in at will.

$$\phi' \left\{\begin{matrix} X.9. \\ 16.13.10.Y \end{matrix}\right\} = \left\{\begin{matrix} X. \\ 19.16.13.Y \end{matrix}\right\}$$

where X, Y represent any licit series of C's and B respectively.

$$'\phi \left\{\begin{matrix} X.9. \\ 16.13.7.Y \end{matrix}\right\} = \left\{\begin{matrix} X.9.6. \\ 13.10.7.Y \end{matrix}\right\} \quad \phi' \left\{\begin{matrix} X.9 \\ 16.13.10.4 \end{matrix}\right\} = \left\{\begin{matrix} X. \\ 19.16.13.4 \end{matrix}\right\}$$

$$\phi' \left\{\begin{matrix} X.9 \\ 7.4.1 \end{matrix}\right\} = \left\{\begin{matrix} X. \\ 10.7.4 \end{matrix}\right\} \quad '\phi \left\{ 10.7.4 \right\} = \left\{\begin{matrix} 9. \\ 7.4.1 \end{matrix}\right\} \quad \phi' \left\{\begin{matrix} 3. \\ 13.7.4.1 \end{matrix}\right\} = \left\{ 16.7.4.1 \right\}$$

The following are examples of paired arrangements of the second species with $a = 2$ and $b = 1$ as usual.

$$'\psi \left\{\begin{matrix} X.12. \\ 7.4.1. \\ Y.2 \end{matrix}\right\} = \left\{\begin{matrix} X.12.9. \\ 4.1 \\ Y \end{matrix}\right\} \quad \psi' \left\{\begin{matrix} X.12. \\ 7.4.1. \\ Y.5 \end{matrix}\right\} = \left\{\begin{matrix} X \\ 10.7.4.1. \\ Y.5.2 \end{matrix}\right\}$$

$$'\psi \left\{\begin{matrix} 7.4.1. \\ Y.5. \end{matrix}\right\} = \left\{\begin{matrix} 12. \\ 4.1. \\ Y \end{matrix}\right\} \quad \psi' \left\{\begin{matrix} X.15 \\ 7.4.1 \\ Y.8 \end{matrix}\right\} = \left\{\begin{matrix} X. \\ 10.7.4.1 \\ Y.8.5 \end{matrix}\right\} \quad \psi' \left\{\begin{matrix} X.9. \\ \ldots. \\ \ldots. \end{matrix}\right\} = \left\{\begin{matrix} X. \\ 1. \\ 8 \end{matrix}\right\}$$

$$\psi' \left\{\begin{matrix} 6. \\ 1. \\ 8 \end{matrix}\right\} = \left\{\begin{matrix} \ldots \\ 4.1. \\ 8.2 \end{matrix}\right\} \quad \psi' \left\{\begin{matrix} X.9. \\ \ldots \\ Y.11 \end{matrix}\right\} = \left\{\begin{matrix} X. \\ 1. \\ Y.11.8 \end{matrix}\right\}$$

We come now to the third species. Here, I think, the reader will find it a great relief to the strain upon his attention if I invite him before attacking the demonstration to consider the annexed diagrammatic cases accommodated to the supposition $a = 2$, $b = 1$. The B's it will be remembered in this species do not exist, and the action neither of $'\vartheta$ nor ϑ' introduces any B into the transformed arrangement. In the examples given below the C and A terms occupy the higher and lower lines respectively—the comma is used in the latter to mark off the ${}_1A$'s from the A_1's.

$$'\vartheta \left\{\begin{matrix} 9.6. \\ 14.11.8.5, \end{matrix}\right\} = \begin{matrix} 9.6.3. \\ 14.11.8,2 \end{matrix} \quad \vartheta' \left\{\begin{matrix} 6.3. \\ 14.11.8,2 \end{matrix}\right\} = \begin{matrix} 6. \\ 14.11.8.5, \end{matrix} \quad '\vartheta\left(17.8.5\right) = \begin{matrix} 3. \\ 17.8,2 \end{matrix}$$

$$'\vartheta\left(17.8.5,\right) = \begin{matrix} 3. \\ 17.8,2 \end{matrix} \quad '\vartheta\left(17,8.5.2\right) = \begin{matrix} 6. \\ ,11.8.5.2 \end{matrix} \quad '\vartheta\left(17.14.8.5.2\right) = \begin{matrix} 3. \\ 17,11.8.5.2 \end{matrix}$$

$$'\vartheta\, 11, = \begin{matrix} 9. \\ ,2 \end{matrix} \quad \vartheta' \left\{\begin{matrix} 12.9.3. \\ ,11.8.5.2 \end{matrix}\right\} = \begin{matrix} 12.9. \\ 14,8.5.2 \end{matrix} \quad \vartheta' \left\{\begin{matrix} 9.6.3. \\ ,11.8.5.2 \end{matrix}\right\} = \begin{matrix} 9.6 \\ 14,8.5.2 \end{matrix}$$

The left-hand accent is used here as elsewhere to signify that the phase of the operator which brings about an increase and the right-hand one a decrease in the number of C's. It will readily be seen that the action of the operator in each of the above examples prepares the arrangement for the action of the contrary one which will restore it to its original value. It is worthy of notice that in any two associated arrangements above, an a (here 2) *may* appear in each and *must* appear in one of them. I will now proceed to the general demonstration.

(67) Let us first suppose $\underset{\cdot}{A}_1 = 0$, then ${}_1\underset{\cdot}{A} > 0$, otherwise we shall be dealing with the antecedent species and ${}'\vartheta$ will be applicable, making $[\underset{\cdot}{A}] = [\dot{A}_1] = a$ $[\underset{\cdot}{C}] = \underset{\cdot}{A} - a < \underset{\cdot}{C}$ and $> (\underset{\cdot}{A} - a)$. Thus the generated arrangement is licit and belongs still to the third species; but now $[\underset{\cdot}{C}] + [\dot{A}_1] = \underset{\cdot}{A}$ and $[{}_1\underset{\cdot}{A}] = 0 > \underset{\cdot}{A}$. Hence the reversing operator ϑ' is applicable to the new arrangement; the remaining cases to consider (in which $\underset{\cdot}{A} = a$ for the arrangement as well before as after being operated upon) may be separated into those where $\underset{\cdot}{C} > 0$, and at the same time either $\underset{\cdot}{C} + \dot{A}_1 < {}_1\dot{A}$ or ${}_1\dot{A} = 0$, which are amenable to the operation ϑ' and the complementary cases which are amenable to ${}'\vartheta$.

In the cases first considered $[\dot{A}_1] = \dot{A}_1 - c$, $[{}_1\dot{A}] = \underset{\cdot}{C} = \underset{\cdot}{A}_1 \vartheta [\underset{\cdot}{C}] + 0$ or $> \underset{\cdot}{C}$ (and *à fortiori* > 0), consequently the new arrangement is licit and still belongs to the third species, and since either $[\underset{\cdot}{C}] = 0$ or else $[\underset{\cdot}{C}] + [\dot{A}_1] > \underset{\cdot}{C} + \dot{A}_1 - \underset{\cdot}{C} = > [{}_1\dot{A}]$ and $[{}_1\dot{A}] > 0$, it is one of the complementary cases and is subject to the reversing operator ${}'\vartheta$.

Again, any arrangement for which $\underset{\cdot}{A} = a$ belonging to the complementary cases is defined by the conditions ${}_1\dot{A} > 0$ and $\underset{\cdot}{C} + \dot{A}_1 = > {}_1\underset{\cdot}{A}$ and is by hypothesis to be subjected to the operator ${}'\vartheta$ which will make $[\dot{A}_1] = \dot{A}_1 + c$, $[{}_1\underset{\cdot}{A}] = 0$ or $> {}_1\underset{\cdot}{A}$ $[\underset{\cdot}{C}] = {}_1\underset{\cdot}{A} - \dot{A}_1 - c$, and since $\underset{\cdot}{C} = > {}_1\underset{\cdot}{A} - \dot{A}_1$ $[\underset{\cdot}{C}] < \underset{\cdot}{C}$, so that the operation leads to a licit new arrangement.

Also $[\underset{\cdot}{C}] + [\dot{A}_1] = {}_1\underset{\cdot}{A}$, and consequently either $[{}_1\underset{\cdot}{A}] = 0$ or $[\underset{\cdot}{C} + \dot{A}_1] < [{}_1\underset{\cdot}{A}]$, which is a condition belonging to the first considered class of cases, subject to the reversing operator ϑ', and thus for the third as for both the antecedent species of the second genus, it has been proved that each designated operator prior to any arrangement being performed does not take away its licit character nor carry it out of the species to which it belongs, and on being repeated brings it back to its original form, and that the effect of any single operation is to maintain the content (or inertia) of the arrangement constant but to reverse each of its characters. This is the thing that was to be proved and brings my wearisome but indispensable task to an end.

(68) Another and perhaps somewhat clearer image of the classification of the numbers of the second Genus may be presented as follows: The combinations of the characters $XCOExcoe$ give rise to eight pairs of groups, say eight classes. Of these classes four belong to Species 2. and may be represented by four indefinite vertical parallelograms, set side to side, and subdivided each of them into four, (say) black, white, grey, and tawny stripes, corresponding to the four varieties of the second species. The other four classes may be similarly represented by four such parallelograms as before, but separated by a transverse horizontal line into eight sub-classes, four corresponding to the first species and four to the second. The upper parallelograms may then be each divided into blue and green, the lower into yellow and red stripes to represent the respective couples of varieties of the first and third species. There will thus be in all thirty-two stripes, viz. four blue, green, yellow and red, and four black, white, grey and tawny, each of which is bifid, representing two groups of opposite sexual characters, which may be fittingly represented by the upper and under sides of the sixteen unlimited single-colored stripes of the first and the eight unlimited double-colored stripes of the second set of parallelograms.

The above logical scheme is not intended to convey any notion of the relative frequency of the three species. The general case is that of the first species. The second is conditioned by $'\underset{\cdot}{B}=b$ or $\underset{\cdot}{B}=0$, and the third by $\underset{\cdot}{B}=0$. When $'\underset{\cdot}{B}=b$ it is about an even chance whether the arrangement is of the second or first species, and when $\underset{\cdot}{B}=0$ of the second or third. Either equality is a particularization of the B series, the latter signifying that there are no B's in the arrangement, the latter that there are B's descending in rational progression down to b: this supposition is apparently infinitely more general than the former, because there is no limit to the number of terms in the progression, and the case of a natural progression of B's of the kind mentioned with any given number of terms as regards the probability of its occurring in an arrangement seems to be on a par with the case of the B's being all wanting. Hence the first species is infinitely more frequent than the second, and the second than the third. According to Prof. Max Müller's theory of the relation of thought to language (if I interpret it rightly) I ought to have thought out my divisions and schemes of operation in language, but I certainly had formed in my mind a dim abstract of them before I had found the language that was competent to give them expression.

In conclusion, I may remark that whilst the experience of the past indicated the probability that there did exist (if one could find it) a method of distributing the arrangements of the second genus into pairs, in such a way that in each pair the total or partial character should be reversed in passing from the one to the other, there was nothing to induce a reasonable degree of assurance that both those characters should be found simultaneously reversed in one and the same distribution; for aught that could have been foreseen to the contrary, it might very well have happened that one mode of distribution might have been needed to prove Jacobi's theorem for the case of only negative signs appearing in the factors on the left-hand side of the equation, and a different one for the other case where only every third factor contains such sign—indeed upon the principle of *divide et impera* or doing one thing at a time (as invaluable a maxim to the algebraist as to the politician) I had completed the proof for the former case without thinking of the latter, and only when on the point of attacking it was agreeably surprised to find that there was nothing left to be done, for that the proof found for the one extended to the other—in familiar phrase, I had hit two birds with one stone. We may now ask whether this was a happily found chance solution or was predestined by the nature of things, and that *simple* necessarily implies *double* enantritropy of conjugation. Probably I think not, and if so, a question arises as to the number of solutions for each of the two sorts of enantritropy and whether the number of each kind of simply-enantritropic conjugations is the same.

Viewed merely as a question of direct multiplication, I think it must be allowed that what I have here called Jacobi's theorem (including Euler's marvellous one, as the ocean a drop of water) is the most surprising revelation that has been made in elementary algebra since the discovery of the general binomial theorem, and that the space devoted to its independent, and so to say, materialistic proof in these pages, although considerable, is not out of proportion to its intrinsic importance.

*Note H. Intuitional Exegesis of Generalized Farey Series.**

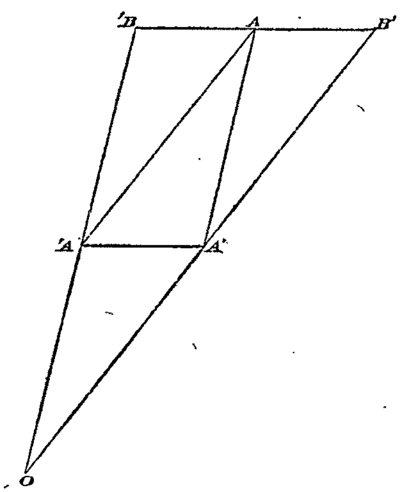
(69) The demands of the press will only admit of a rapid sketch of what appears to me to be the true underlying principles of the theory initiated by Farey, honored by the notice of Cauchy, and to a certain extent generalized by Mr. Glaisher, whose inductive method in the cases treated by him finds its full development in the method of continuous change of boundary, explained in the

* Continued from note *G*, Interact, Part 2.

course of what follows. Let us start from the conception of an infinite cross-grating formed by two orthogonal systems of parallel lines in a plane, the distance between any two parallels being made equal to unity. The intersections of any two lines of the grating may, as heretofore, be termed nodes. A triangle which has nodes at its apices and at no other point on or within its periphery, may be termed an elementary triangle, and the double of the area of any such triangle will be unity. If any finite aggregate of nodes be given it must be possible to pick out a certain number of them which may be formed together by right lines so as to form a sort of ring-fence, within which all the rest are included: the area thus formed, if it admits of being mapped out into elementary triangles, may be termed a *complete* nodal aggregate. Any other contour consisting of lines of any form (curved or straight) drawn outside of this ring-fence in such a manner that no nodes occur between the two, may be termed a regular contour.

If any node O be taken as origin and any nodal lines through O as axes of coordinates, and if $'A$, A' are the nearest nodes to O in the radial lines on which they lie, and if no nodes of the given aggregate are passed over as an indefinite line rotating round O, passes from one of these radial lines to the other, $'AOA$ is an elementary triangle, and if $'p$, $'q$; p, q be the coordinates of $'A$, A respectively, $'pq - p'q = \varepsilon$ where ε is $+1$ or -1 but is fixed in sign when the direction of the rotation is given.

When the aggregate is *complete*, if the values of the coordinates of the successive points passed over by the rotating line be called $\ldots {}''p$, $''q$; $'p$, $'q$; p, q; p', q', p'', q''; $\ldots$, we shall have a Farey series formed by the successive couples p, q, *i. e.* $p''\,'q - p'q'' = \varepsilon$; $p'q - pq' = \varepsilon$; $pq' - p'q = \varepsilon \ldots$ Thus we see that the Farey property is invariantive in the sense of being independent of the position of the origin.



Next I say, that if any contour to a given aggregate is regular, every contour similar thereto in respect to any node of the aggregate regarded as the center of similitude is also regular, provided the boundary is simple; meaning that there are no interior limiting lines giving rise to holes or perforations in aggregate, and no loops formed by the boundary cutting itself.

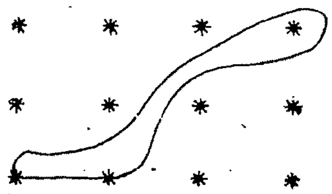
In the above figure $'BOB'$ is any triangle whose sides are bisected in $'A$, A, A'. Suppose O to be the origin $'A$, A', two nodes of greatest proximity to O successively passed over by the rotating line for a given contour. As this contour expands uniformly in all directions through O, the line $'AA'$ remains parallel to itself. Since $'AOA'$ is an elementary triangle so also must the similar triangles $'AAA'$, $A'AB'$ $'AA'B$ be all elementary, consequently A will be the first new node intervening between $'A$, A' brought into the enlarged aggregate as $'AA'$ moves continuously parallel to itself, and $'AOA$, AOA' will be elementary triangles [it may be noticed in order to bring this method into relation with that indicated by Mr. Glaisher, that the coordinates of this new node A are the sums of the coordinates of its neighbors $'A$, A']. If the contour were not supposed to be simple, this condition could not be drawn; for if there were a hole round the middle point of $'AA'$ the node A would be missing in the enlarged aggregate, and if the first node to intervene as the contour went on enlarging be called (A); $'AO(A)$ or $(A)OA'$ or each of them would be a multiple of the elementary triangle so that the constancy of the value of the successive determinants would no longer hold. In like manner it will be seen that on the same supposition as above made, if in consequence of the contour contracting about O as the centre of similitude, two points $'AA'$ which originally are non-contiguous, at any moment become contiguous, at the moment previous to this taking place A (and no other point) must have intervened, and after A has disappeared from the reduced aggregate, no other point can make its appearance between $'A$, A'.

(70) Hence we may contract at pleasure the given contour about any node as origin, and if the contour so contracted contains at least one node besides the origin, it will suffice to determine whether the given contour is or is not regular.

Thus *ex gr.* in the case of a triangle limited by the axes and by the right line $x+y=n$, we may make $n=1$ and the trial-series will then become $\frac{0}{1}\ \frac{1}{1}\ \frac{1}{0}$ which possesses the Farey property. Hence this will hold good for a triangular boundary of any size and wherever the origin is situated: this includes the case of the ordinary Farey series when the origin is taken at either

extremity of the hypothenuse. So again for the area contained within the axes and the hyperbola $xy = n$, we may take $xy = 1$ and the trial series is the same as before.

(71) It is easy to form *unperforated* areas of any magnitude which shall not satisfy the Farey law: *ex gr.* we may as in the annexed figure



draw a curve passing through the origin, the point $(0, 1)$, and the point $(2, 3)$, $\frac{0}{1}$, $\frac{2}{3}$ does not satisfy the Farey law, and consequently the contour obtained by magnifying this in regard to the point (O, O) and may be made to contain an indefinitely large number of nodes will not be a "complete contour," and the successive values of (p, q) obtained by the rotation of a line round the origin will not constitute a Farey series.

The theory will, I believe, admit of being extended to solid reticulations, formed by the intersections of three systems of equidistant parallel planes, determinants of the third order between the three coordinates of successive points, replacing the $pq' - p'q$ of the plane theory. The chief difference will consist in the introduction of a new element in the multiplicity of the "normal orders" in which a given set (of points in a plane or) of radii in *solido* may be taken so that the signs of the determinants between each consecutive determinant shall remain of invariable sign—but as far as I can see this will in no way militate against the existence of the laws of invariance and similitude established for the case of a plane reticulation, but will only introduce a further principle of invariance, viz. that the law of unit-determinants if satisfied by one normal arrangement of the points of the solid reticulation will be satisfied by every other.

An Expression of the Coordinates of a Point on a Bi-Nodal Quartic Curve as Rational Functions of the Elliptic Functions of a Variable Parameter.

BY E. W. DAVIS, *Fellow of Johns Hopkins University.*

The equation of the bi-nodal quartic can always be put in the form

(1) $$x^2y^2+z^2(ax^2+by^2+cz^2+2fyz+2gzx+2hxy)=0,$$

where $z=0$ is the line joining the nodes and $x=0$ and $y=0$ its harmonic conjugates with regard to the tangents at the nodes.

We may otherwise write (1)

$$\frac{y^2}{z^2}\left(\frac{x^2}{z^2}+b\right)+2\frac{y}{z}\left(f+h\frac{x}{z}\right)+\left(a\frac{x^2}{z^2}+2g\frac{x}{z}+c\right)=0.$$

This gives

(2) $$\left(\frac{x^2}{z^2}+b\right)\frac{y}{z}=$$

$$-f-h\frac{x}{z}\pm\sqrt{-(a\frac{x^4}{z^4}+2g\frac{x^3}{z^3}+(c+ab-h^2)\frac{x^2}{z^2}+2(bg-hf)\frac{x}{z}+bc-f^2)}.$$

Suppose that α, β, γ, δ are the roots of the quartic in $\frac{x}{z}$ under the radical sign, and let $\alpha<\beta<\gamma<\delta$.

Assume

(3) $$\frac{x}{z}=\frac{\alpha(\beta-\delta)+\delta(\alpha-\beta)s^2}{\beta-\delta+(\alpha-\beta)s^2}=\frac{A+Bs^2}{C+Ds^2},$$

where s^2 is a new variable.

Also let $$k^2=\frac{\alpha-\beta.\gamma-\delta}{\alpha-\gamma.\beta-\delta},$$

the quartic in $\frac{x}{z}$ then becomes $-a\cdot\frac{x}{z}-\alpha\cdot\frac{x}{z}-\beta\cdot\frac{x}{z}-\gamma\cdot\frac{x}{z}-\delta$,

which is $$a\,\frac{(\alpha-\beta)^2(\alpha-\gamma)(\alpha-\delta)^2(\beta-\delta)^3.s^2(1-s^2)(1-k^2s^2)}{[(\beta-\delta)+(\alpha-\beta)s^2]^4},$$

This vanishes for $\frac{x}{z}=\alpha$, β, γ, δ, that is for $s^2=0,\ 1,\ \frac{1}{k^2},\ \infty$.

By reference to (2) we see that $x=\alpha z$, $x=\beta z$, $x=\gamma z$, $x=\delta z$ are the four tangents from the node (xz) to the curve. k^2 is therefore one of the anharmonic ratios of these tangents.

Put
$$\sqrt{a(\alpha-\beta)^2(\alpha-\gamma)(\alpha-\delta)^2(\beta-\delta)^3}=n$$
$$s^2,\ 1-s^2,\ 1-k^2s^2=\mathrm{sn}^2u,\ \mathrm{cn}^2u,\ \mathrm{dn}^2u=s^2,\ c^2,\ d^2,$$
$$A+Bs^2=M,\quad C+Ds^2=N,\quad scd=P,$$
(2) and (3) then give
$$(M^2+bN^2)\frac{y}{z}=-fN^2-hMN\mp nP,$$
or
$$x:y:z=M(M^2+bN^2):-N(fN^2+hMN\pm nP):M(M^2+bN^2).$$
The double sign before the nP is to be taken as positive or negative according as we take one or the other of the two intersections of a line through (xz) with the curve.

The ratio $x:y:z$, as above given, involves nine constants, A, B, C, D, b, f, h, k^2, n, or the 8 ratios of these constants, which is two more than are needed. Of these n involves a and α, β, γ, δ; a and likewise b can be expressed in terms of h, f, and α, β, γ, δ; and then α, β, γ, δ themselves, in terms of A, B, C, D and k^2.

We have
$$a=-\frac{2g}{\Sigma\alpha}=-\frac{2(bg-hf)}{\Sigma\alpha\beta\gamma}=\frac{2hf}{\Sigma\alpha\beta\gamma-b\Sigma\alpha},$$
also
$$a=\frac{c+ab-h^2}{\Sigma\alpha\beta}=\frac{bc-f^2}{\alpha\beta\gamma\delta}=\frac{c-h^2}{\Sigma\alpha\beta-b}=\frac{bh^2-f^2}{\alpha\beta\gamma\delta-b\Sigma\alpha\beta+b^2},$$
whence
$$0=b^2(2hf+h^2\Sigma\alpha)-b(f^2\Sigma\alpha+2hf\Sigma\alpha\beta+h^2\Sigma\alpha\beta\gamma)+f^2\Sigma\alpha\beta\gamma+2hf\alpha\beta\gamma\delta,$$
where α, β, γ, δ are to be replaced by their values in terms of A, B, C, D, k^2, viz.
$$\alpha=\frac{A}{C},\quad \beta=\frac{A+B}{C+D},\quad \gamma=\frac{k^2A+B}{k^2C+D},\quad \delta=\frac{B}{D}.$$
The value of $x:y:z$ readily gives for the node (xz)
$$s^2=-\frac{A\pm\sqrt{-b}C}{B\pm\sqrt{-b}D}=-\frac{\alpha\pm\sqrt{-b}}{\delta\pm\sqrt{-b}}\cdot\frac{C}{D},$$
and for the node (yz)
$$s^2=-\frac{C}{D}.$$

And since the tangents at the nodes are $x^2+bz^2=0$, and $y^2+az^2=0$, we have for the tangentials of the node (xz)
$$s^2=-\frac{A\pm\sqrt{-b}C}{B\pm\sqrt{-b}D},$$
the same values as for the node itself.

Those of the node (yz) are given by

$$fN^2+hMN\pm nP=\pm(M^2+bN^2)\sqrt{-a}.$$

If a line through (xz) meet the quartic in the points whose parameters are u and u_1, then if s, c, d and s_1, c_1, d_1 are the sn, cn, and dn of those parameters, M, N, P, M_1, N_1, P_1, the corresponding functions of s, c, d and s_1, c_1, d_1, we have

$$0=\begin{vmatrix} x, & z \\ x_1, & z_1 \end{vmatrix}=\begin{vmatrix} M, & N \\ M_1, & N_1 \end{vmatrix}=\begin{vmatrix} A+Bs^2, & C+Ds^2 \\ A+Bs_1^2, & C+Ds_1^2 \end{vmatrix}$$
$$=(BC-AD)(s^2-s_1^2),$$

and therefore $\qquad s=\pm s_1,$

an equation which we have already seen to hold with regard to the tangents at the node (xz). We shall presently show that $s=s_1$ and $s=-s_1$ are true for lines cutting different portions of the curve, the change from one to the other occurring at the points where $s=0$ and $s=\infty$.

The line through the node (yz) is found in precisely the same way to meet the quartic in points whose parameters are connected by the relation

$$0=\begin{vmatrix} -y, & z \\ -y_1, & z_1 \end{vmatrix}=\begin{vmatrix} fN^2+hMN\pm nP, & M^2+bN^2 \\ fN_1^2+hM_1N_1\pm nP_1, & M_1^2+bN_1^2 \end{vmatrix}$$
$$=f\begin{vmatrix} N^2, & M^2 \\ N_1^2, & M_1^2 \end{vmatrix}-h(MM_1-bNN_1)\begin{vmatrix} M, & N \\ M_1, & N_1 \end{vmatrix}\pm n\begin{vmatrix} P, & M^2+bN^2 \\ P_1, & M_1^2+bN_1^2 \end{vmatrix}$$
$$=(s^2-s_1^2)(BC-AD)\begin{bmatrix}[2AC+2BDs^2s_1^2+(BC+AD)(s^2+s_1^2)]f \\ -h(A+Bs^2)(A+Bs_1^2)+hb(C+Ds^2)(C+Ds_1^2)\end{bmatrix}$$
$$\pm n\left[scd\left((A+Bs_1^2)^2+b(C+Ds_1^2)^2\right)-s_1c_1d_1\left((A+Bs^2)^2+b(C+ds^2)^2\right)\right].$$

This result may be easily verified for the line $z=0$ meeting the quartic in the node (xz) counting as two points for which

$$s=\sqrt{\frac{A+\sqrt{-b}\,C}{B+\sqrt{-b}\,D}},\quad s_1=\sqrt{\frac{A-\sqrt{-b}\,C}{B-\sqrt{-b}\,D}}.$$

On substituting these values the quantities multiplying f, h and n separately vanish.

Again, if we put $s_1^2=-\dfrac{C}{D}$, the value of s^2 is that for the tangentials of the node (yz). We get

$$\frac{fN^2+hMN\pm nP}{M^2+bN^2}=\pm\frac{nP_1}{M_1^2}=\pm\sqrt{-a},$$

which is the same as the equation previously obtained for these tangentials.

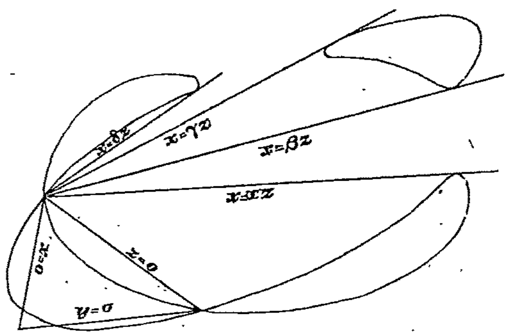
Consider again a line through the node (xz). We had for such a line $s = \pm s_1$; and if the line z were tangent to the quartic, s assumed the values $0,\ 1,\ \frac{1}{k}$, and ∞ as $\frac{x}{z}$ became equal to α, β, γ, and δ successively.

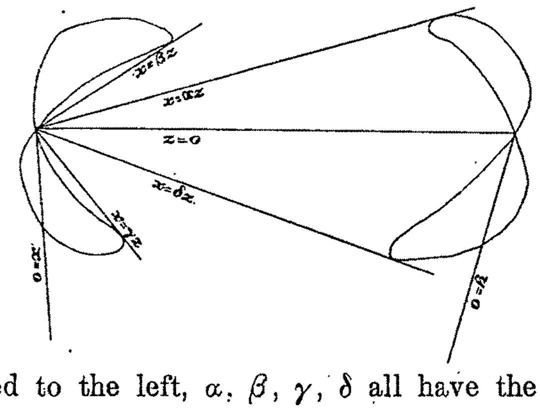
If in (3) we put $\frac{x}{z} > \delta$ or $\frac{x}{z} < \alpha$, we find s a pure imaginary; the argument u is therefore imaginary or differs from a pure imaginary by a half-period. We must then for this portion of the curve have $s = -s_1$ and argument varying from $u = \pm iK'$ at $s = \infty$, to $u = 0$ at $s = 0$. For intermediate points we have $u = +i\lambda$ and $u = -i\lambda$.

Now since for $x = \beta z$ $s = 1$, it must be that for $x = \beta z \pm d$, where d is an infinitesimal, s varies but little from 1; therefore both values of s must be positive, *i. e.* $s = s_1$: the same reasoning shows that s continues equal to s_1 until we come to $s = \infty$ again.

For $s = 1$ we have $u = K$, also for $s = \frac{1}{k}$, $u = K \pm iK'$, and we are enabled to see how u varies as the line through (xz) revolves. From $x = \alpha z$, where $u = 0$ or $2K$, to $x = \beta z$, $u = K \pm \lambda$ becoming K at $x = \beta z$; from $x = \beta z$ to $x = \gamma z$, $u = K \pm i\lambda$ becoming $K \pm iK'$ at $x = \gamma z$; finally, from $x = \gamma z$ to $x = \delta z$, $u = K \pm iK' \pm \lambda$ becoming iK' at $x = \delta z$.

There are two bi-nodal quartics, the tangents at the nodes of which and the four tangents from the nodes of which are real. In one the two nodes are on the same branch of the curve, and in the other on different branches of the curve.





For the first, which I have sketched to the left, α, β, γ, δ all have the same sign. In the drawing I have assumed all negative, since then x and z will both be positive when both are reckoned the same way from the centre of the triangle of reference. For the second form of the curve α and β are negative, and γ and δ positive. In either case (yz) lies upon that portion of the curve for

which the argument and therefore the s is imaginary. This is as it should be, for the s at (xz) is $\sqrt{-\frac{C}{D}}$, and $\frac{C}{D}$ is always positive. In the first case (xz) lies upon the portion with imaginary s, in the second case not. This is also as it should be. For in the case where s of (xz) is imaginary, either α and δ are negative and $-b > \delta^2$ or α and δ are positive and $1-b < \alpha^2$, so that we must have $\frac{\alpha \pm \sqrt{-b}}{\delta \pm \sqrt{-b}}$ positive, and therefore

$$\sqrt{-\frac{\alpha \pm \sqrt{-bC}}{\delta \pm \sqrt{-bD}}} = \sqrt{-\frac{A \pm \sqrt{-bC}}{B \pm \sqrt{-bD}}} \text{ is imaginary.}$$

When, on the other hand, s of (xz) is real, α is negative, δ positive, $-b < \alpha^2$ and $< \delta^2$, giving $\frac{\alpha \pm \sqrt{-b}}{\delta \pm \sqrt{-b}}$ negative and so $\sqrt{-\frac{A \pm \sqrt{-bC}}{B \pm \sqrt{-bD}}}$ is real.

It is obvious, that instead of solving the original equation as one in $\frac{y}{z}$ we could have solved it as one in $\frac{x}{z}$ by putting

$$\frac{y}{z} = \frac{M'}{N'} = \frac{A' + B'\sigma^2}{C' + D'\sigma^2} = \frac{\alpha'(\beta' - \delta') + \delta'(\alpha' - \beta')\sigma^2}{\beta' - \delta' + (\alpha' - \beta')\sigma^2},$$

$y = \alpha' z$, $\beta' z$, $\gamma' z$, $\delta' z$ being the tangents from the node (yz) to the curve and $\alpha' < \beta' < \gamma' < \delta'$. All that we have made out with regard to a line through (xz) now applies equally to a line through (yz), if we change the unaccented for accented letters and the b for a, a for b, and f for g.

The k^2 remains unchanged since the anharmonic ratio of the four tangents from (yz) = that of the four from (xz). The parameters for the tangents from (yz) have then the values 0, K, $K + iK'$, iK'. The intersections of corresponding tangents lie on a conic through the nodes. If we denote by v the parameter of the points of the quartic when we put $\frac{y}{z} = \frac{M'}{N'}$, then $u = v$ means that the line through (xz) and u (say) meets the line through (yz) and v on the above mentioned conic. The condition that lines through (xz) and (yz) should so meet is that they are corresponding rays of the two homographic pencils determined by the tangents from the two nodes. Therefore we have, expressing this condition analytically by writing the general values of $\frac{x}{z}$ and $\frac{y}{z}$ instead of α and α' in the values of k^2 and by equating results,

$$\frac{\left(\frac{\alpha(\beta-\delta) + \delta(\alpha-\beta)s^2}{\beta - \delta + (\alpha-\beta)s^2} - \beta\right)(\gamma - \delta)}{\left(\frac{\alpha(\beta-\delta) + \delta(\alpha-\beta)s^2}{\beta - \delta + (\alpha-\beta)s^2} - \gamma\right)(\beta - \delta)} = \frac{\left(\frac{\alpha'(\beta'-\delta') + \delta'(\alpha'-\beta')\sigma^2}{\beta' - \delta' + (\alpha'-\beta')\sigma^2} - \beta'\right)(\gamma' - \delta')}{\left(\frac{\alpha'(\beta'-\delta') + \delta'(\alpha'-\beta')\sigma^2}{\beta' - \delta' + (\alpha'-\beta')\sigma^2} - \gamma'\right)(\beta' - \delta')}.$$

This gives immediately

$$\frac{(\alpha-\beta)(\beta-\delta)(1-s^2)(\gamma-\delta)}{(\alpha-\gamma)(\beta-\delta)+(\delta-\gamma)(\alpha-\beta)s^2(\beta-\delta)}=\frac{(\alpha'-\beta')(\beta'-\delta')(1-\sigma^2)(\gamma'-\delta')}{(\alpha'-\gamma')(\beta'-\delta')+(\delta'-\gamma')(\alpha'-\beta')\sigma^2(\beta'-\delta')},$$

$$\text{or}\quad \frac{k^2(1-s^2)}{1-k^2s^2}=\frac{k^2(1-\sigma^2)}{1-k^2\sigma^2}\quad i.\ e.\ s^2=\sigma^2.$$

It was not at all necessary to have assumed $\alpha<\beta<\gamma<\delta$; we might have taken them in any order whatsoever and have changed them in 4 different ways without altering the k^2. The same is of course true of the α', β', γ' and δ'. Corresponding to a definite value of k^2 there will then be 4 different conics through the nodes on which corresponding lines through the nodes will meet.

Some Notes on the Numbers of Bernoulli and Euler.

By G. S. Ely, *Fellow in Mathematics, Johns Hopkins University.*

The expansion of $\tan x$, $\cot x$ and $\operatorname{cosec} x$, are known expansions depending on the numbers of Bernoulli; the expansion of $\sec x$ depends upon a series of numbers which are closely allied to the numbers of Bernoulli, and by many continental writers are designated as secant coefficients. Professor Sylvester, however, has named them Euler's numbers. Differentiating the series for $\tan x$ and $\cot x$ we obtain the expansion of $(\sec x)^2$ and $(\operatorname{cosec} x)^2$, and thence $(\tan x)^2$ and $(\cot x)^2$. The problem was therefore suggested to the writer to find expansions for the n^{th} powers of the trigonometrical functions. The functions $(\sin x)^n$ and $(\cos x)^n$ may be expressed in terms of sines and cosines of multiple arcs, and then be expanded by Maclaurin's formula. For the other functions the following formulæ may be used:

$$(\tan x)^n = \frac{1}{n-1}\frac{d}{dx}(\tan x)^{n-1} - (\tan x)^{n-2} \tag{1}$$

$$(\cot x)^n = -\frac{1}{n-1}\frac{d}{dx}(\cot x)^{n-1} - (\cot x)^{n-2} \tag{2}$$

$$(\sec x)^n = \frac{1}{(n-1)(n-2)}\left\{\frac{d^2}{dx^2} + (n-2)^2\right\}(\sec x)^{n-2} \tag{3}$$

$$(\operatorname{cosec} x)^n = \frac{1}{(n-1)(n-2)}\left\{\frac{d^2}{dx^2} + (n-2)^2\right\}(\operatorname{cosec} x)^{n-2} \tag{4}$$

For example, if $t_{2n-1} = \dfrac{2^{2n}(2^{2n}-1)}{2n}B_{2n-1}$

$$\tan x = t_1 x + t_3\frac{x^3}{3!} + t_5\frac{x^5}{5!} + \text{etc.}$$

$$(\tan x)^2 = t_3\frac{x^2}{2!} + t_5\frac{x^4}{4!} + t_7\frac{x^6}{6!} + \text{etc.}$$

$$(\tan x)^3 = \frac{t_5 - 2t_3}{2}\frac{x^3}{3!} + \frac{t_7 - 2t_5}{2}\frac{x^5}{5!} + \frac{t_9 - 2t_7}{2}\frac{x^7}{7!} + \text{etc.}$$

$$(\tan x)^4 = \frac{t_7 - 8t_5}{3!}\frac{x^4}{4!} + \frac{t_9 - 8t_7}{3!}\frac{x^6}{6!} + \frac{t_{11} - 8t_9}{3!}\frac{x^8}{8!} + \text{etc.}$$

Formulæ (1), (2), (4) and (3) when n is even depend on Bernoulli's numbers; when n is odd (3) depends on Euler's numbers. This is the case I shall specially consider.

$$\sec x = 1 + E_2 \frac{x^2}{2!} + E_4 \frac{x^4}{4!} + E_6 \frac{x^6}{6!} + \text{etc.} \tag{5}$$

The values of the E's are readily found by multiplying (5) by

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \text{etc.} \tag{6}$$

and equating coefficients. The expression thus obtained for the E's may be put in the symbolical form

$$(E - i)^{2n} = 0 \tag{7}$$

in which $i = \sqrt{-1}$ and after expansion the representative quantity E^{2m} becomes the actual quantity E_{2m}. Then it is easy to show from (7) that the E's are odd integers, and the proof that they are positive is also an easy matter. Actually computing a few of the E's we find

$$\begin{aligned} E_2 &= 1 & E_8 &= 1385, \\ E_4 &= 5 & E_{10} &= 50521, \\ E_6 &= 61 & E_{12} &= 2702765, \text{ etc.} \end{aligned}$$

Applying formula (3) to (5) we find

$$(\sec x)^3 = \frac{1 + E_2}{2} + \frac{E_2 + E_4}{2} \frac{x^2}{2!} + \frac{E_4 + E_6}{2} \frac{x^4}{4!} + \text{etc.} \tag{8}$$

$$(\sec x)^5 = \frac{9 + 10E_2 + E_4}{4!} + \frac{9E_2 + 10E_4 + E_6}{4!} \frac{x^2}{2!} + \frac{9E_4 + 10E_6 + E_8}{4!} \frac{x^4}{4!} + \text{etc.} \tag{9}$$

From these relations numbers of formulæ may be derived giving relations among the numbers of Euler, or between the numbers of Bernoulli and Euler. I will mention the following: multiplying (8) by (6) we obtain a value for $(\sec x)^2$, but differentiating the value of $\tan x$ we obtain another value of $(\sec x)^2$. Equating coefficients we find the relation

$$\frac{(2n-1)2^{2n}(2^{2n}-1)}{(2n)!} B_{2n-1} = \sum_{i=1}^{i=n} (-)^{n+i} \frac{E_{2i} + E_{2i-2}}{(2i-2)!\,(2n-2i)!},$$

in which $E_0 = 1$.

Again

$$(\cos x)^3 = \frac{1}{4} \cos 3x + \frac{3}{4} \cos x = 1 - \frac{3x^2}{2!} + 21 \frac{x^4}{4!} - 183 \frac{x^6}{6!} + \text{etc.} \tag{10}$$

The law of the coefficients 3, 21, 183 ... being that each is 9 times the preceding less 6. Multiplying (8) by (10) one obtains a relation among the

E's; or expanding $\dfrac{1}{(\cos x)^3} = \dfrac{1}{1 - 3\dfrac{x^2}{2!} + 21\dfrac{x^4}{4!} - \text{etc.}}$

by a known formula (*vide* Hammond, *Proceedings Lon. Math. Soc.*, Vol. VI, p. 69) we find

$$E_{2n+2} + E_{2n} = 2 \begin{vmatrix} 3, & 1, & 0, & 0, & \dots \\ 21, & 6.3, & 1, & 0, & \dots \\ 183, & 15.21, & 15.3, & 1, & \dots \\ 1641, & 28.183, & 70.21, & 28.3, & \dots \\ \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \end{vmatrix} (n^{\text{th}} \textit{ order})$$

But the chief purpose of the present paper is to obtain the expansion of $(\sec x)^p$, where p is an odd prime. To this end observe that

$$(\sec x)^p = \left(1 + E_2\frac{x^2}{2!} + E_4\frac{x^4}{4!} + \text{etc.}\right)^p;$$

whence as the E's are integers the coefficient of $\dfrac{x^{2n}}{(2n)!}$ in the expansion on the right is an integer, and if $n > 0$ is evidently divisible by the prime number p. If we consider the squares of the odd numbers less than p, viz.

$$1^2,\ 3^2,\ 5^2 \dots (p-2)^2$$

and represent their combinations n at a time by S_n, and for the sake of convenience write $\dfrac{p-1}{2} = p'$; it is easy to see that by successive applications of formula (3) we obtain

$$(\sec x)^p = \sum_{n=0}^{n=\infty} \frac{S_{p'}E_{2n} + S_{p'-1}E_{2n+2} + \dots + S_1E_{2n+p-3} + E_{2n+p-1}}{(p-1)!} \frac{x^{2n}}{(2n)!} \quad (11)$$

in which $E_0 = 1$. Since the coefficient of $\dfrac{x^{2n}}{(2n)!}$ is integral and divisible by $p\,(n > 0)$, *à fortiori* the numerator in the last is divisible by p. Thus we have

$$S_{p'}E_{2n} + S_{p'-1}E_{2n+2} + \dots + S_1E_{2n+p-3} + E_{2n+p-1} \equiv 0 \pmod p \quad (12)$$
$$(n < 0).$$

We are thus led to a consideration of the numbers S_m.* Consider the expression

$$1^{2n} + 2^{2n} + 3^{2n} + \dots + (p-1)^{2n} =$$
$$\frac{p^{2n+1}}{2n+1} - \frac{1}{2}p^{2n} + \frac{B_1}{2}2np^{2n-1} + \dots (-)^{n-1}B_{2n-1}p.$$

* I am indebted to Mr. Durfee for the idea of the proof of the following property of the numbers S_m.

This is of course an integral expression; as no B can, by Staudt's theorem, contain the same factor twice in its denominator, and as p^2 multiplies every term but the last the expression will be divisible by p, if, and only if, the last term contains p. Now by Staudt's theorem the first B which can have the factor p in its denominator is B_{p-2}. Therefore

$$1^{2n}+2^{2n}+\ldots+(p-1)^{2n}\equiv 0 \pmod p \tag{13}$$

if $n<\frac{p-1}{2}$. But as p is odd, every number less than p is an odd number or p minus an odd number. Thus (13) is

$$1^{2n}+\{p-(p-2)\}^{2n}+3^{2n}+\{p-(p-4)\}^{2n}+\ldots$$

which is $\equiv 2\{1^{2n}+3^{2n}+5^{2n}+\ldots+(p-2)^{2n}\} \pmod p$, or as p is odd

$$1^{2n}+3^{2n}+5^{2n}+\ldots+(p-2)^{2n}\equiv 0 \pmod p \text{ if } n<\frac{p-1}{2} \tag{14}$$

If $n=\frac{p-1}{2}$ we have by Fermat's theorem

$$1^{2n}+3^{2n}+5^{2n}+\ldots+(p-2)^{2n}\equiv\frac{p-1}{2} \pmod p \tag{15}$$

Further let $\sigma_1, \sigma_2, \sigma_3, \ldots$ represent the sums of the first, second, third, ... powers of the elements $1^2, 3^2, 5^2 \ldots (p-2)^2$. Then by a formula of symmetric functions

$$S_m=\frac{1}{m!}\begin{vmatrix} \sigma_1, & 1, & 0, & \ldots & 0 \\ \sigma_2, & \sigma_1, & 2, & \ldots & 0 \\ \sigma_3, & \sigma_2, & \sigma_1, & \ldots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \sigma_m, & \sigma_{m-1}, & \sigma_{m-2}, & \ldots & \sigma_1 \end{vmatrix}$$

Therefore if $m<\frac{p-1}{2}$ the first column divides by p and we have

$$S_m\equiv 0 \pmod p \tag{16}$$

From (14) and (15),

$$S_p\equiv\frac{1}{p'!}\begin{vmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 2 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & \ldots & p'-1 \\ p' & 0 & 0 & \ldots & 0 \end{vmatrix} \pmod p$$

i. e. $$S_{p'} \equiv (-1)^{p'+1} (\text{mod } p) \tag{17*}$$

Applying these results to (12) we see that

$$E_{2n} \equiv (-)^{\frac{p-1}{2}} E_{2n+p-1} (\text{mod } p) \tag{18†}$$

For the case in which $n=0$ a slight modification is necessary. The coefficient of x^0 in the expansion of $(\sec x)^p$ is

$$\frac{S_{p'} + S_{p'-1}E_2 + S_{p'-2}E_4 + \dots + S_1 E_{p-3} + E_{p-1}}{(p-1)!}$$

which is equal to 1. From what has been proved concerning the numbers S_m, and from Wilson's theorem, by which

$$(p-1)! \equiv -1 \ (\text{mod } p)$$

we have $E_{p-1} \equiv 0$ (mod p) if p is a prime of form $4n+1$

$E_{p-1} \equiv -2$ " " " " $4n+3$.

An immediate consequence of these results is that the E's end alternately in 1 and 5, a point which I believe Scherk was the first to prove.

That the residues of the E's with respect to any prime modulus are periodic has been noticed (Lucas, *l. c.*). We can, however, go further and find the periodic residues of the E's with respect to certain composite moduli. Thus for example in the expansion of $(\sec x)^5$, the coefficient of $\frac{x^{2n}}{(2n)!}$ is

$$\frac{9E_{2n} + 10E_{2n+2} + E_{2n+4}}{24},$$

and since this is an integer

$$9E_{2n} + 10E_{2n+2} + E_{2n+4} \equiv 0 \ (\text{mod } 24),$$

whence
$$E_{8n} \equiv 17 \quad (\text{mod } 24)$$
$$E_{8n+2} \equiv 1 \quad \text{"}$$
$$E_{8n+4} \equiv 5 \quad \text{"}$$
$$E_{8n+6} \equiv 13 \quad \text{"}$$

Similarly from the expansion of $(\sec x)^7$ we find the period of 48 terms

$$E_{48m+4k+2} \equiv 60k + 1 \quad (\text{mod } 720)$$
$$E_{48m+4k} \equiv (13-k)\,60 + 5 \quad \text{"}$$

Ex gr.
$$E_{10} = 50521 \equiv 121 \quad \text{"}$$
$$E_{12} = 2702765 \equiv 605 \quad \text{"}$$

* The results given in formulæ (16) and (17) may be obtained much more simply. Since 1, 2, 3 . . . $(p-1)$ are roots of Fermat's congruence, $x^{p-1} - 1 = (x-1)(x-2)(x-3) \dots (x-p+1) + p\phi(x)$. Then adding p to each of the factors $(x-2m)$ we have

$$x^{p-1} - 1 = (x^2 - 1^2)(x^2 - 3^2)(x^2 - 5^2) \dots \left(x^2 - \left(\tfrac{p-1}{2}\right)^2\right) + pf(x)$$

whence formulæ (16) and (17) follow immediately.

† This result has been given by Lucas (*Messenger*, VII), but obtained in an entirely different manner from that here employed.

Tables for Facilitating the Determination of Empirical Formulæ.

By A. W. Hale.

By making the intervals between observations constant and equal to unity, the calculation of empirical formulæ may be abridged.

Put

$$x = \text{the independent variable}$$

$$y = \text{'' dependent ''}$$

$$x_1, x_2, x_3;\ y_1, y_2, y_3, \text{ \&c.} = \text{observations}$$

$$m = \text{number of observations}$$

$$a_1, a_2, a_3;\ b_1, b_2, b_3, \text{ \&c.} = \text{undetermined coefficients}$$

$$[z] = z_1 + z_2 + z_3, \text{ \&c.}$$

$$[zu] = z_1 u_1 + z_2 u_2 + z_3 u_3, \text{ \&c.}$$

$$[z^2 u] = z_1^2 u_1 + z_2^2 u_2 + z_3^2 u_3, \text{ \&c. \&c.}$$

$$x_2 - x_1 = x_3 - x_2 = x_4 - x_3, \text{ \&c.}$$

$$y = a_1 + a_2 x + a_3 x^2 \ldots + a_{n+1} x^n$$

$$u = y - y_1 \tag{1}$$

$$z = \frac{x - x_1}{x_2 - x_1} \tag{2}$$

$$u = b_1 z + b_2 z^2 + \ldots + b_n z^n \tag{3}$$

$$\left.\begin{array}{l} u_1 = b_1 z_1 + b_2 z_1^2 + \ldots + b_n z_1^n \\ u_2 = b_1 z_2 + b_2 z_2^2 + \ldots + b_n z_2^n \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ u_n = b_1 z_n + b_2 z_n^2 + \ldots + b_n z_n^n \end{array}\right\} \tag{4}$$

Multiplying equations (4) successively by the coefficients of $b_1, b_2 \ldots b_n$ and adding the results of each multiplication, we have

$$\left.\begin{array}{l} [zu] = [z^2] b_1 + [z^3] b_2 + \ldots + [z^{n+1}] b_n \\ [z^2 u] = [z^3] b_1 + [z^4] b_2 + \ldots + [z^{n+2}] b_n \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ [z^n u] = [z^{n+1}] b_1 + [z^{n+2}] b_2 + \ldots + [z^{2n}] b_n \end{array}\right\} \tag{5}$$

Putting $n = 2$ we have from equation (3)

$$u = b_1 z + b_2 z^2 \tag{6}$$

and from equations (5)

$$\left.\begin{aligned}[zu] &= [z^2]\,b_1 + [z^3]\,b_2\\ [z^2u] &= [z^3]\,b_1 + [z^4]\,b_2\end{aligned}\right\} \tag{7}$$

Solving equations (7) with respect to b_1 and b_2 and substituting the values of b_1 and b_2 obtained in equation (6) after putting

$$\frac{[z^3]}{[z^2][z^4]-[z^3]^2} = \alpha,\quad \frac{[z^4]}{[z^3]} = \beta,\quad \frac{[z^2]}{[z^3]} = \gamma,$$

we have

$$u = \alpha\{(\beta\,[zu] - [z^2u])z + (\gamma\,[z^2u] - [zu])z^2\} \tag{8}$$

By equations (1), (2) and (8) and the following table (1) in which the values of α, β and γ are given for values of m from 5 to 26, a value for y may be found corresponding to any assumed value of x.

Putting $n = 3$ we have from equation (3)

$$u = b_1 z + b_2 z^2 + b_3 z^3 \tag{9}$$

and from equations (5)

$$\left.\begin{aligned}[zu] &= [z^2]\,b_1 + [z^3]\,b_2 + [z^4]\,b_3\\ [z^2u] &= [z^3]\,b_1 + [z^4]\,b_2 + [z^5]\,b_3\\ [z^3u] &= [z^4]\,b_1 + [z^5]\,b_2 + [z^6]\,b_3\end{aligned}\right\} \tag{10}$$

Solving equations (10) with respect to b_1, b_2 and b_3, after putting

$$[z^4]([z^2][z^6] + 2\,[z^3][z^5] - [z^4]^2) - ([z^2][z^5]^2 + [z^3]^2[z^6]) = A,$$

$$[z^3][z^5] - [z^4]^2 = B,\quad [z^2][z^5] - [z^3][z^4] = C,\quad [z^2][z^4] - [z^3]^2 = D,$$

$$B/A = \delta,\ ([z^4][z^6] - [z^5]^2)/B = \varepsilon,\ ([z^3][z^6] - [z^4][z^5])/B = \zeta,\ C/A = \eta,$$

$$([z^3][z^6] - [z^4][z^5])/C = \theta,\ ([z^2][z^6] - [z^4]^2)/C = \kappa,\ D/A = \lambda,\ B/D = \mu,\ C/D = \nu,$$

we have

$$\left.\begin{aligned}b_1 &= \delta(\varepsilon\,[zu] - \zeta\,[z^2u] + [z^3u])\\ b_2 &= -\eta(\theta\,[zu] - \kappa\,[z^2u] + [z^3u])\\ b_3 &= \lambda(\mu\,[zu] - \nu\,[z^2u] + [z^3u])\end{aligned}\right\} \tag{11}$$

By equations (1), (2), (9) and (11) and table 2, in which the values of δ, ε, ζ, &c., are given for values of m from 9 to 26, a value for y may be found corresponding to any assumed value of x.

Whether an empirical formula derived from a series of observations will involve the third power of the variable or not, is shown by simply putting the third equation of equations (11)

$$b_3 = \lambda(\mu\,[zu] - \nu\,[z^2u] + [z^3u]) = 0,$$

whence $$\nu\,[z^2u] - \mu\,[zu] = [z^3u] \tag{12}$$

EXAMPLE 1.

1	2	3	4	5	6	7	8
m	y	x	u	z	zu	z^2u	z^3u
1	5	4	0	0	0	0	0
2	10	6	5	1	5	5	5
3	21	8	16	2	32	64	128
4	38	10	33	3	99	297	891
5	61	12	56	4	224	896	3584
6	90	14	85	5	425	2125	10625
7	125	16	120	6	720	4320	25920
8	166	18	161	7	1127	7889	55223
9	213	20	208	8	1664	13312	106496
					4296	28908	202872
					$[zu]$	$[z^2u]$	$[z^3u]$

In column 1 are given the number of observations m; in columns 2 and 3 the observations y_1, y_2, x_1, x_2, &c. The figures in column 4 are calculated from the figures in column 2 by equation (1) by subtracting the value of y for $m = 1$ from the following values of y successively. Column 5 is calculated from column 3 by equation (2) and is in all cases the series of consecutive numbers 0, 1, 2, &c. Columns 6, 7 and 8 are the products of column 5 and columns 4, 6 and 7 respectively.

By equation (12), the footings of columns 6, 7 and 8, example 1, and the values of μ and ν for $m = 9$, Table 2, we have

$$\overset{\nu}{11\tfrac{25}{109}} \times \overset{[z^2u]}{28908} - \overset{\mu}{28\tfrac{37}{109}} \times \overset{[zu]}{4296} = [z^3u]$$

i. e. $$202872 = 202872$$

The empirical formula for example 1 does not, therefore, involve the third power of the variable.

By equation (8), the footings of columns 6 and 7, example 1, and the values of α, β and γ for $m = 9$, Table 1, we have

$$u = \overset{\alpha}{.0118}\{(\overset{\beta}{6.7685} \times \overset{[zu]}{4296} - \overset{[z^2u]}{28908})z + (\overset{\gamma}{.1574} \times \overset{[z^2u]}{28908} - \overset{[zu]}{4296})z^2\}$$

whence $$u = 2z + 3z^2 \tag{13}$$

By equations (1), (2) and (13)

$$y - y_1 = 2\frac{x - x_1}{x_2 - x_1} + 3\left(\frac{x - x_1}{x_2 - x_1}\right)^2 \tag{14}$$

By equation (14), example 1, columns 2 and 3 for $m = 1$ and 2, we have

$$y - 5 = 2\frac{x - 4}{6 - 4} + 3\left(\frac{x - 4}{6 - 4}\right)^2$$

whence

$$y = 13 - 5x + \tfrac{3}{4}x^2 \tag{15}$$

EXAMPLE 2.

1	2	3	4	5	6	7	8
m	y	x	u	z	zu	z^2u	z^3u
1	4	3	0	0	0	0	0
2	4	6	0	1	0	0	0
3	4	9	0	2	0	0	0
4	10	12	6	3	18	54	162
5	28	15	24	4	96	384	1536
6	64	18	60	5	300	1500	7500
7	124	21	120	6	720	4320	25920
8	214	24	210	7	1470	10290	72030
9	340	27	336	8	2688	21504	172032
					5292	38052	279180
					$[zu]$	$[z^2u]$	$[z^3u]$

By equation (12), the footings of columns 6, 7 and 8, example 2, and the values of μ and ν for $m = 9$, Table 2, we have

$$\begin{array}{ccc} \nu \quad [z^2u] & \mu \quad [zu] & \\ 11\tfrac{25}{109} \times 38052, & -\,28\tfrac{37}{109} \times 5292, & = [z^3u] \end{array}$$

i. e.

$$277345\tfrac{17}{109} \qquad < 279180.$$

The empirical formula for example 2 therefore involves the third power of the variable.

By equations (11), table 2, for $m = 9$, and the footings of columns 6, 7 and 8, example 2, we have

$$\left.\begin{array}{l} b_1 = .0153\,(33.5593 \times 5292 - 12.00055 \times 38052 + 279180) \;= 2, \\ b_2 = -\,.0061\,(30.2857 \times 5292 - 11.5357 \times 38052 + 279180) = -\,3, \\ b_3 = .00054\,(28.33945 \times 5292 - 11.2294 \times 38052 + 279180) = 1 \end{array}\right\} \tag{16}$$

Substituting these values of b_1, b_2 and b_3 in equation (9), we have

$$u = 2z - 3z^2 + z^3 \tag{17}$$

By equations (1), (2) and (17) we have

$$y-y_1=2\frac{x-x_1}{x_2-x_1}-3\left(\frac{x-x_1}{x_2-x_1}\right)^2+\left(\frac{x-x_1}{x_2-x_1}\right)^3 \tag{18}$$

By equation (18) and example 2, columns 2 and 3, for $m=1$ and 2, we have

$$y-4=2\frac{x-3}{6-3}-3\left(\frac{x-3}{6-3}\right)^2+\left(\frac{x-3}{6-3}\right)^3$$

whence

$$y=-2+\frac{11}{3}x-\frac{2}{3}x^2+\frac{1}{27}x^3 \tag{19}$$

The first and second differential coefficients of y, equation (19), placed equal to zero give *a* minimum value for y, $y=3.62$, $x=7.73$, and *a* maximum $y=4.39$, $x=4.27$, and the point of inflexion $x=6$, $y=4$.

In the following tables, repeating decimals are indicated by a dot or bar.

In table 2, in column 2 one, in column 5 two, in column 8 three zeros on the left are omitted, and the tabular numbers in these columns are to be divided by 10, 100, and 1000 respectively.

TABLE I.

1	2	3	4
m	α	β	γ
5	.161290	3.54	.3
6	.069876	$4.35\dot{1}$	$.2\dot{4}$
7	.035156	5.15873	$.\overline{206349}$
8	.019607	5.96428	.17857142
9	.011795	6.76851	$.15\overline{740}$
10	.007520	$7.57\overline{185}$	$.1\overline{407}$
1	.005020	$8.37\overline{45}$	$.1\overline{27}$
2	.003478	$9.1\overline{76}$	$.1\overline{16}$
3	.002486	9.97863	$.1\overline{068376}$
4	.001824	10.78022	$.\overline{098901}$
5	.001369	11.58158	$.0\overline{920634}$
6	.001047	$12.382\dot{7}$	$.086\dot{1}$
7	.0008149	13.18382	.08088235
8	.0006435	13.984706	.076252723
9	.0005149	14.785575	.072124756
20	.0004169	15.586315	.068421052
1	.0003412	16.386984	$.0\overline{650793}$
2	.0002819	17.187590	$.\overline{062049}$
3	.0002349	17.988142	.059288537
4	.0002197	18.788647	.056763285
5	.0001670	$19.589\dot{1}$	$.05\dot{4}$
6	.0001423	20.389538	$.05\overline{230769}$

TABLE II.

1	2	3	4	5	6	7	8	9	10	11	12
m	$10\,\delta$	ϵ	ζ	$100\,\eta$	θ	κ	$1000\,\lambda$	μ	μ	ν	ν
9	.152951	33.5593118	12.0005549	$.\overline{60}$	$30\frac{3}{7}$	11.5357142	.5397	$28\frac{37}{109}$	28.3394495	$11\frac{25}{109}$	11.2293577
10	.086340	42.1331622	13.4445088	.30547	38	12.92	.2429	$35\frac{183}{340}$	35.5382352	$12\frac{39}{68}$	12.5735294
1	.051765	51.6587336	14.8854328	.16545	$46\frac{4}{7}$	$14.30\overline{18}$	.1188	$43\frac{223}{415}$	43.5373493	$13\frac{76}{83}$	13.9156626
2	.032578	62.1362457	16.3241478	.09496	56	$15.6\overline{81}$	.06224	$52\frac{67}{199}$	52.3366834	$15\frac{51}{199}$	15.2562814
3	.021338	73.5658340	17.7612013	.05717	$66\frac{2}{7}$	$17.0\overline{604395}$	.03445	$61\frac{44}{47}$	61.9361702	$16\frac{28}{47}$	16.5957446
4	.014452	85.9475854	19.1969727	.03583	$77\frac{3}{7}$	18.4379905	.01997	$72\frac{46}{137}$	72.3357664	$17\frac{128}{137}$	17.9343065
5	.010071	99.2815580	20.6317336	.02323	$89\frac{3}{7}$	19.8146938	.01205	$83\frac{423}{790}$	83.5354433	$19\frac{43}{158}$	19.2721518
6	.007193	113.5677917	22.0656837	.01551	$102\frac{2}{7}$	21.1907142	.007529	$95\frac{966}{1805}$	95.5351800	$20\frac{220}{361}$	20.6094182
7	.005248	128.8063147	23.4989731	.01063	116	22.5661764	.004844	$108\frac{137}{409}$	108.3349633	$21\frac{387}{409}$	21.9462102
8	.003902	144.9971474	24.9317108	.00745	$130\frac{4}{7}$	23.9411764	.003301	$121\frac{4}{115}$	121.0347826	$23\frac{13}{46}$	23.2826086
9	.002951	162.1403177	26.3640047	.00532	146	25.3157894	.002164	$136\frac{86}{257}$	136.3346303	$24\frac{159}{257}$	24.6186770
20	.002264	180.2357981	27.7959078	.00387	$162\frac{2}{7}$	26.9600751	.001494	$151\frac{1526}{2855}$	151.5345008	$25\frac{545}{571}$	25.9544658
1	.001761	199.2836359	29.2274831	.00286	$179\frac{3}{7}$	28.0640816	.001051	$167\frac{1686}{3155}$	167.5343898	$27\frac{183}{631}$	27.2900158
2	.001386	219.2838248	30.6587768	.00215	$197\frac{3}{7}$	29.4378478	.000752	$184\frac{116}{347}$	184.3342939	$28\frac{217}{347}$	28.6253602
3	.001103	240.2363700	32.0898267	.00163	$216\frac{2}{7}$	30.8114059	.0005464	$201\frac{71}{76}$	201.9342105	$29\frac{73}{76}$	29.9605263
4	.000887	262.1412757	33.5206644	.00126	236	32.1847826	.0004026	$220\frac{277}{829}$	220.3341375	$31\frac{245}{829}$	31.2955367
5	.000719	284.9985450	34.9513159	.00098	$256\frac{4}{7}$	33.558	.0003004	$239\frac{2406}{4505}$	239.5340732	$32\frac{568}{901}$	32.6304106
6	.000588	308.8081808	36.3818035	.00077	278	34.9310769	.0002268	$259\frac{1303}{2440}$	259.5340163	$33\frac{471}{488}$	33.9651637

The Tabulation of Symmetric Functions.

BY W. P. DURFEE, *Fellow of Johns Hopkins University.*

M. Faà de Bruno gives, in his *Théorie des Formes Binaires*, tables of the values of symmetric functions, which are symmetrically arranged and at the same time confined to a half square. In these tables the functions of the self-conjugate partitions are placed in the middle. The remainder of the functions are separated into pairs, a pair consisting of a function and the function of its conjugate partition, and the members of each pair are disposed symmetrically about the middle. How he obtains a suitable arrangement of these pairs he does not say. My object is to show that such an arrangement is always possible, and at the same time to indicate how it can be obtained.

I shall first show that such an arrangement is possible in the tables giving the values of the combinations of the simple symmetric functions in terms of the general symmetric functions.

Let a_1, a_2, etc. denote the partitions of any number arranged in natural or dictionary order; P_1, P_2, etc. the combinations, and ϕ_1, ϕ_2, etc. the symmetric functions corresponding to these partitions. I shall represent the conjugate of the partition a_λ by $a_{\lambda'}$, and the coefficient of ϕ_μ in the value of P_λ by $(\lambda\mu)$.

Professor Cayley has shown (*Phil. Trans.* 1857) that $(\lambda\mu)=0$, if a_μ is prior to $a_{\lambda'}$, the conjugate of a_λ. We have then

$$(\lambda\mu)=0 \quad \mu<\lambda'. \tag{1}$$

M. Betti proved (*Tortolini*, 1858)

$$(\lambda\mu)=(\mu\lambda)$$

therefore

$$(\lambda\mu)=0 \quad \lambda<\mu'. \tag{2}$$

If a_λ is a self-conjugate partition, $\lambda=\lambda'$ and

$$(\lambda\mu)=0 \quad \begin{matrix}\mu<\lambda\\ \lambda<\mu',\end{matrix} \tag{3}$$

if in addition a_μ is a self-conjugate partition

$$(\lambda\mu)=0 \quad \begin{matrix}\mu<\lambda\\ \lambda<\mu.\end{matrix} \tag{4}$$

If instead of arranging the partitions in the natural order we had arranged them as follows: $a\ b_1,\ b_2,\ b_3 \ldots c_1,\ c_2,\ c_3 \ldots$ where a is the partition of one part, those in the group b the partitions of two parts in dictionary order, etc., the same conditions would obtain, and it is upon this latter arrangement that the accompanying table is based.

Let us now separate the partitions, which are, say, $n-1$ in number, into pairs consisting of a partition and its conjugate. Designate the prior partition of a pair by b_ν and the other by $b_{n-\nu}$. Call the unpaired (self-conjugate) partitions $c_1,\ c_2 \ldots c_k$.

If now we arrange the partitions b_ν in the order in which they occur in the first arrangement, and place after them the $c_1,\ c_2 \ldots c_k$, I say the order so obtained will give the desired form to the table.

We have by eq. (2)

$$(\nu . n-\pi)=0 \quad \pi > \nu,$$

since the conjugate of $b_{n-\pi}$, *i. e.* b_π is subsequent in natural order to b_ν. The order of the c's is arbitrary, since by (4) the P of a self-conjugate partition can contain the ϕ of no other self-conjugate partition. If then we arrange the ϕ functions across the top of our table in the order last named, and the P functions down the side in the same order, there can be no coefficients above the sinister diagonal except in the case of the self-conjugates. The coefficients on the sinister diagonal will be units since (Cayley, *l. c.*) $(\nu . n-\nu)=1$. The self-conjugates will, for the same reason, have unit coefficients on the principal diagonal, and these unit coefficients will be symmetrically placed with reference to the sinister diagonal. Since $(\lambda\mu)=(\mu\lambda)$, the coefficients similarly placed with reference to the principal diagonal will be equal, *i. e.* the table will be symmetrical.

It is evident that the table of the values of the ϕ's in terms of the P's will be similar in form except that the coefficients will occupy the part *above* the sinister diagonal. This diagonal will consist of units as before, and if we agree to consider these units as belonging to both tables we may write both tables in the same square.

The accompanying tables of the twelfthic were published in a different form in a former number of this Journal, but it is thought that the new arrangement is of enough interest to warrant reprinting. These tables have also been calculated by M. Řehořovsky, and appeared in the Transactions of the Royal Academy of Vienna.

On a Θ-Function Formula.

By Thomas Craig, *Johns Hopkins University.*

The notation adopted in the following is that used by Clifford in his paper "Algebraic Introduction to Elliptic Functions," contained in the volume of his "Mathematical Papers." I may just give the definitions of the four different Θ-functions in this notation: they are

$$\begin{aligned}
\Theta(u, a) &= \Sigma e^{n^2 a + 2nu}, \\
\Theta'(u, a) &= \Sigma^{(-)^n} e^{n^2 a + 2nu}, \\
\Theta_1(u, a) &= \Sigma e^{(n+\frac{1}{2})^2 a + 2(n+\frac{1}{2})u}. \\
\Theta'_1(u, a) &= \Sigma^{(-)^n} e^{(n+\frac{1}{2})^2 a + 2(n+\frac{1}{2})u}.
\end{aligned}$$

The relations connecting these are of course

$$\Theta\left(u + \frac{\pi i}{2}, a\right) = \Theta'(u, a) \qquad \Theta'\left(u + \frac{\pi i}{2}, a\right) = \Theta(u, a)$$

$$\Theta\left(u + \frac{a}{2}, a\right) = e^{-n-\frac{a}{4}}\Theta_1(u, a) \qquad \Theta_1\left(u + \frac{a}{2}, a\right) = e^{-n-\frac{a}{4}}\Theta(u, a)$$

$$\Theta\left(u + \frac{\pi i}{2} + \frac{a}{2}, a\right) = e^{-n-\frac{a}{4}}\Theta'_1(u, a) \quad \Theta'_1\left(u + \frac{\pi i}{2} + \frac{a}{2}, a\right) = ie^{-n-\frac{a}{4}}\Theta(u, a).$$

The quantities $\frac{\pi i}{2}$, $\frac{a}{2}$, $\frac{\pi i}{2} + \frac{a}{2}$ may be conveniently spoken of as the first, second and third quadrants.

On page 448 of the "Mathematical Papers" occurs the well-known formula

$$\begin{aligned}
\Theta(u)\dot{\Theta}(v) - \dot{\Theta}(u)\Theta(v) = {} & 2\Theta'(v-u, 2a)\Theta(v+u, 2a) \\
& + 2\dot{\Theta}_1(v-u, 2a)\Theta_1(v+u, 2a)
\end{aligned}$$

(whenever the modulus is not written it will be understood to be a, when it has any other value it will always be written down). From this follow, by giving special values to u and v, certain well-known and important formulae. The form of the left-hand number of this equation suggests writing it as

$$\begin{vmatrix} \Theta(u), & \dot{\Theta}(u) \\ \Theta(v) & \dot{\Theta}(v) \end{vmatrix} = 2\dot{\Theta}(v-u, 2a)\dot{\Theta}(v+u, 2a) + 2\dot{\Theta}_1(v-u, 2a)\Theta_1(v+u, 2a).$$

In what follows I examine the determinant

$$\begin{vmatrix} \Theta(u), & \dot{\Theta}(u), & \ddot{\Theta}(u), & \dddot{\Theta}(u) \\ \Theta(v), & \dot{\Theta}(v), & \ddot{\Theta}(v), & \dddot{\Theta}(v) \\ \Theta(w), & \dot{\Theta}(w), & \ddot{\Theta}(w), & \dddot{\Theta}(w) \\ \Theta(t), & \dot{\Theta}(t), & \ddot{\Theta}(t), & \dddot{\Theta}(t) \end{vmatrix}, \text{ say } \Delta.$$

The dots of course denote differentiation with respect to the argument.

We have

$$\Theta(u) = \sum_{\alpha=-\infty}^{\alpha=\infty} e^{\alpha^2 a + 2\alpha u},$$

$$\Theta(v) = \sum_{\beta=-\infty}^{\beta=\infty} e^{\beta^2 a + 2\beta v},$$

$$\Theta(w) = \sum_{\gamma=-\infty}^{\gamma=\infty} e^{\gamma^2 a + 2\gamma w},$$

$$\Theta(t) = \sum_{\delta=-\infty}^{\delta=\infty} e^{\delta^2 a + 2\delta t},$$

It is unnecessary to write down hereafter the limits of the summation which is always understood to extend from $-\infty$ to $+\infty$. On substituting the values of $\Theta(u)$, $\Theta(v)$, $\Theta(w)$, $\Theta(t)$ in the above determinant and making some easy reductions it becomes

$$\begin{aligned}\Delta = 64 \sum_\alpha \sum_\beta \sum_\gamma \sum_\delta \{ & (\alpha^2\beta^2 + \gamma^2\delta^2)(\beta-\alpha)(\delta-\gamma) + (\beta^2\delta^2 + \alpha^2\gamma^2)(\gamma-\alpha)(\beta-\delta) \\ & + (\beta^2\gamma^2 + \delta^2\alpha^2)(\delta-\alpha)(\gamma-\beta)\} \text{ exp. } [(\alpha^2+\beta^2+\gamma^2+\delta^2)a \\ & + 2\alpha u + 2\beta v + 2\gamma w + 2\delta t]\end{aligned}$$

where exp. $[H]$ stands for e^H. Write this as

$$\Delta = \sum_\alpha \sum_\beta \sum_\gamma \sum_\delta 64(\alpha, \beta, \gamma, \delta) \text{ exp. } [H].$$

It is very easy to see that

$$(\alpha, \beta, \gamma, \delta) = \begin{vmatrix} 1, & 1, & 1, & 1 \\ \alpha, & \beta, & \gamma, & \delta \\ \alpha^2, & \beta^2, & \gamma^2, & \delta^2 \\ \alpha^3, & \beta^3, & \gamma^3, & \delta^3 \end{vmatrix}$$

or, using the ordinary notation for this,

$$(\alpha, \beta, \gamma, \delta) = \zeta^{\frac{1}{2}}(\alpha, \beta, \gamma, \delta),$$

and we know that

$$\zeta^{\frac{1}{2}}(\alpha, \beta, \gamma, \delta) = (\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\beta-\gamma)(\beta-\delta)(\gamma-\delta).$$

Introduce four new quantities $\alpha_1, \beta_1, \gamma_1, \delta_1$, defined by the equations

$$\begin{aligned} \alpha_1 &= \alpha-\beta-\gamma-\delta, \\ \beta_1 &= -\alpha+\beta-\gamma-\delta, \\ \gamma_1 &= -\alpha-\beta+\gamma-\delta, \\ \delta_1 &= -\alpha-\beta-\gamma+\delta; \end{aligned}$$

from these we have the relations

$$\alpha^2+\beta^2+\gamma^2+\delta^2 = \tfrac{1}{4}(\alpha_1^2+\beta_1^2+\gamma_1^2+\delta_1^2)$$

and

$$\begin{aligned} \alpha &= \tfrac{1}{4}(\alpha_1-\beta_1-\gamma_1-\delta_1), \\ \beta &= \tfrac{1}{4}(-\alpha_1+\beta_1-\gamma_1-\delta_1), \\ \gamma &= \tfrac{1}{4}(-\alpha_1-\beta_1+\gamma_1-\delta_1), \\ \delta &= \tfrac{1}{4}(-\alpha_1-\beta_1-\gamma_1+\delta_1). \end{aligned}$$

It is clear by inspection of the above values for $\alpha_1, \beta_1, \gamma_1, \delta_1$, that whatever (integral) values $\alpha, \beta, \gamma, \delta$ may have. these are either all even or all odd, and so there are only two cases to examine, viz.

$$\text{I.} \quad \alpha_1, \beta_1, \gamma_1, \delta_1 = 2f,\ 2g,\ 2h,\ 2k.$$

$$\text{II.} \quad \alpha_1, \beta_1, \gamma_1, \delta_1 = 2f+1,\ 2g+1,\ 2h+1,\ 2k+1.$$

The exponent H expressed in terms of the new quantities α_1, etc., is

$$\begin{aligned} H = {} & \tfrac{1}{2}\alpha_1(u-v-w-t) + \tfrac{1}{2}\beta_1(-u+v-w-t) \\ & + \tfrac{1}{2}\gamma_1(-u-v+w-t) + \tfrac{1}{2}\delta_1(-u-v-w+t); \end{aligned}$$

also the factor which multiplies exp. $[H]$, viz.

$$64\zeta^{\frac{1}{2}}(\alpha, \beta, \gamma, \delta)$$

becomes

$$\zeta^{\frac{1}{2}}(\alpha_1, \beta_1, \gamma_1, \delta_1).$$

Substituting these values in the expression for Δ this becomes

$$\begin{aligned} \Delta = \sum_{\alpha_1}\sum_{\beta_1}\sum_{\gamma_1}\sum_{\delta_1}\zeta^{\frac{1}{2}}(\alpha_1, \beta_1, \gamma_1, \delta_1) \text{ exp. } [\tfrac{1}{4}(\alpha_1^2+\beta_1^2+\gamma_1^2+\delta_1^2) + \tfrac{1}{2}\alpha_1(u-v-w-t) \\ + \tfrac{1}{2}\beta_1(-u+v-w-t) + \tfrac{1}{2}\gamma_1(-u-v+w-t) + \tfrac{1}{2}\delta_1(-u-v-w+t)] \end{aligned}$$

where

$$\zeta^{\frac{1}{2}}(\alpha_1, \beta_1, \gamma_1, \delta_1) = (\alpha_1-\beta_1)(\alpha_1-\gamma_1)(\alpha_1-\delta_1)(\beta_1-\gamma_1)(\beta_1-\delta_1)(\gamma_1-\delta_1).$$

There are two cases to be examined, viz. case I. of the above where $\alpha_1, \beta_1, \gamma_1, \delta_1$ are all even, and case II. where they are all odd. The results obtained for the two separate cases will give on being added together the complete value of Δ.

In analogy with the relations connecting α, β, etc. with α_1, β_1, etc., it will be convenient to write

$$\begin{aligned} u_1 &= \quad u - v - w - t \\ v_1 &= -u + v - w - t \\ w_1 &= -u - v + w - t \\ t_1 &= -u - v - w + t \end{aligned}$$

from which follow

$$u = \tfrac{1}{4}(u_1 - v_1 - w_1 - t_1) \text{ etc.}$$
$$\Sigma u^2 = \tfrac{1}{4}\Sigma u_1^2.$$

Now we have

$$\Delta = \sum_{\alpha_1}\sum_{\beta_1}\sum_{\gamma_1}\sum_{\delta_1}\zeta^{\frac{1}{2}}(\alpha_1, \beta_1, \gamma_1, \delta_1) \exp.\left[\tfrac{1}{4}(\alpha_1^2 + \beta_1^2 + \gamma_1^2 + \delta_1^2)a + \tfrac{1}{2}\alpha_1 u_1 + \tfrac{1}{2}\beta_1 v_1 + \tfrac{1}{2}\gamma_1 w_1 + \tfrac{1}{2}\delta_1 t_1\right].$$

Give now the even values to α_1, β_1, γ_1, δ_1, and we have clearly

$$\zeta^{\frac{1}{2}}(\alpha_1, \beta_1, \gamma_1, \delta_1) = 64\zeta^{\frac{1}{2}}(f, g, h, k);$$

further the exponential factor becomes

$$\exp.\left[(f^2 + g^2 + h^2 + k^2)a + fu_1 + gv_1 + hw_1 + kt_1\right].$$

Combining these we have

$$\Delta_1 = \sum_f\sum_g\sum_h\sum_k 64\zeta^{\frac{1}{2}}(f, g, h, k) \exp.\left[(f^2 + g^2 + h^2 + k^2)a + fu_1 + gv_1 + hw_1 + kt_1\right]$$

or

$$\Delta_1 = \begin{vmatrix} \Theta\left(\frac{u_1}{2}\right) & \dot\Theta\left(\frac{u_1}{2}\right) & \ddot\Theta\left(\frac{u_1}{2}\right) & \dddot\Theta\left(\frac{u_1}{2}\right) \\ \Theta\left(\frac{v_1}{2}\right) & \dot\Theta\left(\frac{v_1}{2}\right) & \ddot\Theta\left(\frac{v_1}{2}\right) & \dddot\Theta\left(\frac{v_1}{2}\right) \\ \Theta\left(\frac{w_1}{2}\right) & \dot\Theta\left(\frac{w_1}{2}\right) & \ddot\Theta\left(\frac{w_1}{2}\right) & \dddot\Theta\left(\frac{w_1}{2}\right) \\ \Theta\left(\frac{t_1}{2}\right) & \dot\Theta\left(\frac{t_1}{2}\right) & \ddot\Theta\left(\frac{t_1}{2}\right) & \dddot\Theta\left(\frac{t_1}{2}\right) \end{vmatrix}.$$

Again take the odd values of α_1, β_1, γ_1, δ_1, these give

$$\zeta^{\frac{1}{2}}(\alpha_1, \beta_1, \gamma_1, \delta_1) = 64\zeta^{\frac{1}{2}}(f, g, h, k).$$

The exponential factor now is

$$\exp.\left[\left((f + \tfrac{1}{2})^2 + (g + \tfrac{1}{2})^2 + (h + \tfrac{1}{2})^2 + (k + \tfrac{1}{2})^2\right)a + (f + \tfrac{1}{2})u_1 + (g + \tfrac{1}{2})v_1 + (h + \tfrac{1}{2})w_1 + (k + \tfrac{1}{2})t_1\right].$$

Combining these we have

$$\Delta_2 = \sum_f\sum_g\sum_h\sum_k 64\zeta^{\frac{1}{2}}(f, g, h, k) \exp.\left[\left((f + \tfrac{1}{2})^2 + (g + \tfrac{1}{2})^2 + (h + \tfrac{1}{2})^2 + (k + \tfrac{1}{2})^2\right)a + (f + \tfrac{1}{2})u_1 + (g + \tfrac{1}{2})v_1 + (h + \tfrac{1}{2})w_1 + (k + \tfrac{1}{2})t_1\right],$$

and since

$$\zeta^{\frac{1}{2}}(f+\tfrac{1}{2},\, g+\tfrac{1}{2},\, h+\tfrac{1}{2},\, k+\tfrac{1}{2}) = \zeta^{\frac{1}{2}}(f,\, g,\, h,\, k)$$

this becomes

$$\Delta_2 = \begin{vmatrix} \Theta_1\left(\frac{u_1}{2}\right), & \dot{\Theta}_1\left(\frac{u_1}{2}\right), & \ddot{\Theta}_1\left(\frac{u_1}{2}\right), & \dddot{\Theta}_1\left(\frac{v_1}{2}\right) \\ \Theta_1\left(\frac{v_1}{2}\right), & \dot{\Theta}_1\left(\frac{v_1}{2}\right), & \ddot{\Theta}_1\left(\frac{v_1}{2}\right), & \dddot{\Theta}_1\left(\frac{v_1}{2}\right) \\ \Theta_1\left(\frac{w_1}{2}\right), & \dot{\Theta}_1\left(\frac{w_1}{2}\right), & \ddot{\Theta}_1\left(\frac{w_1}{1}\right), & \dddot{\Theta}_1\left(\frac{w_1}{2}\right) \\ \Theta_1\left(\frac{t_1}{2}\right), & \dot{\Theta}_1\left(\frac{t_1}{2}\right), & \ddot{\Theta}_1\left(\frac{t_1}{2}\right), & \dddot{\Theta}_1\left(\frac{t_1}{1}\right) \end{vmatrix}$$

Now $\Delta = \Delta_1 + \Delta_2$, and so we have the final formula

$$\begin{vmatrix} \Theta(u) & \dot{\Theta}(u) & \ddot{\Theta}(u) & \dddot{\Theta}(u) \\ \Theta(v) & \dot{\Theta}(v) & \ddot{\Theta}(v) & \dddot{\Theta}(v) \\ \Theta(w) & \dot{\Theta}(w) & \ddot{\Theta}(w) & \dddot{\Theta}(w) \\ \Theta(t) & \dot{\Theta}(t) & \ddot{\Theta}(t) & \dddot{\Theta}(t) \end{vmatrix} = \begin{vmatrix} \Theta\left(\frac{u_1}{2}\right) & \dot{\Theta}\left(\frac{u_1}{2}\right) & \ddot{\Theta}\left(\frac{u_1}{2}\right) & \dddot{\Theta}\left(\frac{u_1}{2}\right) \\ \Theta\left(\frac{v_1}{2}\right) & \dot{\Theta}\left(\frac{v_1}{2}\right) & \ddot{\Theta}\left(\frac{v_1}{2}\right) & \dddot{\Theta}\left(\frac{v_1}{2}\right) \\ \Theta\left(\frac{w_1}{2}\right) & \dot{\Theta}\left(\frac{w_1}{2}\right) & \ddot{\Theta}\left(\frac{w_1}{2}\right) & \dddot{\Theta}\left(\frac{w_1}{2}\right) \\ \Theta\left(\frac{t_1}{2}\right) & \dot{\Theta}\left(\frac{t_1}{2}\right) & \ddot{\Theta}\left(\frac{t_1}{2}\right) & \dddot{\Theta}\left(\frac{t_1}{2}\right) \end{vmatrix}$$

$$+ \begin{vmatrix} \Theta_1\left(\frac{u_1}{2}\right) & \dot{\Theta}_1\left(\frac{u_1}{2}\right) & \ddot{\Theta}_1\left(\frac{u_1}{2}\right) & \dddot{\Theta}_1\left(\frac{u_1}{2}\right) \\ \Theta_1\left(\frac{v_1}{2}\right) & \dot{\Theta}_1\left(\frac{v_1}{2}\right) & \ddot{\Theta}_1\left(\frac{v_1}{2}\right) & \dddot{\Theta}_1\left(\frac{v_1}{2}\right) \\ \Theta_1\left(\frac{w_1}{2}\right) & \dot{\Theta}_1\left(\frac{w_1}{2}\right) & \ddot{\Theta}_1\left(\frac{w_1}{2}\right) & \dddot{\Theta}_1\left(\frac{w_1}{2}\right) \\ \Theta_1\left(\frac{t_1}{2}\right) & \dot{\Theta}_1\left(\frac{t_1}{2}\right) & \ddot{\Theta}_1\left(\frac{t_1}{2}\right) & \dddot{\Theta}_1\left(\frac{t_1}{2}\right) \end{vmatrix}$$

The functions Θ and Θ_1 are even functions, so in this equation we might replace $\frac{u_1}{2}$, $\frac{v_1}{2}$, $\frac{w_1}{2}$, $\frac{t_1}{2}$ by $-\frac{u_1}{2}$, $-\frac{v_1}{2}$, $-\frac{w_1}{2}$, $-\frac{t_1}{2}$ without altering the values of either Θ or Θ_1. Assume now the particular values

$$\begin{aligned} u &= u, \\ v &= u + \frac{\pi i}{2}, \\ w &= u + \frac{a}{2}, \\ t &= u + \frac{\pi i}{2} + \frac{a}{2}. \end{aligned}$$

We find now
$$\frac{u_1}{2} = -u - \frac{\pi i}{2} - \frac{a}{2},$$
$$\frac{v_1}{2} = -u - \frac{a}{2},$$
$$\frac{w_1}{2} = -u - \frac{\pi i}{2},$$
$$\frac{t_1}{2} = -u \cdot$$

Substitute these values in the above formula, observing the known relations

$$\Theta\left(u + \frac{\pi i}{2}\right) = \Theta'(u) \qquad \Theta_1\left(u + \frac{\pi i}{2}\right) = i\Theta'_1(u)$$
$$\Theta\left(u + \frac{a}{2}\right) e^{-n-\frac{a}{4}}\Theta_1(u) \qquad \Theta_1\left(u + \frac{a}{2}\right) = e^{-n-\frac{a}{4}}\Theta(u)$$
$$\Theta\left(u + \frac{\pi i}{2} + \frac{a}{2}\right) = e^{-n-\frac{a}{4}}\Theta'_1(u) \qquad \Theta_1\left(u + \frac{\pi i}{2} + \frac{a}{2}\right) = -ie^{-n-\frac{a}{4}}\Theta'(u)$$

and we find without difficulty

$$\begin{vmatrix} \Theta, & \dot{\Theta} & \ddot{\Theta} & \dddot{\Theta} \\ \Theta', & \dot{\Theta}' & \ddot{\Theta}' & \dddot{\Theta}' \\ \Theta_1, & \dot{\Theta}_1 - \Theta_1, & \ddot{\Theta}_1 - 2\dot{\Theta}_1 + \Theta_1, & \dddot{\Theta}_1 - 3\ddot{\Theta}_1 + 3\dot{\Theta}_1 - \Theta_1 \\ \Theta'_1, & \dot{\Theta}'_1 - \Theta'_1, & \ddot{\Theta}'_1 - 2\dot{\Theta}'_1 + \Theta'_1, & \dddot{\Theta}'_1 - 3\ddot{\Theta}'_1 + 3\dot{\Theta}'_1 - \Theta'_1 \end{vmatrix}$$
$$= \begin{vmatrix} \Theta'_1, & \dot{\Theta}'_1 - \Theta'_1, & \ddot{\Theta}'_1 - 2\dot{\Theta}'_1 + \Theta'_1, & \dddot{\Theta}'_1 - 3\ddot{\Theta}'_1 + 3\dot{\Theta}'_1 - \Theta'_1 \\ \Theta_1, & \dot{\Theta}_1 - \Theta_1, & \ddot{\Theta}_1 - 2\dot{\Theta}_1 + \Theta_1, & \dddot{\Theta}_1 - 3\ddot{\Theta}_1 + 3\dot{\Theta}_1 - \Theta_1 \\ \Theta', & \dot{\Theta}', & \ddot{\Theta}', & \dddot{\Theta}' \\ \Theta, & \dot{\Theta}, & \ddot{\Theta}, & \dddot{\Theta} \end{vmatrix}$$
$$+ \begin{vmatrix} \Theta', & \dot{\Theta}' - \Theta', & \ddot{\Theta}' - 2\dot{\Theta}' + \Theta', & \dddot{\Theta}' - 3\ddot{\Theta}' + 3\dot{\Theta}' - \Theta' \\ \Theta, & \dot{\Theta} - \Theta, & \ddot{\Theta} - 2\dot{\Theta} + \Theta, & \dddot{\Theta} - 3\ddot{\Theta} + 3\dot{\Theta} - \Theta \\ \Theta'_1, & \dot{\Theta}'_1, & \ddot{\Theta}'_1, & \dddot{\Theta}'_1 \\ \Theta_1, & \dot{\Theta}_1, & \ddot{\Theta}_1, & \dddot{\Theta}_1 \end{vmatrix}$$

A common factor $e^{-2n-\frac{a}{2}}$ has been divided out of both sides of this equation.

Arranging the rows in the first determinant on the right-hand side of this equation so that they shall be in the same order as those of the determinant on the left, we see that this determinant is identical with that on the left-hand side,

and so we arrive at the remarkable result

$$\begin{vmatrix} \Theta', & \dot{\Theta}' - \Theta', & \ddot{\Theta}' - 2\dot{\Theta}' + \Theta', & \dddot{\Theta}' - 3\ddot{\Theta}' + 3\dot{\Theta}' - \Theta' \\ \Theta, & \dot{\Theta} - \Theta, & \ddot{\Theta} - 2\dot{\Theta} + \Theta, & \dddot{\Theta} - 3\ddot{\Theta} + 3\dot{\Theta} - \Theta \\ \Theta'_1, & \dot{\Theta}'_1, & \ddot{\Theta}'_1, & \dddot{\Theta}'_1, \\ \Theta_1, & \dot{\Theta}_1, & \ddot{\Theta}_1, & \dddot{\Theta}_1, \end{vmatrix} = 0$$

Interchange the first and second rows of this, also the third and fourth; add the first column to the second; add the first column plus twice the second to the third; and finally add the first column plus three times the second plus three times the third to the fourth, and this becomes

$$\begin{vmatrix} \Theta, & \dot{\Theta}, & \ddot{\Theta}, & \dddot{\Theta}, \\ \Theta', & \dot{\Theta}', & \ddot{\Theta}', & \dddot{\Theta}', \\ \Theta_1, & \dot{\Theta}_1 + \Theta_1, & \ddot{\Theta}_1 + 2\dot{\Theta}_1 + \Theta_1, & \dddot{\Theta}_1 + 3\ddot{\Theta}_1 + 3\dot{\Theta}_1 + \Theta_1 \\ \Theta'_1, & \dot{\Theta}'_1 + \Theta'_1, & \ddot{\Theta}'_1 + 2\dot{\Theta}'_1 + \Theta'_1, & \dddot{\Theta}'_1 + 3\ddot{\Theta}'_1 + 3\dot{\Theta}'_1 + \Theta'_1 \end{vmatrix} = 0$$

In all of these the u has been omitted, simply for convenience in writing.

By decomposing this last into the sum of two determinants and partially expanding the second this may be written in the form

$$\begin{vmatrix} \Theta & \Theta' & \Theta_1 & \Theta'_1 \\ \dot{\Theta} & \dot{\Theta}' & \dot{\Theta}_1 & \dot{\Theta}'_1 \\ \ddot{\Theta} & \ddot{\Theta}' & \ddot{\Theta}_1 & \ddot{\Theta}'_1 \\ \dddot{\Theta} & \dddot{\Theta}' & \dddot{\Theta}_1 & \dddot{\Theta}'_1 \end{vmatrix} = \Theta' \begin{vmatrix} \Theta_1 & \Theta'_1 & \dot{\Theta} \\ 2\dot{\Theta}_1 + \Theta_1 & 2\dot{\Theta}'_1 + \Theta'_1 & \ddot{\Theta} \\ 3\ddot{\Theta}_1 + 3\dot{\Theta}_1 + \Theta_1, & 3\ddot{\Theta}'_1 + 3\dot{\Theta}'_1 + \Theta'_1, & \dddot{\Theta} \end{vmatrix}$$

$$- \Theta \begin{vmatrix} \dot{\Theta}', & \Theta_1, & \Theta'_1 \\ \ddot{\Theta}', & 2\dot{\Theta}_1 + \Theta_1, & 2\dot{\Theta}'_1 + \Theta'_1 \\ \dddot{\Theta}', & 3\ddot{\Theta}_1 + 3\dot{\Theta}_1 + \Theta_1, & 3\ddot{\Theta}'_1 + 3\dot{\Theta}'_1 + 3\dot{\Theta}'_1 + \Theta'_1 \end{vmatrix}$$

The left-hand side of this equation is the Wronskian of Θ, Θ', Θ_1, Θ'_1, or say (using Muir's notation) $W(\Theta, \Theta', \Theta_1, \Theta'_1)$.

By choosing other sets of values of v, w and t, a number of other formulæ may be obtained; it is, however, scarcely worth while writing them down.

I do not know that the following relation has ever been noticed, viz. writing sn, cn, dn, for sn u, cn u, dn u, and denoting by accents the successive differentiations of these functions with respect to u, we have

$$\begin{aligned} \mathrm{sn}' &= \mathrm{cn}\,\mathrm{dn}, & \mathrm{sn}'' &= -\,\mathrm{sn}\,(\mathrm{dn}^2 + k^2\,\mathrm{cn}^2) \\ \mathrm{cn}' &= -\,\mathrm{sn}\,\mathrm{dn}, & \mathrm{cn}'' &= -\,\mathrm{cn}\,(\mathrm{dn}^2 - k^2\,\mathrm{sn}^2) \\ \mathrm{dn}' &= -\,k^2\,\mathrm{sn}\,\mathrm{cn}, & \mathrm{dn}'' &= -\,\mathrm{dn}\,(k^2\,\mathrm{cn}^2 - k^2\,\mathrm{sn}^2) \end{aligned}$$

and substituting these values in the determinant

$$\begin{vmatrix} \mathrm{sn} & \mathrm{sn}' & \mathrm{sn}'' \\ \mathrm{cn} & \mathrm{cn}' & \mathrm{cn}'' \\ \mathrm{dn} & \mathrm{dn}' & \mathrm{dn}'' \end{vmatrix}$$

we find after some very simple reductions which need not be given,

$$\begin{vmatrix} \mathrm{sn} & \mathrm{sn}' & \mathrm{sn}'' \\ \mathrm{cn} & \mathrm{cn}' & \mathrm{cn}'' \\ \mathrm{dn} & \mathrm{dn}' & \mathrm{dn}'' \end{vmatrix} = -k'^2, \text{ or } W(\mathrm{sn}\, u, \mathrm{cn}\, u, \mathrm{dn}\, u) = -k'^2$$

If instead of sn, cn, dn we write sn^α, cn^α, dn^α, we have

$$(\mathrm{sn}^\alpha u)' = \alpha\, \mathrm{sn}^{\alpha-1} u\, \mathrm{cn}\, u\, \mathrm{dn}\, u, \text{ etc.}$$

$$(\mathrm{sn}^\alpha u)'' = \alpha(\alpha-1)\, \mathrm{sn}^{\alpha-2} u\, \mathrm{cn}^2 u\, \mathrm{dn}^2 u - \alpha\, \mathrm{sn}^\alpha u(\mathrm{dn}^2 u + k^2 \mathrm{cn}^2 u)$$

$$(\mathrm{cn}^\alpha u)'' = \alpha(\alpha-1)\, \mathrm{cn}^{\alpha-2} u\, \mathrm{dn}^2 u\, \mathrm{sn}^2 u - \alpha\, \mathrm{cn}^\alpha u(\mathrm{dn}^2 u - k^2 \mathrm{sn}^2 u)$$

$$(\mathrm{dn}^\alpha u)'' = \alpha(\alpha-1)\, \mathrm{dn}^{\alpha-2} u\, k^4 \mathrm{sn}^2 u\, \mathrm{cn}^2 u - \alpha\, \mathrm{dn}^\alpha u(k^2 \mathrm{cn}^2 u - k^2 \mathrm{sn}^2 u)$$

and consequently

$$\begin{vmatrix} \mathrm{sn}^\alpha u & (\mathrm{sn}^\alpha u)', & (\mathrm{sn}^\alpha u)'' \\ \mathrm{cn}^\alpha u & (\mathrm{cn}^\alpha u)', & (\mathrm{cn}^\alpha u)'' \\ \mathrm{dn}^\alpha u & (\mathrm{dn}^\alpha u)', & (\mathrm{dn}^\alpha u)'' \end{vmatrix}$$

$$= \alpha^2\, \mathrm{sn}^{\alpha-1} u\, \mathrm{cn}^{\alpha-1} u\, \mathrm{dn}^{\alpha-1} u \begin{vmatrix} \mathrm{sn}^2 u, & 1, & (\alpha-1)\, \mathrm{cn}^2 u,\ \mathrm{dn}^2 u - \mathrm{sn}^2 u\, (\mathrm{dn}^2 u + k^2 \mathrm{cn}^2 u) \\ \mathrm{cn}^2 u, & -1, & (\alpha-1)\, \mathrm{dn}^2 u,\ \mathrm{sn}^2 u - \mathrm{cn}^2 u\, (\mathrm{dn}^2 u - k^2 \mathrm{sn}^2 u) \\ \mathrm{dn}^2 u, & -k^2, & (\alpha-1)\, \mathrm{sn}^2 u,\ \mathrm{cn}^2 u - \mathrm{dn}^2 u\, (k^2 \mathrm{cn}^2 u - k^2 \mathrm{sn}^2 u) \end{vmatrix}$$

Decomposing the determinant on the right into the sum of two this is

$$= \alpha^2(\alpha-1)\, \mathrm{sn}^{\alpha-1} u\, \mathrm{cn}^{\alpha-1} u\, \mathrm{dn}^{\alpha-1} u \begin{vmatrix} \mathrm{sn}^2 u, & 1, & \mathrm{cn}^2 u\, \mathrm{dn}^2 u \\ \mathrm{cn}^2 u, & -1, & \mathrm{dn}^2 u\, \mathrm{sn}^2 u \\ \mathrm{dn}^2 u, & -k^2, & \mathrm{sn}^2 u\, \mathrm{cn}^2 u \end{vmatrix}$$

$$+ \alpha^2 \mathrm{sn}^{\alpha-1} u\, \mathrm{cn}^{\alpha-1} u\, \mathrm{dn}^{\alpha-1} u \begin{vmatrix} \mathrm{sn}\, u, & \mathrm{cn}\, u\, \mathrm{dn}\, u, & -\mathrm{sn}\, u\, (\mathrm{dn}^2 u + k^2 \mathrm{cn}^2 u) \\ \mathrm{cn}\, u, & -\mathrm{dn}\, u\, \mathrm{sn}\, u, & -\mathrm{cn}\, u(\mathrm{dn}^2 u - k^2 \mathrm{sn}^2 u) \\ \mathrm{dn}\, u, & -k^2 \mathrm{sn}\, u\, \mathrm{cn}\, u, & -\mathrm{dn}\, u(k^2 \mathrm{cn}^2 u - k^2 \mathrm{sn}^2 u) \end{vmatrix}$$

The value of the first of these determinants is easily seen to be k'^2, and of the second $-k'^2$, so we have finally

$$\begin{vmatrix} \mathrm{sn}^\alpha u, & (\mathrm{sn}^\alpha u)', & (\mathrm{sn}^\alpha u)'' \\ \mathrm{cn}^\alpha u, & (\mathrm{cn}^\alpha u)', & (\mathrm{cn}^\alpha u)'' \\ \mathrm{dn}^\alpha u, & (\mathrm{dn}^\alpha u)', & (\mathrm{dn}^\alpha u)'' \end{vmatrix} \text{ or } W(\mathrm{sn}^\alpha u, \mathrm{cn}^\alpha u, \mathrm{dn}^\alpha u) = \alpha^2(\alpha-2) k'^2\, \mathrm{sn}^{\alpha-1} u\, \mathrm{cn}^{c-1} u\, \mathrm{dn}^{\alpha-1} u$$

For $\alpha = 1$ the right-hand side becomes $= -k'^2$ as found above; for $\alpha = 2$ the determinant is identically equal zero. A more general form is readily found by writing in the above determinant $\mathrm{sn}^\alpha u$, $\mathrm{cn}^\beta u$, $\mathrm{dn}^\gamma u$, instead of $\mathrm{sn}^\alpha u$, $\mathrm{cn}^\alpha u$, $\mathrm{dn}^\alpha u$.

On Non-Euclidean Properties of Conics.

By William E. Story.

In this paper I apply Professor Cayley's projective measurement,* generalized by Professor Klein† and still farther extended by me,‡ to a conic; the complete theory of this application is, of course, the complete theory of the projective properties of a conic in its relation to an arbitrary fixed conic, here called the absolute.

Corresponding to the ordinary classification of conics as ellipses, hyperbolas, parabolas and circles, we have here a classification of real conics according to their relation to the absolute as

Ellipses cutting the absolute in four imaginary points,

Hyperbolas cutting the absolute in four real points,

Semi-Hyperbolas cutting the absolute in two real and two imaginary points,

Elliptic Parabolas meeting the absolute in two coincident points and cutting it in two other imaginary points,

Hyperbolic Parabolas meeting the absolute in two coincident points and cutting it in two other real points,

Semi-Circular Parabolas meeting the absolute in three coincident points and cutting it in one other real point,

Circular Parabolas meeting the absolute in four coincident points,

Circles having double contact with the absolute.

Either the outside of the absolute is to be regarded as citra-infinite and the inside ultra-infinite or *vice versâ.* Of ellipses there are then two varieties, one citra-infinite and one ultra-infinite, each having a single closed branch; of hyperbolas two varieties each having two citra-infinite and two ultra-infinite branches alternating, one being cut by every entirely finite real straight line in

---

* Sixth Memoir upon Quantics. *Phil. Trans.*, 1859.

† Ueber die sogenannte Nicht-Euklidische Geometrie. *Math. Ann.*, Vol. IV.

‡ This volume, pp. 180–211.

real points, the other cut in real points only by some entirely finite real straight lines; of semi-hyperbolas one variety having one citra-infinite and one ultra-infinite branch; of elliptic parabolas, two varieties, one citra-infinite and one ultra-infinite, each having one closed branch; of hyperbolic parabolas two varieties, one having the points adjacent to the contact with the absolute ultra-infinite and one having the adjacent points citra-infinite, each having one citra-infinite and one ultra-infinite branch (the relations of these two varieties to the citra-infinite portion of the plane are quite different, one goes to infinity in two and the other in three different directions, some entirely finite real straight lines will cut the one in two imaginary points and others will cut it in two real points, whereas, every entirely finite real straight line cuts the second in two real points); of semi-circular parabolas one variety having one citra-infinite and one ultra-infinite branch; of circular parabolas two varieties, one citra-infinite and one ultra-infinite, each having a single closed branch; of circles four varieties, one citra-infinite with real contacts, one ultra-infinite with real contacts, one citra-infinite with imaginary contacts, and one ultra-infinite with imaginary contacts, each having a single closed branch. Circular parabolas may be regarded as special cases of hyperbolic parabolas or of circles.

This classification is useful only when the absolute is real. If the absolute is imaginary, every conic is an ellipse or circle with respect to it, and there is no real ultra-infinite portion of the plane.

Every conic has four intersections with the absolute, say the *absolute points* of the conic, and four tangents in common with the absolute, say the *absolute tangents* of the conic; the four absolute points lie by twos on six straight lines, say the *focal lines* of the conic; the four absolute tangents pass by twos through six points, say the *foci* of the conic; the tangents to the conic at its absolute points are its four *asymptotes* or *asymptotic tangents*; the contacts of the absolute tangents with the conic are its four *asymptotic points*; the six intersections of the asymptotic tangents, also poles of the focal lines with respect to the conic, are the *director points* or simply *directors* of the conic; the six junctions of the asymptotic points, also polars of the foci with respect to the conic, are the *directrices* of the conic; there is one self-conjugate triangle common to the conic and the absolute, whose sides are the three *axes*, and whose vertices are the three *centres* of the conic. The six focal lines, six foci, six directors, and six directrices may be conveniently grouped in pairs, namely each focal line is the junction of two absolute points, and the other focal line

of the same pair is the junction of the other two absolute points; each focus is the intersection of two absolute tangents, and the other focus of the same pair is the intersection of the other two absolute tangents; each director is the intersection of two asymptotes, and the other director of the same pair is the intersection of the other two asymptotes; each directrix is the junction of two asymptotic points, and the other directrix of the same pair is the junction of the other two asymptotic points. Moreover, to each focal line corresponds a definite director which is its pole with respect to the conic, and to each focus corresponds a definite directrix which is its polar with respect to the conic. Finally, the poles and polars of these characteristic lines and points with respect to the absolute may be considered as themselves characteristic of the conic, but their introduction here seems unnecessary.

It is evident that the focal lines pass by twos through the centres, the foci lie by twos on the axes, the asymptotes intersect by twos on the axes, the asymptotic points lie by twos on lines through the centres, the directors lie by twos on the axes, and the directrices pass by twos through the centre. Each axis cuts the conic in two points, its *vertices*, and through each centre pass two tangents, its *vertical tangents*, whose points of contact are the vertices on the opposite axis.

Just as we have employed the terms citra-infinite and ultra-infinite to denote on this side and on the other side of the absolute, so it will be convenient to use *intra-absolute* and *extra-absolute* to denote inside and outside the absolute. The former distinction is conventional, the latter actual. We may call a real straight line *semi-infinite* or *finite* according as it does or does not cut the absolute in real points. Then, in the Euclidean geometry, every real straight line is semi-infinite; in the non-Euclidean geometry with an imaginary absolute every real straight line is finite; and in the non-Euclidean geometry with a real absolute (true conic) some straight lines are finite and some semi-infinite, the limit between the two is a tangent to the absolute, which might properly be called an *infinite* straight line, since it makes an infinite angle with any other straight line; then, in the Euclidean geometry, every straight line is infinite.

It may be assumed for convenience that, if either the conic under consideration or the absolute is imaginary, the coefficients of its equation referred to any real system of coordinates are nevertheless real. Then the three axes and the three centres of any conic other than a semi-hyperbola are real. It will be

readily seen in which of the results obtained in this paper this assumption may be dropped.

We may call a real axis *transverse* or *conjugate* according as the vertices on it are real or imaginary, and a real centre *exterior* or *interior* according as the vertical tangents through it are real or imaginary. Then an interior centre is always opposite a conjugate axis, and an exterior centre is opposite a transverse axis. It is evident that any imaginary conic has three real interior centres and three real conjugate axes, and a real conic other than a semi-hyperbola has one real interior and two real exterior centres and one real conjugate and two real transverse axes. Also, if the absolute is imaginary, all three centres of any conic are intra-absolute and all three axes finite; but if the absolute is real and a true conic, one centre of any conic other than a semi-hyperbola is intra-absolute, two centres extra-absolute, one axis finite and two axes semi-infinite. In the Euclidean geometry, an ellipse has one finite interior centre, two infinite exteriors centres, two transverse infinite axes, and one conjugate axis situated altogether at infinity; while an hyperbola has one finite exterior centre, one infinite exterior centre, one infinite interior centre, one infinite conjugate axis, one infinite transverse axis, and one transverse axis situated altogether at infinity. In the non-Euclidean geometry with a real absolute, every conic other than a semi-hyperbola has either one intra-absolute interior centre, two extra-absolute exterior centres, one finite conjugate axis, and two semi-infinite transverse axes; or one extra-absolute interior centre, one extra-absolute exterior centre, one intra-absolute exterior centre, one semi-infinite conjugate axis, one semi-infinite transverse axis, and one finite transverse axis. This might be made the basis of a subdivision of the classification given above.

Most of the results contained in this paper may be obtained readily by purely geometrical methods based on the definitions of distances involving anharmonic ratios, but I have preferred to treat the subject analytically, in order to illustrate better the nature of the non-Euclidean geometry as a metrical geometry.

I employ the same notation, with some slight modifications, as in the paper above cited, namely

Ω is the absolute,

$\Omega_{00}=0$ is the equation of the absolute,

$\Omega_{11}=0$ is the condition that the point 1 lies on Ω,

$\Omega_{10}=0$ is the equation of the polar of the point 1 with respect to Ω,

$\Omega_{12}=0$ is the condition that the point 2 lies on the polar of the point 1 with respect to Ω,

$\overline{12}$ is the projective (non-Euclidean) distance between the points 1 and 2, or what is the same thing, between the polars of 1 and 2 with respect to Ω,

$\overline{\overset{\perp}{12}}$ is the projective distance between the point 1 and the polar of the point 2 with respect to Ω; further, I take for convenience, the constants involved in the projective measures so that

$$2ik = 2ik' = 2ik'' = \ldots = 1,$$

which will cause no ambiguity, as we may always pass back to the general case by dividing each distance by the constant $2ik$ belonging to that species of measurement. Then

$$(1) \qquad \overline{12} = \cos^{-1}\left(\frac{\Omega_{12}}{\sqrt{\Omega_{11}\Omega_{22}}}\right),$$

$$(2) \qquad \overline{\overset{\perp}{12}} = \sin^{-1}\left(\frac{\Omega_{12}}{\sqrt{\Omega_{11}\Omega_{22}}}\right).$$

Let also S be the conic under consideration, to which S_{00}, S_{11}, S_{10}, S_{12}, bear the same relation as Ω_{00}, Ω_{11}, Ω_{10}, Ω_{12}, to Ω.

If the points 1 and 2 are the poles, *quâ* Ω, of the two focal lines of S of the same pair, we may write

$$(3) \qquad S_{00} \equiv \lambda\Omega_{12}\Omega_{00} - 2\Omega_{10}\Omega_{20},$$

where the signification of the parameter λ is determined by the condition that for every point 0 of S

$$\lambda = 2\frac{\Omega_{10}\Omega_{20}}{\Omega_{12}\Omega_{00}} = 2\frac{\sqrt{\Omega_{11}\Omega_{22}}}{\Omega_{12}} \cdot \frac{\Omega_{10}}{\sqrt{\Omega_{10}\Omega_{11}}} \cdot \frac{\Omega_{20}}{\sqrt{\Omega_{22}\Omega_{00}}}$$

$$= 2\frac{\cos\overline{10}\,.\cos\overline{20}}{\cos\overline{12}},$$

so that

$$(4) \qquad \cos\overline{10}\,.\cos\overline{20} = \sin\overline{\overset{\perp}{10}}\,.\sin\overline{\overset{\perp}{20}} = \tfrac{1}{2}\lambda\cos\overline{12} = \text{const.},$$

i. e. the product of the sines of the distances of any point of a conic from its focal lines of either pair is constant. It is to be noticed that, with the projective measurement, every theorem, descriptive or metrical, has a perfect reciprocal, hence *the product of the sines of the distances of any tangent to a conic from its foci of either pair is constant.*

If 1 is a focus of S and 2 the pole, *quâ* Ω, of the corresponding directrix, we may write

$$(5) \qquad S_{00} \equiv \Omega_{22}(\Omega_{11}\Omega_{00} - \Omega_{10}^2) - \mu\Omega_{11}\Omega_{20}^2,$$

hence for every point 0 of S

$$\frac{\Omega_{11}\Omega_{00}-\Omega_{10}^2}{\Omega_{11}\Omega_{00}}=\mu\frac{\Omega_{20}^2}{\Omega_{22}\Omega_{00}},$$

i. e.

$$\sin^2\overline{10}=\mu\sin^2\overline{20},$$

whence

(6)
$$\frac{\sin\overline{10}}{\sin\overline{20}}=\sqrt{\mu}$$

i. e. the ratio of the sines of the distances of any point of a conic from either focus and the corresponding directrix is constant, and reciprocally *the ratio of the sines of the distances of any tangent of a conic from either focal line and the corresponding director is constant.* The values of these constants, which correspond to the eccentricity in the ordinary theory, depend on the pairs to which the focus and focal line belong.

If 1 and 2 are the foci of S of either pair, the directrix corresponding to 1 will pass through the intersection of $\Omega_{10}=0$ and $\Omega_{20}=0$, so that its equation will be of the form $\nu\Omega_{10}\sqrt{\Omega_{22}}-\Omega_{20}\sqrt{\Omega_{11}}=0$,
and we may write

$$S_{00}\equiv\lambda\Omega_{22}(\Omega_{11}\Omega_{00}-\Omega_{10}^2)-\mu(\nu\Omega_{10}\sqrt{\Omega_{22}}-\Omega_{20}\sqrt{\Omega_{11}})^2,$$

but the tangents to Ω from 2, whose equation is

$$\Omega_{22}\Omega_{00}-\Omega_{20}^2=0,$$

are also the tangents to S, hence the result of substituting the value of Ω_{00} from this equation in the left-hand member of the preceding must be a perfect square,

i. e.
$$\lambda\Omega_{11}\Omega_{20}^2-\lambda\Omega_{22}\Omega_{10}^2-\mu(\nu\Omega_{10}\sqrt{\Omega_{22}}-\Omega_{20}\sqrt{\Omega_{11}})^2$$
$$=(\lambda-\mu)\Omega_{11}\Omega_{20}^2+2\mu\nu\sqrt{\Omega_{11}\Omega_{22}}\,\Omega_{10}\Omega_{20}-(\lambda+\mu\nu^2)\Omega_{22}\Omega_{10}^2$$

must be a perfect square, and hence

$$\mu^2\nu^2+(\lambda-\mu)(\lambda+\mu\nu^2)=0,$$

i. e.
$$\lambda:\mu=1-\nu^2:1,$$

and S_{00} becomes

(7)
$$S_{00}\equiv(1-\nu^2)\Omega_{11}\Omega_{22}\Omega_{00}-(\Omega_{22}\Omega_{10}^2+\Omega_{11}\Omega_{20}^2)+2\nu\sqrt{\Omega_{11}\Omega_{22}}\,\Omega_{10}\Omega_{20},$$

which may also be written

$$S_{00}\equiv\Omega_{11}\Omega_{22}\Omega_{00}\{(1-\nu^2)-(\cos^2\overline{10}+\cos^2\overline{20})+2\nu\cos\overline{10}\cos\overline{20}\}$$
$$\equiv-\Omega_{11}\Omega_{22}\Omega_{00}[\cos(\overline{10}+\overline{20})-\nu][\cos(\overline{10}-\overline{20})-\nu];$$

i. e. for any point of S,

(8)
$$\overline{10}+\overline{20}=\pm\cos^{-1}\nu\quad\text{or}\quad\overline{10}-\overline{20}=\pm\cos^{-1}\nu,$$

hence *either the sum or difference of the distances of any point of a conic from the two foci of either pair is constant,* and reciprocally *either the sum or difference*

of the distances (angles) of any tangent of a conic from the two focal lines of either pair is constant.

If 1, 2, 3 are the centres of S as above defined,

$$\Omega_{12}=0,\ \Omega_{13}=0,\ \Omega_{23}=0,$$

and the equations of the axes are

$$\Omega_{10}=0,\ \Omega_{20}=0,\ \Omega_{30}=0,$$

and we may write (provided S has not contact with Ω)

(9) $$S_{00}\equiv\lambda\Omega_{10}^2+\mu\Omega_{20}^2+\nu\Omega_{30}^2;$$

if then 4, 5 are the intersections of S with any chord through 1, we may write

$$x_5=x_1+\rho x_4,\ y_5=y_1+\rho y_4,\ z_5=z_1+\rho z_4,$$
$$\Omega_{55}=\Omega_{11}+2\rho\Omega_{14}+\rho^2\Omega_{44},$$
$$\Omega_{15}=\Omega_{11}+\rho\Omega_{14},\ \Omega_{25}=\rho\Omega_{24},\ \Omega_{35}=\rho\Omega_{34},$$

and, since 4, 5 are points of S,

$$\lambda\Omega_{14}^2+\mu\Omega_{24}^2+\nu\Omega_{34}^2=0,$$
$$\lambda(\Omega_{11}+\rho\Omega_{14})^2+\mu\rho^2\Omega_{24}^2+\nu\rho^2\Omega_{34}^2=0,$$

and, by subtraction of the first from the second, and division by $\lambda\Omega_{11}$,

$$\Omega_{11}+2\rho\Omega_{14}=0,\ i.\,e.\ \rho=-\frac{\Omega_{11}}{2\Omega_{14}},$$

then $$\cos\overline{15}=\frac{\Omega_{11}+\rho\Omega_{14}}{\sqrt{\Omega_{11}(\Omega_{11}+2\rho\Omega_{14}+\rho^2\Omega_{44})}}=\pm\frac{\Omega_{11}}{2\rho\sqrt{\Omega_{11}\Omega_{44}}}$$
$$=\mp\frac{\Omega_{14}}{\sqrt{\Omega_{11}\Omega_{44}}}=\mp\cos\overline{14},$$

i. e., neglecting multiples of the length of the whole line and noticing that in general 4 and 5 are different points,

(10) $$\overline{15}=-\overline{14},$$

hence *either centre of a conic bisects any chord through it*, and reciprocally *either axis of a conic bisects the distance (angle) between the tangents from any point of it.* This justifies the name *centre.*

In general, if 6 is the bisector of the line joining 4 and 5, we may write

$$x_6=x_4+\rho x_5,\ y_6=y_4+\rho y_5,\ z_6=z_4+\rho z_5,$$
$$\Omega_{66}=\Omega_{44}+2\rho\Omega_{45}+\rho^2\Omega_{55},$$
$$\Omega_{46}=\Omega_{44}+\rho\Omega_{45},\ \Omega_{56}=\Omega_{45}+\rho\Omega_{55},$$

and the condition for ρ is

$$\cos\overline{56}=\pm\cos\overline{46},$$

i. e. $$\frac{\Omega_{56}}{\sqrt{\Omega_{55}\Omega_{66}}}=\pm\frac{\Omega_{46}}{\sqrt{\Omega_{44}\Omega_{66}}},$$

or
$$\Omega_{44}\Omega_{56}^2 - \Omega_{55}\Omega_{46}^2 = \Omega_{44}(\Omega_{45} + \rho\Omega_{55})^2 - \Omega_{55}(\Omega_{44} + \rho\Omega_{45})^2$$
$$= (\Omega_{44}\Omega_{55} - \Omega_{45}^2)(\rho^2\Omega_{55} - \Omega_{44}) = 0,$$

i. e.
$$\rho = \pm\sqrt{\frac{\Omega_{44}}{\Omega_{55}}},$$

and there are two bisectors

$$(11)\qquad \begin{cases}(x_4\sqrt{\Omega_{55}} + x_5\sqrt{\Omega_{44}},\ y_4\sqrt{\Omega_{55}} + y_5\sqrt{\Omega_{44}},\ z_4\sqrt{\Omega_{55}} + z_5\sqrt{\Omega_{44}})\\ (x_4\sqrt{\Omega_{55}} - x_5\sqrt{\Omega_{44}},\ y_4\sqrt{\Omega_{55}} - y_5\sqrt{\Omega_{44}},\ z_4\sqrt{\Omega_{55}} - z_5\sqrt{\Omega_{44}}),\end{cases}$$

namely, these bisect respectively the two segments of the line between 4 and 5, and are distant from each other by half the length of the whole line, *i. e.* each lies on the polar of the other, *quâ* Ω; in other words, the bisectors of $\overline{45}$ are the foci of the involution of which 4, 5 and the intersections of the line 45 with Ω are pairs of conjugates. Hence if a point bisects every chord of S through it, its polar with respect to S is at the same time its polar with respect to Ω, *i. e. the vertices of the self-conjugate triangle common to S and Ω are the only centres of S.*

There exists an identical relation between the distances of any point 0 from three arbitrary points not in one straight line, say 1, 2, 3, or from three arbitrary straight lines not passing through one point, say the polars of 1, 2, 3. Namely, if 1, 2, 3 are not collinear, we may write

$$x_0 = \lambda x_1 + \mu x_2 + \nu x_3,\ y_0 = \lambda y_1 + \mu y_2 + \nu y_3,\ z_0 = \lambda z_1 + \mu z_2 + \nu z_3,$$

then

$$\begin{aligned}\Omega_{00} &\equiv \lambda^2\Omega_{11} + \mu^2\Omega_{22} + \nu^2\Omega_{33} + 2\mu\nu\Omega_{23} + 2\nu\lambda\Omega_{31} + 2\lambda\mu\Omega_{12};\\ \Omega_{10} &\equiv \lambda\Omega_{11} + \mu\Omega_{12} + \nu\Omega_{13},\\ \Omega_{20} &\equiv \lambda\Omega_{21} + \mu\Omega_{22} + \nu\Omega_{23},\\ \Omega_{30} &\equiv \lambda\Omega_{31} + \mu\Omega_{32} + \nu\Omega_{33},\end{aligned}$$

so that λ, μ, ν can be expressed as homogeneous linear functions of Ω_{10}, Ω_{20}, Ω_{30}, and these values give by substitution Ω_{00} as a homogeneous quadratic function of Ω_{10}, Ω_{20}, Ω_{30}, *i. e.* there exists a (non-homogeneous) quadratic relation between $\cos\overline{10}$, $\cos\overline{20}$, $\cos\overline{30}$, *i. e.* also between $\sin\overline{10}^{\perp}$, $\sin\overline{20}^{\perp}$, $\sin\overline{30}^{\perp}$. These relations are easily seen to be

$$(12)\qquad \begin{vmatrix}\Omega_{11} & \Omega_{12} & \Omega_{13} & \Omega_{10}\\ \Omega_{21} & \Omega_{22} & \Omega_{23} & \Omega_{20}\\ \Omega_{31} & \Omega_{32} & \Omega_{33} & \Omega_{30}\\ \Omega_{01} & \Omega_{02} & \Omega_{03} & \Omega_{00}\end{vmatrix} \equiv 0,$$

$$(13)\quad \begin{vmatrix} 1 & \cos\overline{12} & \cos\overline{13} & \cos\overline{10} \\ \cos\overline{21} & 1 & \cos\overline{23} & \cos\overline{20} \\ \cos\overline{31} & \cos\overline{32} & 1 & \cos\overline{30} \\ \cos\overline{10} & \cos\overline{20} & \cos\overline{30} & 1 \end{vmatrix} \equiv \begin{vmatrix} 1 & \sin\overset{\perp}{\overline{12}} & \sin\overset{\perp}{\overline{13}} & \sin\overset{\perp}{\overline{10}} \\ \sin\overset{\perp}{\overline{21}} & 1 & \sin\overset{\perp}{\overline{23}} & \sin\overset{\perp}{\overline{20}} \\ \sin\overset{\perp}{\overline{31}} & \sin\overset{\perp}{\overline{32}} & 1 & \sin\overset{\perp}{\overline{30}} \\ \sin\overset{\perp}{\overline{10}} & \sin\overset{\perp}{\overline{20}} & \sin\overset{\perp}{\overline{30}} & 1 \end{vmatrix} = 0.$$

If $1, 2, 3$ are vertices of any self-conjugate triangle, *quâ* Ω, $\Omega_{23} = \Omega_{31} = \Omega_{12} = 0$, and the relations are

$$(14)\qquad \Omega_{11}\Omega_{22}\Omega_{33}\Omega_{00} \equiv \Omega_{22}\Omega_{33}\Omega_{10}^2 + \Omega_{33}\Omega_{11}\Omega_{20}^2 + \Omega_{11}\Omega_{22}\Omega_{30}^2,$$

$$(15)\qquad \begin{cases} \cos^2\overline{10} + \cos^2\overline{20} + \cos^2\overline{30} \equiv \sin^2\overset{\perp}{\overline{10}} + \sin^2\overset{\perp}{\overline{20}} + \sin^2\overset{\perp}{\overline{30}} \equiv 1; \\ \text{or} \\ \sin^2\overline{10} + \sin^2\overline{20} + \sin^2\overline{30} \equiv \cos^2\overset{\perp}{\overline{10}} + \cos^2\overset{\perp}{\overline{20}} + \cos^2\overset{\perp}{\overline{30}} \equiv 2. \end{cases}$$

By virtue of these relations equation (9) gives, for any point on S,

$$\lambda\Omega_{11}\cos^2\overline{10} + \mu\Omega_{22}\cos^2\overline{20} + \nu\Omega_{33}\cos^2\overline{30}$$

$$(16)\qquad \begin{cases} = (\lambda\Omega_{11} - \nu\Omega_{33})\cos^2\overline{10} + (\mu\Omega_{22} - \nu\Omega_{33})\cos^2\overline{20} + \nu\Omega_{33} \\ = (\lambda\Omega_{11} - \nu\Omega_{33})\sin^2\overset{\perp}{\overline{10}} + (\mu\Omega_{22} - \nu\Omega_{33})\sin^2\overset{\perp}{\overline{20}} + \nu\Omega_{33} = 0, \\ \text{or} \\ (\lambda\Omega_{11} - \nu\Omega_{33})\sin^2\overline{10} + (\mu\Omega_{22} - \nu\Omega_{33})\sin^2\overline{20} - (\lambda\Omega_{11} + \mu\Omega_{22} - \nu\Omega_{33}) = 0, \end{cases}$$

i. e. there exists a linear relation between the squares of the sines or cosines of the distances of any point of a conic from any two centres or axes, and reciprocally *a similar relation exists between the squares of the sines or cosines of the distances of any tangent of a conic from any two axes or centres.*

Let $1, 2$ be two directors of S of one pair, then, just as S_{00} was written in the form (3), so Ω_{00} can now be written in the form

$$(17)\qquad \Omega_{00} \equiv \lambda S_{12} S_{00} - 2 S_{10} S_{20},$$

where $S_{10} = 0$ and $S_{20} = 0$ are the focal lines of S corresponding to 1 and 2 respectively. Take any point 3 on S, the tangent at this point meets S_{10} and S_{20} in 4 and 5 respectively, say. Then

$$S_{33} = 0,\ S_{34} = 0,\ S_{35} = 0,\ S_{14} = 0,\ S_{25} = 0,$$

and by (17)

$$\Omega_{34} = -S_{13}S_{24},\ \Omega_{44} = \lambda S_{12}S_{44},\ \Omega_{35} = -S_{23}S_{15},\ \Omega_{55} = \lambda S_{12}S_{55},$$
$$\Omega_{33} = -2S_{13}S_{23};$$

and by (12) applied to $1, 2, 3, 4$ and S,

$$0 \equiv \begin{vmatrix} S_{11} & S_{12} & S_{13} & 0 \\ S_{21} & S_{22} & S_{23} & S_{24} \\ S_{31} & S_{32} & 0 & 0 \\ 0 & S_{42} & 0 & S_{44} \end{vmatrix} \equiv \begin{vmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & 0 \end{vmatrix} S_{44} + S_{13}^2 S_{24}^2,$$

$$i.\,e. \begin{vmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & 0 \end{vmatrix} S_{44} \equiv - S_{13}^2 S_{24}^2,$$

and similarly

$$\begin{vmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & 0 \end{vmatrix} S_{55} \equiv - S_{23}^2 S_{15}^2;$$

from these equations follow

$$\cos^2 \overline{34} = \frac{\Omega_{34}^2}{\Omega_{33}\Omega_{44}} = -\frac{S_{13}S_{24}^2}{2\lambda S_{12}S_{23}S_{44}} = \frac{1}{2\lambda S_{12}S_{13}S_{23}} \begin{vmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & 0 \end{vmatrix},$$

$$\cos^2 \overline{35} = \frac{\Omega_{35}^2}{\Omega_{33}\Omega_{55}} = -\frac{S_{23}S_{15}^2}{2\lambda S_{12}S_{13}S_{55}} = \frac{1}{2\lambda S_{12}S_{13}S_{23}} \begin{vmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & 0 \end{vmatrix},$$

hence
$$\cos^2 \overline{35} = \cos^2 \overline{34},$$
and evidently

(18) $$\overline{35} = -\overline{34},$$

i. e. any point of a conic is equally distant from the intersections of the focal lines of either pair with the tangent at the point, and reciprocally *any tangent to a conic makes equal angles with the junctions of the foci of either pair to its point of contact.*

Still using equation (17), let any transversal cut S in 3 and 4, S_{10} in 5, S_{20} in 6, then

$$S_{33} = 0,\ S_{44} = 0,\ S_{15} = 0,\ S_{26} = 0,$$
$$x_5 = \rho x_3 + \sigma x_4,\ y_5 = \rho y_3 + \sigma y_4,\ z_5 = \rho z_3 + \sigma z_4,$$
$$S_{15} = \rho S_{13} + \sigma S_{14} = 0,\ \therefore\ \rho : \sigma = S_{14} : - S_{13},$$
$$x_5 = S_{14}x_3 - S_{13}x_4,\ y_5 = S_{14}y_3 - S_{13}y_4,\ z_5 = S_{14}z_3 - S_{13}z_4,$$

similarly $\quad x_6 = S_{24}x_3 - S_{23}x_4,\ y_6 = S_{24}y_3 - S_{23}y_4,\ z_6 = S_{24}z_3 - S_{23}z_4,$

and hence

$$S_{25} = S_{14}S_{23} - S_{13}S_{24},\ S_{35} = -S_{13}S_{34},\ S_{55} = -2S_{13}S_{14}S_{34},$$
$$S_{16} = S_{13}S_{24} - S_{14}S_{23},\ S_{46} = S_{24}S_{34},\ S_{66} = -2S_{23}S_{24}S_{34},$$
$$\Omega_{33} = -2S_{13}S_{23},\ \Omega_{55} = \lambda S_{12}S_{55} = -2\lambda S_{12}S_{13}S_{14}S_{34},$$
$$\Omega_{35} = \lambda S_{12}S_{35} - S_{13}S_{25} = -S_{13}(\lambda S_{12}S_{34} + S_{14}S_{23} - S_{13}S_{24})$$
$$\Omega_{44} = -2S_{14}S_{24},\ \Omega_{66} = \lambda S_{12}S_{66} = -2\lambda S_{12}S_{23}S_{24}S_{34},$$
$$\Omega_{46} = \lambda S_{12}S_{46} - S_{16}S_{24} = S_{24}(\lambda S_{12}S_{34} - S_{13}S_{24} + S_{14}S_{23}),$$
$$\cos^2 \overline{35} = \frac{\Omega_{35}^2}{\Omega_{33}\Omega_{55}} = \frac{(\lambda S_{12}S_{34} + S_{14}S_{23} - S_{13}S_{24})^2}{4\lambda S_{12}S_{14}S_{23}S_{34}},$$

$$\cos^2\overline{46} = \frac{\Omega_{46}^2}{\Omega_{44}\Omega_{66}} = \frac{(\lambda S_{12}S_{34} - S_{13}S_{24} + S_{14}S_{23})^2}{4\lambda S_{12}S_{14}S_{23}S_{34}} = \cos^2\overline{35},$$

$$\therefore \quad \overline{46} = \pm\,\overline{35},$$

where the sign can be determined by the case of a tangent, for which evidently the positive sign cannot be taken, hence in general

(19) $$\overline{46} = -\,\overline{35},$$

i. e. on any transversal to a conic, the intercepts between the curve and the two focal lines of either pair are of equal lengths and opposite signs, and reciprocally *the two tangents from any point to a conic make equal (opposite) angles with the junctions of the point to the (different) foci of either pair.* A more accurate statement of the theorem is this: *any straight line intersects a conic in two points and its focal lines of either pair in two points, and either point of the conic is just as far in one direction from the point of either focal line as the other point of the conic is in the opposite direction from the point of the other focal line;* and reciprocally *through any point pass two tangents to a conic and two junctions of the point to its foci of either pair, and either tangent to the conic makes the same angle in one direction with the junction to either focus as the other tangent to the conic makes in the opposite direction with the junction to the other focus.*

Again use equation (17) and let 3, 4 be any fixed points of S, and 5 a variable point of S, and let 6, 7 be the points in which the lines 35 and 45 respectively meet $S_{10} = 0$; then, as in the proof of the preceding theorem, since $S_{16} = 0$ and $S_{17} = 0$,

$$\begin{aligned}
&x_6 = S_{13}x_5 - S_{15}x_3,\ y_6 = S_{13}y_5 - S_{15}y_3,\ z_6 = S_{13}z_5 - S_{15}z_3,\\
&x_7 = S_{14}x_5 - S_{15}x_4,\ y_7 = S_{14}y_5 - S_{15}y_4,\ z_7 = S_{14}z_5 - S_{15}z_4,\\
&S_{33} = 0,\ S_{44} = 0,\ S_{55} = 0,\\
&S_{66} = -\,2S_{13}S_{15}S_{35},\ S_{77} = -\,2S_{14}S_{15}S_{45},\\
&S_{67} = -\,S_{15}(S_{13}S_{45} + S_{14}S_{35} - S_{15}S_{34}),\\
&\Omega_{66} = \lambda S_{12}S_{66},\ \Omega_{77} = \lambda S_{12}S_{77},\ \Omega_{67} = \lambda S_{12}S_{67},
\end{aligned}$$

$$\cos^2\overline{67} = \frac{\Omega_{67}^2}{\Omega_{66}\Omega_{77}} = \frac{S_{67}^2}{S_{66}S_{77}} = \frac{(S_{13}S_{45} + S_{14}S_{35} - S_{15}S_{34})^2}{4S_{13}S_{14}S_{35}S_{45}},$$

but, by (12) applied to 1, 3, 4, 5, and S,

$$0 = \begin{vmatrix} S_{11} & S_{13} & S_{14} & S_{15} \\ S_{31} & 0 & S_{34} & S_{35} \\ S_{41} & S_{43} & 0 & S_{45} \\ S_{51} & S_{53} & S_{54} & 0 \end{vmatrix} = (S_{13}S_{45} + S_{14}S_{35} - S_{15}S_{34})^2 - 2S_{35}S_{45}(2S_{13}S_{14} - S_{11}S_{34}),$$

hence
$$\cos^2 \overline{67} = \frac{2S_{13}S_{14} - S_{11}S_{34}}{2S_{13}S_{14}},$$

(20)
$$\sin^2 \overline{67} = \frac{S_{11}S_{34}}{2S_{13}S_{14}},$$

which is independent of 5, *i. e. the segments of any focal line cut out by the junctions of a variable point of the cônic with two fixed points of the same is constant,* and reciprocally *the angle subtended at either focus by the segment cut out of a variable tangent of the conic by two fixed tangents of the same is constant.*

The theorems already proved show the inaccuracy of the assumption usually made, explicitly or implicitly, that the principal of duality or reciprocity is inapplicable to all cases in which metrical relations are involved. It is, for instance, noticeable that in the chapter on "reciprocal polars" in Salmon's Conic Sections no example is given of reciprocation between a theorem involving distances between points and one similarly involving angles between lines. The fact is *the method of reciprocal polars can be applied to all theorems involving distances between points and angles between lines in which all the points and lines can be constructed by purely descriptive processes.* It will be observed that the properties of the focal lines of a conic above proved are known properties of the asymptotes of a conic in the Euclidean geometry, *i. e. the asymptotes have these properties in the ordinary theory by virtue of their coincidence with focal lines, and not as tangents at infinity.*

Let us consider more carefully the equation (9) of the conic S referred to its axes. This form of equation can be used, of course, only when S is an ellipse or hyperbola. In either centre, say 3, meet two axes $\Omega_{10}=0$, $\Omega_{20}=0$; $\Omega_{10}=0$ meets S in two *vertices*, of which 4 may be one; similarly $\Omega_{20}=0$ meets S in two vertices of which 5 may be one; $\overline{34}$ and $\overline{35}$ are then the lengths of two semi-axes meeting in the centre 3; and we have

$$\Omega_{14}=0,\ \Omega_{25}=0,\ \mu\Omega_{24}^2+\nu\Omega_{34}^2=0,\ \lambda\Omega_{15}^2+\nu\Omega_{35}^2=0,$$

and by (14)

$$\Omega_{00} \equiv \frac{\Omega_{10}^2}{\Omega_{11}} + \frac{\Omega_{20}^2}{\Omega_{22}} + \frac{\Omega_{30}^2}{\Omega_{33}},$$

hence

$$\Omega_{44} = \frac{\Omega_{24}^2}{\Omega_{22}} + \frac{\Omega_{34}^2}{\Omega_{33}} = \left[-\frac{\nu}{\mu\Omega_{22}} + \frac{1}{\Omega_{33}}\right]\Omega_{34}^2,$$

$$\cos^2 \overline{34} = \frac{\Omega_{34}^2}{\Omega_{33}\Omega_{44}} = \frac{\mu\Omega_{22}}{\mu\Omega_{22} - \nu\Omega_{33}},$$

or, putting $\tan \overline{34} = b$, and similarly $\tan \overline{35} = a$,

$$b^2 = -\frac{\nu \Omega_{33}}{\mu \Omega_{22}}, \quad a^2 = -\frac{\nu \Omega_{33}}{\lambda \Omega_{11}},$$

i. e.

$$\lambda : \mu : \nu = \frac{1}{a^2 \Omega_{11}} : \frac{1}{b^2 \Omega_{22}} : -\frac{1}{\Omega_{33}},$$

so that S_{00} can be written

$$(21) \qquad S_{00} \equiv \frac{\Omega_{10}^2}{a^2 \Omega_{11}} + \frac{\Omega_{20}^2}{b^2 \Omega_{22}} - \frac{\Omega_{30}^2}{\Omega_{33}},$$

where a, b are the tangents of the lengths of the semi-axes meeting in 3.

Equation (21) may be made the basis of an investigation of the properties of conjugate diameters through the centre 3. Let 4 be the extremity of any semi-diameter 34 whose equation is

$$\rho \sqrt{\Omega_{22}}\, \Omega_{10} - \sqrt{\Omega_{11}}\, \Omega_{20} = 0;$$

now

$$\Omega_{00} \equiv \frac{\Omega_{10}^2}{\Omega_{11}} + \frac{\Omega_{20}^2}{\Omega_{22}} + \frac{\Omega_{30}^2}{\Omega_{33}},$$

and hence for any point 0 of S,

$$\Omega_{00} \equiv \left(\frac{1}{a^2} + 1\right) \frac{\Omega_{10}^2}{\Omega_{11}} + \left(\frac{1}{b^2} + 1\right) \frac{\Omega_{20}^2}{\Omega_{22}};$$

hence

$$\Omega_{24} = \rho \sqrt{\frac{\Omega_{22}}{\Omega_{11}}}\, \Omega_{14},$$

$$\Omega_{44} = \left[\frac{1}{a^2} + 1 + \rho^2 \left(\frac{1}{b^2} + 1\right)\right] \frac{\Omega_{14}^2}{\Omega_{11}},$$

$$\frac{\Omega_{34}^2}{\Omega_{33}} = \frac{1}{a^2} \frac{\Omega_{14}^2}{\Omega_{11}} + \frac{1}{b^2} \frac{\Omega_{24}^2}{\Omega_{22}} = \left(\frac{1}{a^2} + \rho^2 \frac{1}{b^2}\right) \frac{\Omega_{14}^2}{\Omega_{11}},$$

$$\cos^2 \overline{34} = \frac{\Omega_{34}^2}{\Omega_{33} \Omega_{44}} = \frac{\rho^2 a^2 + b^2}{\rho^2 a^2 (1 + b^2) + b^2 (1 + a^2)},$$

or, putting $\tan \overline{34} = p$,

$$(22) \qquad p^2 = \frac{(\rho^2 + 1) a^2 b^2}{\rho^2 a^2 + b^2}.$$

If $\overline{34}$ and $\overline{35}$ are two conjugate semi-diameters of S whose equations are respectively

$$\rho \sqrt{\Omega_{22}}\, \Omega_{10} - \sqrt{\Omega_{11}}\, \Omega_{20} = 0 \text{ and } \sigma \sqrt{\Omega_{22}}\, \Omega_{10} - \sqrt{\Omega_{11}}\, \Omega_{20} = 0,$$

i. e. if the pole 6 of 34, *quâ* S, lies on 35,

$$S_{60} = \frac{\Omega_{16} \Omega_{10}}{a^2 \Omega_{11}} + \frac{\Omega_{26} \Omega_{20}}{b^2 \Omega_{22}} - \frac{\Omega_{36} \Omega_{30}}{\Omega_{33}}$$

must be proportional to $\rho\sqrt{\Omega_{22}}\,\Omega_{10}-\sqrt{\Omega_{11}}\,\Omega_{20}$, *i. e.*

$$\Omega_{36}=0,\quad b^2\sqrt{\Omega_{22}}\,\Omega_{16}+\rho a^2\sqrt{\Omega_{11}}\,\Omega_{26}=0,\ \text{and}$$
$$\sigma\sqrt{\Omega_{22}}\,\Omega_{16}-\sqrt{\Omega_{11}}\,\Omega_{26}=0,$$

whence

(23) $$\rho\sigma a^2+b^2=0,\ \text{or}\ \sigma=-\frac{b^2}{\rho a^2},$$

which is then the condition that $\rho\sqrt{\Omega_{22}}\,\Omega_{10}-\sqrt{\Omega_{11}}\,\Omega_{20}$ and $\sigma\sqrt{\Omega_{22}}\,\Omega_{10}-\sqrt{\Omega_{11}}\,\Omega_{20}$ are conjugate diameters.

If we put $\tan\overline{35}=q$ we obtain, as in (22),

$$q^2=\frac{(\sigma^2+1)a^2b^2}{\sigma^2a^2+b^2}=\frac{\rho^2a^4+b^4}{\rho^2a^2+b^2};$$

hence

(24) $$p^2+q^2=a^2+b^2,$$

i. e. the sum of the squares of the tangents of the lengths of any two conjugate semi-diameters through either centre is constant, and reciprocally *the sum of the squares of the tangents of the angles which two tangents, one from each of two conjugate points on either axis, make with that axis is constant.*

Let now ϑ be the angle between the two conjugate semi-diameters $\overline{34}$ and $\overline{35}$, then

$$\cos^2\vartheta=\frac{(\rho\sigma+1)^2}{(\rho^2+1)(\sigma^2+1)}=\frac{\rho^2(a^2-b^2)^2}{(\rho^2+1)(\rho^2a^4+b^4)},$$

$$\sin^2\vartheta=\frac{(\rho-\sigma)^2}{(\rho^2+1)(\sigma^2+1)}=\frac{(\rho^2a^2+b^2)^2}{(\rho^2+1)(\rho^2a^4+b^4)}=\frac{a^2b^2}{p^2q^2};$$

hence

(25) $$pq\sin\vartheta=ab;$$

i. e. the product of the tangents of the lengths of two conjugate semi-diameters through either centre into the sine of the angle between them is constant, and reciprocally *the product of the tangents of the angles which two tangents, one from each of two conjugate points on either axis, make with that axis into the sine of the distance between the points is constant.*

The equation of the tangent to S at 4 is $S_{40}=0$, where

$$S_{40}\equiv\frac{\Omega_{14}\Omega_{10}}{a^2\Omega_{11}}+\frac{\Omega_{24}\Omega_{20}}{b^2\Omega_{22}}-\frac{\Omega_{34}\Omega_{30}}{\Omega_{33}}$$
$$\equiv\frac{\Omega_{14}}{\sqrt{\Omega_{11}}}\left(\frac{\Omega_{10}}{a^2\sqrt{\Omega_{11}}}+\rho\frac{\Omega_{20}}{b^2\sqrt{\Omega_{22}}}-\frac{\sqrt{\rho^2a^2+b^2}\,\Omega_{30}}{ab\sqrt{\Omega_{33}}}\right),$$

and the angle ϕ which this tangent makes with the diameter 35 conjugate to 34

is given by

$$\cos^2\phi = \frac{\left(\frac{\sigma}{a^2} - \frac{\rho}{b^2}\right)^2}{(\sigma^2+1)\left(\frac{1}{a^4} + \frac{\rho^2}{b^4} + \frac{\rho^2 a^2 + b^2}{a^2 b^2}\right)} = \frac{(\rho a^2 - \sigma b^2)^2}{(\sigma^2+1)[\rho^2 a^4(1+b^2) + b^4(1+a^2)]}$$

$$= \frac{\sigma^2+1}{\sigma^2(1+a^2)+(1+b^2)};$$

hence

$$\tan^2\phi = \frac{\sigma^2 a^2 + b^2}{\sigma^2+1} = \frac{a^2 b^2}{q^2},$$

or

$$q \tan\phi = ab, \tag{26}$$

i. e. the tangent of the length of a semi-diameter through either centre into the tangent of the angle which it makes with the tangent at the extremity of the conjugate semi-diameter through the same centre is constant, and reciprocally *the tangent of the angle which a tangent from a point on either axis makes with that axis into the tangent of the distance of that point from the point of contact of either tangent from the conjugate point on the same axis is constant.* The corresponding theorem in the Euclidean geometry will be found by introducing the constants k and k' with the values ∞ and $\frac{i}{2}$ respectively; then (26) becomes

$$\frac{q}{2ik}\tan\phi = \frac{a}{2ik}\,\frac{b}{2ik},$$

i. e.

$$\tan\phi = \frac{ab}{2ikq} = 0,$$

hence $\phi = 0$, *i. e.* any diameter is parallel to the tangents at the extremities of the conjugate diameter. In the general case this is not so, it is not even true that the tangents at the extremities of a diameter are parallel, in fact these tangents meet in the point where the conjugate diameter through the same centre intersects the axis opposite that centre; the angle between either tangent and the conjugate diameter is the angle ϕ in (26), and the angle between the two tangents is 2ϕ. In the Euclidean case the reciprocal of the theorem above stated has a real interpretation only for conjugate points on the *conjugate* axis of an hyperbola. Then any point on the conjugate axis has the same ordinate as the point of contact of a tangent from the conjugate point on the same axis, and the theorem may be expressed thus: the intercept on the conjugate axis of an hyperbola between any point of that axis and the perpendicular let fall from a point of the hyperbola having the same ordinate as the given point upon the tangent from the given point is constant.

These results, so far as they refer to centres and axes, only apply in general to ellipses and hyperbolas. The equation of a conic of any other species cannot be put into the form (21). It can, however, be put into an almost equally simple form. The forms which I give below in equations (27)–(30) are substantially those given by Clebsch ("Vorlesungen über Geometrie," I[ter] Band, 2[te] Abtheilung, VI). Clebsch has given also in the same place a method by which the species to which a conic belongs can be determined. If

$$\Omega_{00} \equiv \alpha x^2 + \beta y^2 + \gamma z^2 + 2\phi yz + 2\chi zx + 2\psi xy,$$
$$S_{00} \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy,$$

the species to which S belongs depends upon the cubic equation

$$\Delta(\lambda) \equiv \begin{vmatrix} \lambda\alpha - a, & \lambda\psi - h, & \lambda\chi - g \\ \lambda\psi - h, & \lambda\beta - b, & \lambda\phi - f \\ \lambda\chi - g, & \lambda\phi - f, & \lambda\gamma - c \end{vmatrix} = 0;$$

namely, if the roots of $\Delta(\lambda) = 0$ are all different, S meets Ω in four distinct points; if two roots are equal, S has contact with Ω; if two roots are equal and for them all the first minors of $\Delta(\lambda)$ vanish, S has double contact with Ω; if all the roots are equal, S meets Ω in three consecutive points; and if all the roots are equal and for them all the first minors of $\Delta(\lambda)$ vanish, S meets Ω in four consecutive points. Let $\Delta(\lambda)$ be developed and the terms arranged according to powers of λ, say

$$\Delta(\lambda) \equiv \begin{vmatrix} \alpha & \psi & \chi \\ \psi & \beta & \phi \\ \chi & \phi & \gamma \end{vmatrix} (\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3),$$

then the discriminant of $\Delta(\lambda)$ is

$$D = 4(3s_2 - s_1^2)(3s_1s_3 - s_2^2) - (9s_3 - s_1s_2)^2;$$

it is assumed that the system of coordinates is real, and that the coefficients in the equations of Ω and S are all real.

I. Ω imaginary.

a) If $D > 0$, S is an ellipse;

b) if $D = 0$, S is a circle.

II. Ω real, S real.

a) If $D > 0$, S is an ellipse or an hyperbola;

b) if $D < 0$, S is a semi-hyperbola;

c) if $D = 0$, S is an elliptic or hyperbolic parabola;

d) if $D = 0$ and all the first minors of $\Delta(\lambda)$ vanish for the double root of $\Delta(\lambda) = 0$, S is a circle;

e) if $\frac{s_1}{3} = \frac{s_2}{s_1} = \frac{3s_3}{s_2}$ (and therefore $D=0$), S is a semi-circular parabola;

f) if $\frac{s_1}{3} = \frac{s_2}{s_1} = \frac{3s_3}{s_2}$ and all the first minors of $\Delta(\lambda)$ vanish for the triple root of $\Delta(\lambda)=0$, S is a circular parabola.

III. Ω real, S imaginary.

a) If $D>0$, S is an imaginary ellipse;

b) if $D=0$, S is an imaginary circle.

It is hardly necessary to state the extra and intra-absolute positions of the centres and foci, and the finite and semi-infinite positions of the axes and focal lines, as they are geometrically evident. It may, however, be worth while to remark that a semi-hyperbola has only one real centre, one real axis, one pair of real foci and one pair of real focal lines; the elliptic or hyperbolic parabola has a double centre and a double axis coincident with the point of contact with Ω and the tangent at that point respectively, two double foci and two double focal lines; a circle has one isolated centre (*axial centre*), every line through which is an axis, and one isolated axis (*central axis*), every point of which is a centre, the axial centre is a quadruple focus, and the central axis is a quadruple focal line; a semi-circular parabola has an infinite triple centre, a triple axis tangent to Ω, two triple foci of which one is the triple centre, and two triple focal lines of which one is the triple axis; a circular parabola has an infinite double axial centre and a double central axis tangent to Ω, a sextuple focus coincident with the axial centre and a sextuple focal line coincident with the central axis.

The equations above given for Ω and S may be regarded as equations in trilinear coordinates x, y, z, where these are constant multiples of Ω_{10}, Ω_{20}, Ω_{30}. The forms of equation given by Clebsch, for what we have called parabolas of various species and circles are substantially then the following: for an hyperbolic or elliptic parabola, let 1 be the single centre, 2 the double centre, and 3 the pole (*quâ* Ω) of the single focal line other than the double axis, then $\Omega_{12}=0$, $\Omega_{22}=0$, $\Omega_{13}=0$,

$$(27) \qquad \Omega_{00} \equiv \frac{\Omega_{10}^2}{\Omega_{11}} - \frac{\Omega_{33}\Omega_{20}^2}{\Omega_{23}^2} + 2\frac{\Omega_{20}\Omega_{30}}{\Omega_{23}}, \quad S_{00} \equiv \frac{\Omega_{10}^2}{\Omega_{11}} - \frac{\Omega_{33}\Omega_{20}^2}{\Omega_{23}^2} + 2\lambda\frac{\Omega_{20}\Omega_{30}}{\Omega_{23}};$$

for a circle, let 1 be the axial centre, 2 and 3 the single foci, then

$$\Omega_{22}=0,\ \Omega_{33}=0,\ \Omega_{12}=0,\ \Omega_{13}=0,$$

$$(28) \qquad \Omega_{00} \equiv \frac{\Omega_{10}^2}{\Omega_{11}} + 2\frac{\Omega_{20}\Omega_{30}}{\Omega_{23}}, \quad S_{00} \equiv \frac{\Omega_{10}^2}{\Omega_{11}} + 2\lambda\frac{\Omega_{20}\Omega_{30}}{\Omega_{23}};$$

for a semi-circular parabola, let 1 be the triple centre, 3 the pole (*quâ* Ω) of the triple focal line other than the axis, and 2 the pole (*quâ* Ω) of the anharmonic conjugate of $\Omega_{30}=0$ with respect to the tangents to Ω and S at their single intersection, then $\Omega_{11}=0$, $\Omega_{12}=0$, $\Omega_{22}\Omega_{33}-\Omega_{23}^2=0$,

$$(29)\quad \Omega_{00}\equiv\frac{\Omega_{20}^2}{\Omega_{22}}-2\frac{\Omega_{23}\Omega_{10}\Omega_{20}}{\Omega_{22}\Omega_{13}}+2\frac{\Omega_{10}\Omega_{30}}{\Omega_{13}},\qquad S_{00}\equiv\lambda\frac{\Omega_{20}^2}{\Omega_{22}}-2\frac{\Omega_{23}\Omega_{10}\Omega_{20}}{\Omega_{22}\Omega_{13}}-2\frac{\Omega_{10}\Omega_{30}}{\Omega_{13}};$$

for a circular parabola, let 1 be the axial centre, 2 any point on the central axis, and 3 any point not on the central axis, then $\Omega_{11}=0$, $\Omega_{12}=0$,

$$(30)\quad \begin{cases}\Omega_{00}\equiv-\dfrac{(\Omega_{22}\Omega_{33}-\Omega_{23}^2)\Omega_{10}^2}{\Omega_{22}\Omega_{13}^2}+\dfrac{\Omega_{20}^2}{\Omega_{22}}-2\dfrac{\Omega_{23}\Omega_{10}\Omega_{20}}{\Omega_{22}\Omega_{13}}+2\dfrac{\Omega_{10}\Omega_{30}}{\Omega_{13}},\\[2ex] S_{00}\equiv-\lambda\dfrac{(\Omega_{22}\Omega_{33}-\Omega_{23}^2)\Omega_{10}^2}{\Omega_{22}\Omega_{13}^2}+\dfrac{\Omega_{20}^2}{\Omega_{22}}-2\dfrac{\Omega_{23}\Omega_{10}\Omega_{20}}{\Omega_{22}\Omega_{13}}+2\dfrac{\Omega_{10}\Omega_{30}}{\Omega_{13}}.\end{cases}$$

There is, however, a simple form of equation which applies to all conics excepting semi-circular parabolas, which I give below as (31). Every ellipse, hyperbola, semi-hyperbola, elliptic, hyperbolic or circular parabola, or circle, has at least one real finite (*i. e.* not situated on Ω) centre, such that the opposite axis is real and transverse (both to Ω and S), not a chord of double contact of S and Ω. In the case of the circle the centre in question is any point of the central axis, and the axis in question passes through the axial centre. That such a real centre exists for all the conics named excepting the semi-hyperbola is evident, and for the semi-hyperbola it is only necessary to prove that the real axis is transverse. To prove this let $z=0$ be the chord joining the real intersections of S and Ω (both real conics), $y=0$ the real axis, $x=0$ the polar, *quâ* Ω, of $(1,0,0)$; then Ω_{00} and S_{00} can evidently be written

$$\Omega_{00}\equiv x^2\pm y^2-z^2,\quad S_{00}\equiv x^2+\lambda y^2-z^2+\mu xy.$$

The intersections of Ω and S lie on the pair of lines

$$(\lambda\mp1)y^2+\mu xy=0,$$

i. e. the imaginary intersections lie on

$$(\lambda\mp1)y+\mu x=0,$$

and for these intersections then

$$\{(\lambda\mp1)^2\pm\mu^2\}y^2-\mu^2z^2=0,$$

hence $(\lambda\mp1)^2\pm\mu^2<0$, and therefore the lower sign must be taken, and

$$\Omega_{00}\equiv x^2-y^2-z^2,\quad S_{00}\equiv x^2+\lambda y^2-z^2+\mu xy,$$

where $(\lambda+1)^2-\mu^2<0$; the intersections of S with the real axis are given by

$$z=0,\ x^2+\lambda y^2+\mu xy=0,$$

and by virtue of the condition for λ and μ the factors of the last equation are real; hence the axis is transverse to S, and it is evidently transverse to Ω.

Returning now to the general case for all the conics named, let $y=0$ be the real axis, $x=0$ the tangent at either vertex on $y=0$ (other than the point of contact with Ω, if such exist), $z=0$ the polar, *quâ* Ω, of $(0, 0, 1)$, then Ω_{00} and S_{00} may be written

(31) $$\Omega_{00}\equiv x^2-y^2-z^2, \quad S_{00}\equiv ax^2+by^2+cxz,$$

where $c^2+4(a+b)b \gtreqless 0$ (this condition is only necessary when $c^2<a^2$). The conditions for the different species of conics are easily seen to be the following:

for an hyperbola, $c^2<a^2$, $\dfrac{b}{a}<-\dfrac{1}{2}$, and $c^2+4(a+b)b>0$;

" " ellipse, $c^2<a^2$, $\dfrac{b}{a}>-\dfrac{1}{2}$, $c^2+4(a+b)b>0$;

" a semi-hyperbola, $c^2>a^2$;

" an hyperbolic parabola, $c^2=a^2$ and $\dfrac{b}{a}<-\dfrac{1}{2}$;

" " elliptic parabola, $c^2=a^2$ and $\dfrac{b}{a}>-\dfrac{1}{2}$;

" a circular parabola, $c^2=a^2$ and $\dfrac{b}{a}=-\dfrac{1}{2}$;

for a circle having real contacts with Ω, $c^2+4(a+b)b=0$ and $\dfrac{a}{b}>-2$;

for a circle having imaginary contacts with Ω, $c^2+4(a+b)b=0$ and $\dfrac{a}{b}<-2$.

If S is a semi-circular parabola we may write

(32) $$\Omega_{00}\equiv 2xy-z^2, \quad S_{00}\equiv 2xy+2byz-z^2,$$

where $y=0$ is the triple axis (and focal line) of S, $z=0$ the other triple focal line, and $x=0$ the tangent to Ω at the single absolute point of S.

It may be remarked that (31) may be written

(33) $$\Omega_{00}\equiv\frac{\Omega_{10}^2}{\Omega_{11}}+\frac{\Omega_{20}^2}{\Omega_{22}}+\frac{\Omega_{30}^2}{\Omega_{33}}, \quad S_{00}\equiv a\frac{\Omega_{10}^2}{\Omega_{11}}-b\frac{\Omega_{20}^2}{\Omega_{22}}+c\frac{\Omega_{10}\Omega_{30}}{\sqrt{-\Omega_{11}\Omega_{33}}},$$

where 1 is intra-absolute, and 2 and 3 extra-absolute; and (32) may be written

(34) $$\Omega_{00}\equiv 2\frac{\Omega_{10}\Omega_{20}}{\Omega_{12}}+\frac{\Omega_{30}^2}{\Omega_{33}}, \quad S_{00}\equiv 2\frac{\Omega_{10}\Omega_{20}}{\Omega_{12}}+\frac{\Omega_{30}^2}{\Omega_{33}}+2b\frac{\Omega_{20}\Omega_{30}}{\sqrt{-\Omega_{12}\Omega_{33}}},$$

where 2 is the triple centre of S, 1 the single absolute point, and 3 the pole *quâ* Ω of the junction of the triple and single absolute points.

The circle is particularly interesting, on account of the simplicity of its metrical properties. Equation (3) becomes, for a circle,

(35) $$S_{00} \equiv \lambda \Omega_{11} \Omega_{00} - \Omega_{10}^2,$$

where 1 is the axial centre, and $\Omega_{10} = 0$ the central axis. From this equation follows, for any point of S,

$$\frac{\Omega_{10}^2}{\Omega_{11} \Omega_{00}} = \lambda,$$

i. e.

(36) $$\overline{10} = \cos^{-1} \sqrt{\lambda},$$

i. e. the distance of any point of a circle from its axial centre is constant, and reciprocally *the angle which any tangent of a circle makes with its central axis is constant,* for the reciprocal, *quâ* Ω, of a conic having double contact with Ω is another conic having double contact with Ω at the same points, *i. e.* a circle concentric with the former one, and every theorem concerning a circle has its exact reciprocal.

Instead of (35) we may evidently write

(37) $$\Omega_{00} \equiv \mu S_{11} S_{00} - S_{10}^2,$$

whence, if $S_{00} = 0$,

$$\Omega_{00} = - S_{10}^2, \; \Omega_{10} = (\mu - 1) S_{11} S_{10}, \; \Omega_{11} = (\mu - 1) S_{11}^2,$$

$$\frac{\Omega_{10}^2}{\Omega_{11} \Omega_{00}} = 1 - \mu;$$

hence, by comparison with the previous values of the left-hand member,

(38) $$\lambda = 1 - \mu, \; \overline{10} = \sin^{-1} \sqrt{\mu}.$$

Let 2 be any point on the central axis of S, and 3 any point of S, the line 23 intersects S again in a point 4; then

$$x_4 = \rho x_2 + \sigma x_3, \; y_4 = \rho y_2 + \sigma y_3, \; z_4 = \rho z_2 + \sigma z_3, \; S_{12} = 0, \; S_{33} = 0, \; S_{44} = 0,$$

but $$S_{44} \equiv \rho^2 S_{22} + 2 \rho \sigma S_{23},$$

and therefore $\rho : \sigma = 2 S_{23} : - S_{22}$, and say

$$x_4 = 2 S_{23} x_2 - S_{22} x_3, \; y_4 = 2 S_{23} y_2 - S_{22} y_3, \; z_4 = 2 S_{23} z_2 - S_{22} z_3,$$

$$S_{14} = - S_{22} S_{13}, \quad S_{24} = S_{22} S_{23},$$

$$\Omega_{23} = \mu S_{11} S_{23}, \quad \Omega_{24} = \mu S_{11} S_{24} = \mu S_{11} S_{22} S_{23},$$

$$\Omega_{22} = \mu S_{11} S_{22}, \quad \Omega_{33} = - S_{13}^2, \quad \Omega_{44} = - S_{14}^2,$$

$$\frac{\Omega_{23}^2}{\Omega_{22} \Omega_{33}} = - \mu \frac{S_{11} S_{23}^2}{S_{22} S_{13}^2} = \frac{\Omega_{24}^2}{\Omega_{22} \Omega_{44}},$$

i. e. $$\overline{24} = - \overline{23},$$

but $\overline{24}$ is not independent of the position of 2; hence *every chord of a circle is bisected by the central axis,* and reciprocally *the angle between any two tangents of a*

circle is bisected by the junction of the axial centre with their intersection. In this sense, then, every point of the central axis is a centre, and every line through the axial centre is an axis.

Still using (37), let 2 be any point in the plane and 3 the point of contact of either tangent from 2 to S; then

$$S_{23}=0, \quad S_{33}=0,$$

$$\Omega_{23}=-S_{12}S_{13}, \quad \Omega_{22}=\mu S_{11}S_{22}-S_{12}^2, \quad \Omega_{33}=-S_{13}^2,$$

$$\frac{\Omega_{23}^2}{\Omega_{22}\Omega_{33}}=-\frac{S_{12}^2}{\mu S_{11}S_{22}-S_{12}^2}=\frac{\Omega_{12}^2}{(1-\mu)\Omega_{11}\Omega_{22}}=\frac{\Omega_{12}^2}{\lambda\Omega_{11}\Omega_{22}},$$

i. e. $\cos\overline{23}=\frac{1}{\sqrt{\lambda}}\cos\overline{12}$, or

(39) $$\tan\overline{23}=\frac{\sqrt{-\mu S_{11}S_{22}}}{S_{12}}=\frac{\sqrt{(1-\mu)\Omega_{11}\Omega_{22}-\Omega_{12}^2}}{\Omega_{12}},$$

which is independent of any choice between the two tangents from the point 2, *i. e. the two tangents to a circle from any point are of equal length,* and reciprocally *any two tangents of a circle make equal angles with the chord joining their points of contact.*

The tangent to S, given by (35), at any point 2 has the equation

$$S_{20}\equiv\lambda\Omega_{11}\Omega_{20}-\Omega_{12}\Omega_{10}=0,$$

whose pole, *quâ* Ω, evidently lies on the radius 12, and hence *any tangent of a circle is perpendicular to the radius to its point of contact from the axial centre,* and reciprocally *any point of a circle is perpendicular to the intersection of the tangent at that point with the central axis.*

The theory of the radical axes of two or three circles may be deduced geometrically (as in Salmon's Conic Sections, Art. 306), or we may proceed analytically. Let S', S'', S''' be three circles, where

(40) $$S'_{00}\equiv\lambda'\Omega_{11}\Omega_{00}-\Omega_{10}^2, \quad S''_{00}\equiv\lambda''\Omega_{22}\Omega_{00}-\Omega_{20}^2, \quad S'''_{00}\equiv\lambda'''\Omega_{33}\Omega_{00}-\Omega_{30}^2.$$

Through the intersections of S' and S'' pass three pairs of straight lines of which one pair is

(41) $$\lambda''\Omega_{22}\Omega_{10}^2-\lambda'\Omega_{11}\Omega_{20}^2=0,$$

and the peculiarity of this pair is that its lines intersect in the intersection of the central axes of S' and S''. The common chords of two circles which pass through the intersection of the central axes of the circles may be called their *radical axes.* Every pair of circles has then two radical axes, and similarly every pair of circles has two *radical centres,* those intersections of their common

tangents which lie on the junction of their axial centres. From (41) follows directly, for any point of either radical axis,

$$(42)\qquad \frac{\Omega_{10}^2}{\lambda'\,\Omega_{11}\,\Omega_{00}}=\frac{\Omega_{20}^2}{\lambda''\,\Omega_{22}\,\Omega_{00}},$$

i. e., by comparison with (39), *the four tangents to two circles from any point of either of their radical axes are of equal length*, and reciprocally *any line through either radical centre of two circles makes equal angles with the tangents to them at the four points where it intersects them.* The three pairs of radical axes of the circles S', S'', S''', taken two at a time, are

$$\lambda'''\,\Omega_{33}\Omega_{20}^2-\lambda''\,\Omega_{22}\Omega_{30}^2=0\,,\ \ \lambda'\,\Omega_{11}\Omega_{30}^2-\lambda'''\,\Omega_{33}\Omega_{10}^2=0\,,\ \ \lambda''\,\Omega_{22}\Omega_{10}^2-\lambda'\,\Omega_{11}\Omega_{20}^2=0\,,$$

i. e. separately

$$\sqrt{\lambda'''\,\Omega_{33}}\,\Omega_{20}\pm\sqrt{\lambda''\,\Omega_{22}}\,\Omega_{30}=0\,,\ \ \sqrt{\lambda'\,\Omega_{11}}\,\Omega_{30}\pm\sqrt{\lambda'''\,\Omega_{33}}\,\Omega_{10}=0\,,$$
$$\sqrt{\lambda''\,\Omega_{22}}\,\Omega_{10}\pm\sqrt{\lambda'\,\Omega_{11}}\,\Omega_{20}=0\,,$$

from which it is evident that *the six lines forming the three pairs of radical axes pass by threes through four points, the "orthogonal centres" of the three circles*, namely through each orthogonal centre passes one radical axis of each pair; and reciprocally *the six radical centres of three circles lie by threes on four straight lines, the "orthostatic axes" of the three circles*, namely on each orthostatic axis lies one radical centre of each pair. The six tangents to the three circles from either orthogonal centre are of equal length, and hence their points of contact lie on the same circle, whose centre is the orthogonal centre and which cuts the given circles perpendicularly in these points, *i. e.* it is *orthogonal* to the given circles. Any three circles have then four *orthogonal circles.* Reciprocally either orthostatic axis makes equal angles with the tangents to the three circles at its six intersections with them, hence these tangents all touch the same circle, whose central axis is the orthostatic axis, and whose contacts with these tangents are perpendicular to the contacts of the given circles with the same, *i. e.* this circle is *orthostatic* to the given circles. Thus any three circles have four *orthostatic circles.* It may be remarked that two circles intersect in four points and have four common tangents, but two and only two of the intersections can be orthogonal, and the contacts of two and only two of the tangents can be orthostatic. Let $\lambda'\Omega_{11}=\alpha'^2$, $\lambda''\Omega_{22}=\alpha''^2$, $\lambda'''\Omega_{33}=\alpha'''^2$; then (giving any possible combination of signs to α', α'', α''') $\alpha'''\Omega_{20}-\alpha''\Omega_{30}=0$, $\alpha'\Omega_{30}-\alpha'''\Omega_{10}=0$, $\alpha''\Omega_{10}-\alpha'\Omega_{20}=0$, are three radical axes meeting in the same orthogonal centre 4, for which then

$$\Omega_{14}:\Omega_{24}:\Omega_{34}=\alpha':\alpha'':\alpha''',$$

say

(43) $$\Omega_{14} = ta', \ \Omega_{24} = ta'', \ \Omega_{34} = ta''',$$

and for the contacts of tangents from A to S' we have

$$S'_{00} = 0, \ S'_{40} = 0,$$

i. e. $$\alpha'^2\Omega_{00} - \Omega_{10}^2 = 0, \ \alpha'^2\Omega_{40} - \Omega_{14}\Omega_{10} \equiv \alpha'(\alpha'\Omega_{40} - t\Omega_{10}) = 0,$$

from which follows

(44) $$t^2\Omega_{00} - \Omega_{40}^2 = 0,$$

which is symmetrical with respect to S', S'' and S''', and hence is the equation of the orthogonal circle corresponding to the given combination of signs of α', α'' and α'''. The ratios $x_4 : y_4 : z_4 : t$ are determined by (43) and these are to be substituted in (44).

Similarly let $(1-\lambda')\Omega_{11} = \beta'^2$, $(1-\lambda'')\Omega_{22} = \beta''^2$, $(1-\lambda''')\Omega_{33} = \beta'''^2$; then the radical centres of S' and S'' are those points of the line 12 from which tangents to S and S' will coincide, say one of these points has the coordinates $\rho x_1 + \sigma x_2$, $\rho y_1 + \sigma y_2$, $\rho z_1 + \sigma z_2$, and the equation of the pair of tangents from this point to S' is found to be

$$[\sigma^2(\Omega_{11}\Omega_{22} - \Omega_{12}^2) - (\rho^2\Omega_{11} + 2\rho\sigma\Omega_{12} + \sigma^2\Omega_{22})\beta'^2]\Omega_{00} - (\sigma^2\Omega_{22} - \rho^2\beta'^2)\Omega_{10}^2$$
$$- \sigma^2(\Omega_{11} - \beta'^2)\Omega_{20}^2 + 2\sigma(\sigma\Omega_{12} - \rho\beta'^2)\Omega_{10}\Omega_{20} = 0,$$

i. e. $$\sigma^2[(\Omega_{11}\Omega_{22} - \Omega_{12}^2)\Omega_{00} - \Omega_{22}\Omega_{10}^2 - \Omega_{11}\Omega_{20}^2 + 2\Omega_{12}\Omega_{10}\Omega_{20}]$$
$$- \beta'^2[(\rho^2\Omega_{11} + 2\rho\sigma\Omega_{12} + \sigma^2\Omega_{22})\Omega_{00} - \rho^2\Omega_{10}^2 - \sigma^2\Omega_{20}^2 + 2\rho\sigma\Omega_{10}\Omega_{20}] = 0,$$

and in order that these shall coincide with the tangents to S'' it is necessary and sufficient that $\rho^2 : \sigma^2 = \beta''^2 : \beta'^2$, *i. e.* $\rho : \sigma = \beta'' : \pm\beta'$ and the equation of the polar (*quâ* Ω) of the radical centre is

(45) $$\beta''\Omega_{10} \pm \beta'\Omega_{20} = 0;$$

the combined equation of the polars, *quâ* Ω, of the two radical centres of S and S' is then $(1-\lambda'')\Omega_{22}\Omega_{10}^2 - (1-\lambda')\Omega_{11}\Omega_{20}^2$.

If $\Omega_{50} = 0$ is the equation of an orthostatic axis of S', S'' and S''', then, if any combination of signs is given to β', β'' and β''', by (45)

$$\Omega_{15} : \Omega_{25} : \Omega_{35} = \beta' : \beta'' : \beta''',$$

i. e., say

(46) $$\Omega_{15} = t'\beta', \ \Omega_{25} = t'\beta'', \ \Omega_{35} = t'\beta'''.$$

The locus of poles, *quâ* Ω, of tangents to the orthostatic circle of S, S' and S'', say to $$\mu\Omega_{55}\Omega_{00} - \Omega_{50}^2 = 0$$

is given by the equation

(47) $$(1-\mu)\Omega_{55}\Omega_{00} - \Omega_{50}^2 = 0;$$

and if 6 is either point of intersection of the orthostatic axis $\Omega_{50}=0$ with S', we have

(48) $$\Omega_{56}=0,\ \lambda'\Omega_{11}\Omega_{66}-\Omega_{16}^2=0,$$

and the tangent to S' at 6 is

$$\lambda'\Omega_{11}\Omega_{60}-\Omega_{16}\Omega_{10}=0,\ \textit{i. e.}\ \Omega_{16}\Omega_{60}-\Omega_{66}\Omega_{10}=0,$$

i. e. the pole of this tangent, *quâ* Ω, has the coordinates

$$\Omega_{16}x_6-\Omega_{66}x_1,\ \Omega_{16}y_6-\Omega_{66}y_1,\ \Omega_{16}z_6-\Omega_{66}z_1,$$

and this pole must then lie on the circle (47), whence follows, on reduction by (48),

$$(1-\mu)(1-\lambda')\Omega_{11}\Omega_{55}-\Omega_{15}^2=0,$$

i. e. $$\mu=\frac{\Omega_{55}-t'^2}{\Omega_{55}},$$

and the equation of the orthostatic circle corresponding to the given combination of signs of β', β'', β''' is

(49) $$(\Omega_{55}-t'^2)\Omega_{00}-\Omega_{50}^2=0,$$

and the locus of poles, *quâ* Ω, of its tangents is given by the equation

(50) $$t'^2\Omega_{00}-\Omega_{50}^2=0,$$

where the ratios $x_5:y_5:z_5:t'$ are determined by (46).

BALTIMORE, *June*, 1883.

*Di un nuovo teorema relativo alla rotazione di un corpo ad un asse.**

Nota del Prof. Dominico Turazza.

Si riferisca il sistema a tre assi rettangolari passandi pel baricentro e per brevità di scrittura pongasi

$$\Sigma x^2 \Delta m = m.e^2, \quad \Sigma y^2 \Delta m = m.f^2, \quad \Sigma z^2 \Delta m = m.g^2,$$
$$\Sigma xy \Delta m = m.ch, \quad \Sigma xz \Delta m = m.ck, \quad \Sigma yz \Delta m = m.cl$$

essendo m la massa del corpo.

Ora nella mia memoria "di alcune proprietà relative agli assi di rotazione di un sistema rigido" io ho dimostrato che per imprimere al sistema un moto di rotazione con velocità angolari w intorno ad asse parallelo all' asse z e distante δ dal baricentro è mestieri impiegare una forza $mF = -mw.d$ ed un giratore $mG = -m.cl$ applicati al centro di giratore minimo, che è quel punto del piano XZ che a per coordinati

$$x_m = -\frac{e^2+f^2}{\delta}, \quad z_m = -\frac{ck}{\delta}$$

diretti perpendicularmenti al piano XZ e in senso contrario ad Y se il corporota da X verso Y.

Problema. Abbiasi un sistema rigido di massa m fisso a due perni A e B intorno cie quali possa liberamente girare: a questo sistema venga trasmessa una determinata quantità di moto mF in data direzioni; si domanda la velocità angolari iniziale che prenderà il sistema e le pressioni che sopporteranno i cardini.

Si riferisca il sistema a tre assi rettangolari passandi pel baricentro; l'uno (Z) parallelo all' asse di rotazione; l'altro (Y) perpendicolare al piano che passa per l'asse e pel baricentro; e il terzo (X) perpendicolare a questo piano; sieno α e β le coordinate del punto in cui la quantità di moto trasmessa incontra il piano XZ, ed a e b le distanze dei due cardini dal piano XY. Si decomponga la forza mF in tre, l'una mf_x parallela all' asse x; l'altra mf_y parallela ad Y e la terza mf_z parallela ad z, e rimossi i cardini si sostituiscano in loro luogo due forze equali e direttamente contrarie alle pressioni sopportate dai medisimi, e decomposte queste in tre secondo i tre assi sieno mp_x; mp_y; mp_z le tre relative

---

* Owing to some oversight, this article, which was received about three years ago, was mislaid and has only just come to light.—Ed.

al cardine A, ed mp'_x; mp'_y; mp'_z le tre relative al cardini B; sia finalmente w la velocità angolare iniziale intorno all' asse AB. Trasportando tutte le forze al centro di giratore minimo, queste dovranno ridursi unicamente alla forza $-mwd$, ed al giratore $-mcl$, e quindi, dividendo tutto per w dovranno sussistere le equazioni sequenti:

$$\text{(I)} \qquad f_x+p_x+p'_x=0\,;\ f_y+p_y+p'_y=-w.\delta\,;\ f_z+p_z+p'_z=0,$$

$$\text{(II)} \left\{\begin{array}{l} \left\{\beta+\frac{ck}{\delta}\right\}f_y+\left\{a+\frac{ck}{\delta}\right\}p_y+\left\{b+\frac{ck}{\delta}\right\}p'_y=0 \\ \left\{\beta+\frac{ck}{\delta}\right\}f_y+\left\{a+\frac{ck}{\delta}\right\}p_x+\left\{b+\frac{ck}{\delta}\right\}p'_x-\left\{\alpha+\frac{e^2+f^2}{\delta}\right\}f_z \\ \qquad\qquad -\left\{\delta+\frac{e^2+f^2}{\delta}\right\}\{p_z+p'_z\}=-w.cl \\ \left\{\alpha+\frac{e^2+f^2}{\delta}\right\}f_y+\left\{\delta+\frac{e^2+f^2}{\delta}\right\}\{p_y+p'_y\}=0. \end{array}\right.$$

Donde ponendo per brevità

$$\delta^2+e^2+f^2=i^2.$$

$$\text{(III)} \qquad w=\frac{(\alpha-\delta).f_y}{i^2},$$

$$\text{(IV)} \left\{\begin{array}{l} p_x=\ \ \frac{(\alpha-\delta).c.l}{(b-a)i^2}.f_y-\frac{\alpha-\delta}{b-a}.f_z-\frac{b-\beta}{b-a}.f_x \\ p'_x=-\frac{(\alpha-\delta).c.l}{(b-a).i^2}.f_y+\frac{\alpha-\delta}{b-a}.f_z+\frac{a-\beta}{b-a}.f_x \\ p_y=-\left\{\frac{(\alpha-\delta)(b.\delta+ck)}{(b-a)i^2}+\frac{b-\beta}{b-a}\right\}.f_y \\ p'_y=\ \ \left\{\frac{(\alpha-\delta)(a.\delta+ck)}{(b-a)i^2}+\frac{a-\beta}{b-a}\right\}.f_y \\ p_z+p'_z=-f_z \end{array}\right.$$

le quali equazioni risolvono completamente il problema proposto.

Se la direzione della quantita di moto trasmessa incontra il piano XZ nel centro di giratore minimo allora sarà

$$\alpha=x_m=-\frac{e^2+f^2}{\delta}\,;\ \beta=z_m=-\frac{ck}{\delta},$$

medianti i quali valori le equazioni superiori diventano.

$$\text{(V)} \left\{\begin{array}{l} w=-\frac{f_y}{\delta}\,;\ p_y=0\,;\ p'_y=0\,;\ p_z+p'_z=-f_z \\ p_x=-\frac{cl}{\delta(b-a)}.f_y+\frac{i^2}{\delta(b-a)}.f_z-\frac{b.\delta+ck}{\delta(b-a)}.f_x\,; \\ p'_x=\frac{cl}{\delta(b-a)}.f_y-\frac{i^2}{\delta(b-a)}.f_z+\frac{a\delta+ck}{\delta(b-a)}.f_x, \end{array}\right.$$

dalle quali equazioni risulta che, essendo essenzialmente a differente di b, non possono p_x e p'_x essere zero a meno che non sia in primo luogo $f_x = 0$, ossia a meno che la quantità di moto trasmessa non giacia in un piano perpendicolare all' asse X, cioè parallelo all' asse dato e perpendicolare al piano che passa per l'asse stesso e pel baricentro; quando ciò avvenga allora tanto p_x quanto p'_x saronno nullo cosi se sia

$$cl.f_y - i^2 f_z = 0$$

come se sia $f_z = 0$, e $cl = 0$ in quest' ultimo caso è pure $p_z + p'_z = 0$ il caso in cui sia $f_z = 0$, e $cl = 0$ corrisponde al caso già noto, ed essendo pure $p_z + p'_z = 0$ l'asse non striscione; se invece sia $cl.f_y = i^2 f_z$ ossia quando la direzione della quantità di moto trasmessa, passando pel centro di giratore minimo, giace in un piano parallelo all' asse e perpendicolare al piano che passa per l'asse e pel baricentro, e formi con questo piano un angolo la cui tangente trigonometrica è espressa da $\frac{i^2}{cl}$, allora pure l'asse non soffre percossa; il corpo prende spontaneamente a girare intorno all' asse desso, solo l'asse puo strisciare lungo sua direzione.

ERRATA.

VOL. V.—*The Intersection of Circles, etc.*

p. 34, last half of third line should read "*a case which*," instead of "the case in which they do intersect."

Asterisk omitted to note on p. 34 by Prof. Cayley, answering to the * on line ninth.